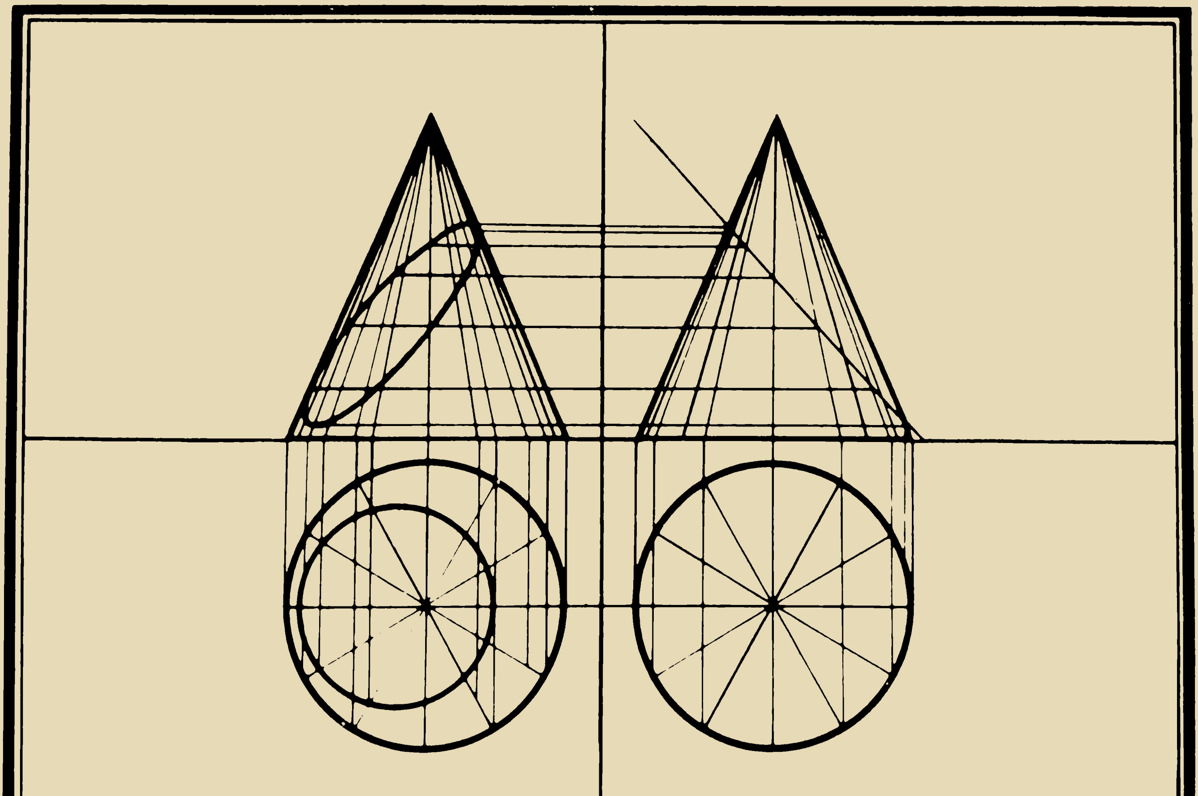
Kh. A. ARUSTAMOV

# PROBLEMS IN DESCRIPTIVE GEOMETRY



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**KHRISTOFOR ARTEMYEVICH ARUSTAMOV,** Professor at the Bauman Higher Technical School in Moscow, has been heading the chair of descriptive geometry and drawing for over forty years. He has written a number of manuals on solving problems in descriptive geometry.

ИЗДАТЕЛЬСТВО · МИР ·

*Х. А. АРУСТАМОВ*

# *СБОРНИК ЗАДАЧ ПО НАЧЕРТАТЕЛЬНОЙ ГЕОМЕТРИИ*

ИЗДАТЕЛЬСТВО «МАШИНОСТРОЕНИЕ» · МОСКВА

Kh. ARUSTAMOV

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*TO THE READER*

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# CONTENTS

# PART ONE

## CHAPTER I

## ORTHOGONAL PROJECTION

### Check-Up Questions

Construct two planes taken as the projection planes, give the names and designations of the planes, half-planes, four quadrants thus obtained (Figs. 1 and 2), and give full answers to the following questions:

1. What is the coordinate axis (the axis of projection)?
2. What half-planes form the first (I), second (II), third (III), and fourth (IV) quadrants?
3. What serves as a boundary between the following pairs of quadrants: I and II, III and IV, I and IV, II and III?

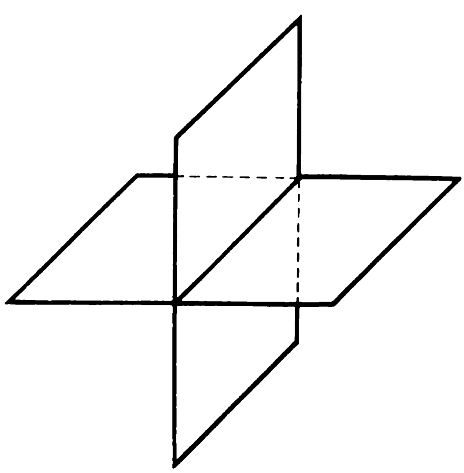


FIG. 1.

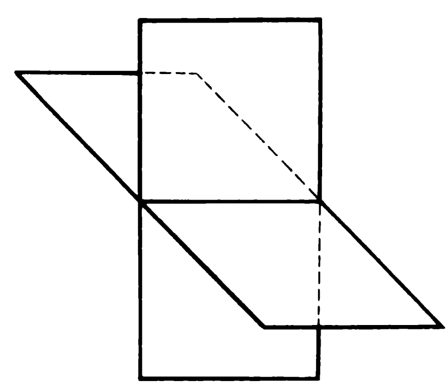


FIG. 2.

4. What quadrants are situated *above* the horizontal plane of projection, *below* the horizontal plane of projection, *in front of* the vertical plane of projection, *behind* the vertical plane of projection?
5. What is the position (relative to the projection planes) of a point situated in the first, second, third, and fourth quadrants, respectively?
6. Where is a point found if it is located between the following pairs of quadrants: I and IV, II and III, I and II, III and IV?

7. Where is a point found if it is located on the common boundary of all the four quadrants?

8. What is the orthogonal projection of a point onto a plane?

9. What is the horizontal projection of a point, the vertical projection of a point?

10. In what half-planes do the projections of a point lie if it is situated in the first, second, third, and fourth quadrant, respectively?

11. Where are the projections of a point found if it lies in the front half-plane of the horizontal plane of projection, in the rear half-plane of the horizontal plane of projection, in the upper half-plane of the vertical plane of projection, in the lower half-plane of the vertical plane of projection, on the coordinate axis?

12. What is common to all points situated in the horizontal plane of projection, in the vertical plane of projection?

13. What is the position of a point if its horizontal projection lies in the front half-plane of the horizontal plane of projection, in the rear half-plane of the horizontal plane of projection; if its vertical projection lies in the upper half-plane of the vertical plane of projection, in the lower half-plane of the vertical plane of projection?

14. What is an orthographic drawing of a point and how can it be made from a projection drawing?

15. What half-planes of projection, when brought into coincidence, will be found above the coordinate axis, below the coordinate axis?

16. Where is a point found if its horizontal projection on the orthographic drawing is located above the coordinate axis, below the coordinate axis; if its vertical projection is located above the coordinate axis, below the coordinate axis?

CHAPTER II

# THE POINT

Points of space are usually denoted by capital letters $A$, $B$, $C$, $D$, etc., and their projections by the corresponding lower-case letters $a$, $b$, $c$, $d$, etc. for horizontal projections, and $a'$, $b'$, $c'$, $d'$, etc. for vertical projections.

Both projections (horizontal and vertical) of a point are always situated on straight lines perpendicular to the axis and intersecting this axis at one and the same point.

| *If in space* | *then on an orthographic drawing* |
|---|---|
| (1) a point is in the first quadrant, | its horizontal projection lies below the axis, while its vertical projection is above the axis; |
| (2) a point is in the second quadrant, | its horizontal and vertical projections lie above the axis; |
| (3) a point is in the third quadrant, | its horizontal projection lies above the axis, while its vertical projection is below the axis; |
| (4) a point is in the fourth quadrant, | its horizontal and vertical projections lie below the axis.[1] |

Any point in the horizontal plane of projection has its vertical projection on the coordinate axis.

[1] And vice versa, *if on an orthographic drawing* the horizontal projection of a point lies below the coordinate axis and its vertical projection is above the axis, *then in space* the point is found in the first quadrant.

*If on an orthographic drawing* both projections of a point are above the axis, *then in space* the point is found in the second quadrant, and so on.

Any point in the vertical plane of projection has its horizontal projection on the coordinate axis.

If both projections of a point coincide and lie on the coordinate axis, then the point is located on the coordinate axis.

The distance $y$ from the horizontal projection of a point to the coordinate axis is equal to the distance from the point to the vertical plane of projection.

The distance $z$ from the vertical projection of a point to the coordinate axis is equal to the distance from the point to the horizontal plane of projection.

The coordinate $z$ is positive for points located above the horizontal plane of projection and negative for those situated below that plane.

The coordinate $y$ is positive for points located in front of the vertical plane of projection and negative for those situated behind that plane.

## Check-Up Questions

Give full answers to the following questions:

1. How are points of space usually denoted?
2. How are the projections of a point of space denoted and how are they distinguished?
3. How are the projections of one and the same point of space on an orthographic drawing situated relative to the coordinate axis?
4. Does an orthographic drawing make any sense if the perpendiculars dropped from the projections of a point onto the coordinate axis do not meet?
5. What is the meaning of the expression "Given: a point in space"?
6. Where are the projections of a point located on an orthographic drawing if the point is found in the first, second, third, fourth quadrant, respectively?
7. When can the horizontal and vertical projections of a point of space coincide outside the coordinate axis?
8. How can the position of a point in space be found from its projections?
9. Where are the projections of a point found on an orthographic drawing if the point is located in the front half-plane of the horizontal plane of projection, in the rear half-plane of the horizontal plane of projection, in the upper half-plane of the vertical plane of projection, in the lower half-plane of the vertical plane of projection?
10. How do we designate the distance from a point of space to the horizontal plane of projection, to the vertical plane of projection?
11. How do we determine, on an orthographic drawing, the distance from a point of space to the horizontal plane of projection, to the vertical plane of projection?
12. In what quadrants is the coordinate $z$ positive, negative?
13. In what quadrants is the coordinate $y$ positive, negative?
14. What signs have the coordinates $y$ and $z$ of a point lying in the first, second, third, and fourth quadrants, respectively?
15. What coordinate is determined on an orthographic drawing by the horizontal projection of a point, by the vertical projection of a point?
16. How is a line segment determining the coordinate $z$ or $y$ drawn on an orthographic drawing if the coordinate is positive, negative?

## EXAMPLES

**Example 1**

Construct an orthographic drawing of the point $A$ located in the second quadrant at a distance of 32 mm from the horizontal plane of projection and 18 mm from the vertical plane of projection (Fig. 3).

*Solution.* Assume an arbitrary point ($a_x$) on the coordinate axis and erect a perpendicular to the axis at this point. Both projections ($a$ and $a'$) of the point $A$ will lie on this perpendicular above the coordinate axis. To observe the given distances

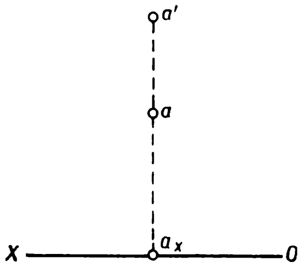
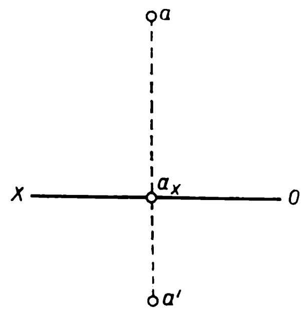



FIG. 3.

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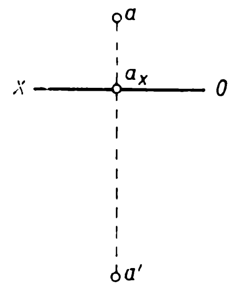
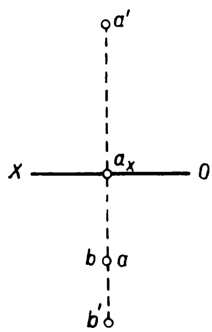



FIG. 5.

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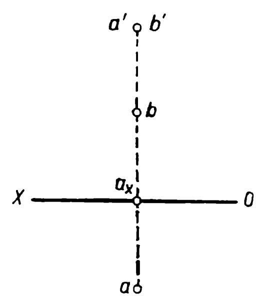


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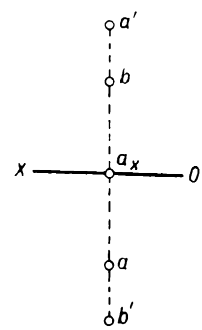



FIG. 8.

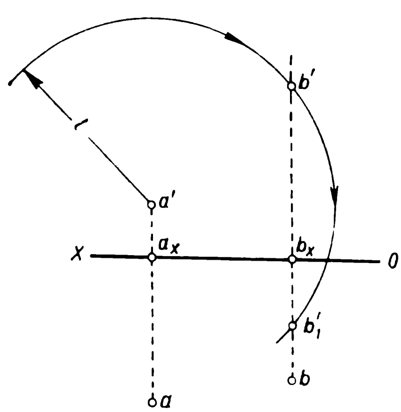



FIG. 9.

from the point to the corresponding planes of projection, make sure that the distance from the horizontal projection of the point to the coordinate axis is 18 mm (i.e. the distance from the point to the vertical plane of projection) and the distance from the vertical projection of the point to the coordinate axis is 32 mm (i.e. the distance from the point to the horizontal plane of projection). Now lay off on the perpendicular a segment 18 mm long *upward* from the point $a_x$, thus obtaining the horizontal projection ($a$) of the point. Then mark off a length of 32 mm *in the same direction* to obtain the vertical projection ($a'$) of the point.

**Example 2**

Construct an orthographic drawing of the point $A$ (—24, —13) (Fig. 4).

*Solution*. Since both coordinates of the point ($a$, $a'$) are negative, it is situated behind the vertical plane of projection and below the horizontal plane of projection, that is in the third quadrant. Assume an arbitrary point $a_x$ on the coordinate axis and draw through it a line perpendicular to the axis. Then lay off on the perpendicular a length $a_xa = 24$ mm (the coordinate $y$) *upward* and a length $a_xa' = 13$ mm (the coordinate $z$) *downward* from the point $a_x$. The projections of the point $A$ thus obtained show that the point is actually located in the third quadrant.

**Example 3**

Given: the horizontal projection ($a$) of the point $A$ located in the third quadrant. Construct its vertical projection ($a'$) observing the following condition: $z = y + 15$ mm (Fig. 5).

*Solution*. The vertical projection of the desired point must lie below the coordinate axis on a straight line passing through the given horizontal projection of the point perpendicular to the coordinate axis. The distance $y$ from the point $A$ to the vertical plane of projection is measured by the distance from the horizontal projection of the point to the coordinate axis. Hence, to obtain the vertical projection ($a'$) of the point, drop a perpendicular from the horizontal projection $a$ onto the coordinate axis and mark off on it a length $a_xa'$ equal to $a_xa + 15$ mm from the point $a_x$ *downward*.

**Example 4**

Given: the point $A$ (12, 20). Construct an orthographic drawing of the point $B$, which is situated symmetrically to the point $A$ relative to: the horizontal plane of projection (Fig. 6), the vertical plane of projection (Fig. 7), the coordinate axis (Fig. 8).

*Solution*. The given point ($a$, $a'$) lies in the first quadrant.

1. The point symmetrical to it relative to the horizontal plane of projection is found in the fourth quadrant, i.e. $B$ (12, —20). Construct an orthographic drawing of the point ($a$, $a'$) and mark off, on the common perpendicular, lengths $a_xb = 12$ mm (the coordinate $y$) and $a_xb' = 20$ mm (the coordinate $z$) from the point $a_x$ downward.

2. The point symmetrical to it relative to the vertical plane of projection is found in the second quadrant, i.e. $B$ (—12, 20). Construct an orthographic drawing of the point ($a$, $a'$) and mark off, on the common perpendicular, lengths $a_xb = 12$ mm ($y$) and $a_xb' = 20$ mm ($z$) from the point $a_x$ upward.

3. The point symmetrical to it relative to the coordinate axis is found in the third quadrant, i.e. $B$ (—12, —20). Construct an orthographic drawing of the point ($a$, $a'$) and mark off, on the common perpendicular, a length $a_xb = 12$ mm ($y$) from the point $a_xb$ upward, and $a_xb' = 20$ mm ($z$), downward.

**Example 5**

Given: the point $A$ and the horizontal projection of the point $B$. In what quadrant is the point $B$ if the distance between the vertical projections of $A$ and $B$ equals 25 mm (Fig. 9)?

*Solution*. First find the vertical projection ($b'$) of the point $B$. Since the distance between the vertical projections of the points must be equal to 25 mm, the point

$b'$ must lie somewhere on a circle of radius 25 mm described from the point $a'$ as centre. At the same time the point $b'$ must lie on a perpendicular dropped from the point $b$ onto the coordinate axis. Thus, the point $b'$ lies at the intersection of the perpendicular with the circle. Here we have two alternative solutions: points $b'$ and $b'_1$. So the point $B$ may lie either in the first or in the fourth quadrant.

In certain cases only one solution is possible (when?), or there is no solution at all (when?).

PROBLEMS

1. Draw the projections of a point $A$ located in the first, second, third and fourth quadrants, respectively, and construct its orthographic drawings (Figs. 10 to 13).

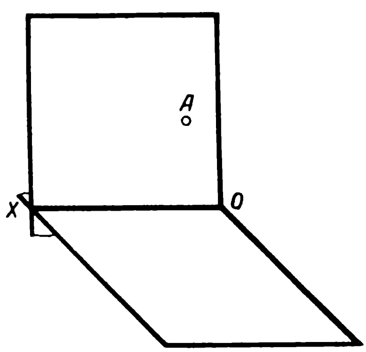



FIG. 10.

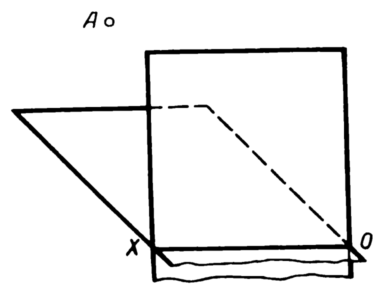



FIG. 11.

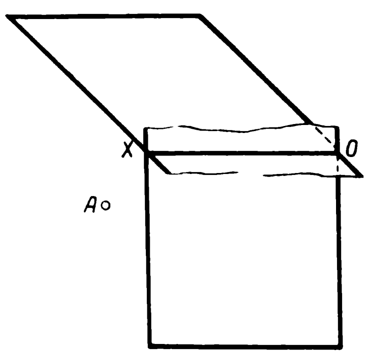



FIG. 12.

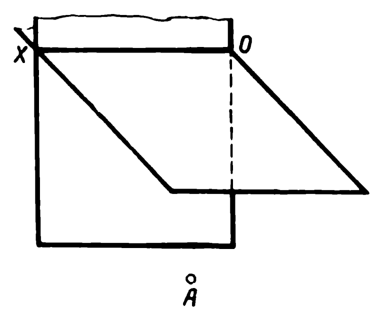



FIG. 13.

2. Draw the projections of a point $A$ with $z = 0$ (Figs. 10 and 12); with $y = 0$ (Figs. 11 and 13) and construct their orthographic drawings.

3. Construct orthographic drawings of a point $A$ using the following coordinates: $y$ 15, 25, 25, −25, −20, −30, 35, 0, −30, 0; $z$ 25, −40, −25, −15, 35, 30, 0, −30, 0, 30.

4. Construct the orthographic drawing of a point $A$ located in a known quadrant, given one of its projections and the relationship between its coordinates ($y = z + n$) (Figs. 14 to 17).

5. Construct the orthographic drawing of a point $A$ located in a known quadrant, given one of its projections and the ratio of the distances from this point to the corresponding planes of projection $\left(\frac{z}{y} = m\right)$ (Figs. 18 to 21).

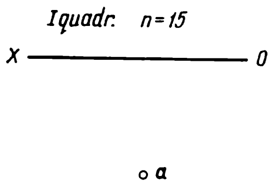
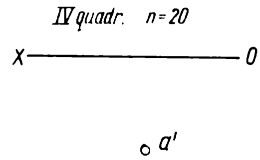


FIG. 14.



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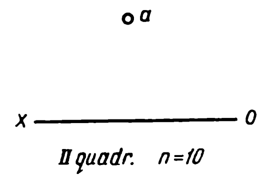

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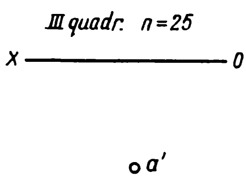

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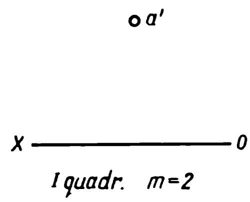

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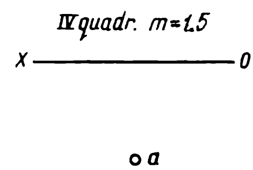

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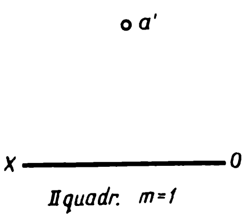


FIG. 20.

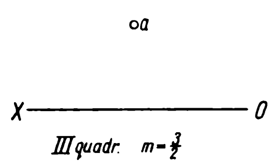


FIG. 21.

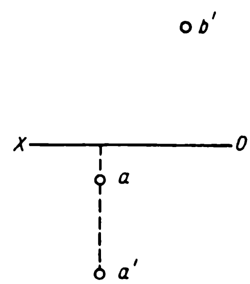


FIG. 22.

6. Construct an orthographic drawing of a point *B* symmetrical to a point *A* (—25, 30) relative to the horizontal plane of projection, to the vertical plane of projection, to the coordinate axis.

7. Given: a point *A* and the vertical projection of a point *B*. In what quadrant is the point *B* situated if the distance between the horizontal projections of *A* and *B* equals 25 mm (Fig. 22)?

## CHAPTER III

# THE STRAIGHT LINE

| *If in space* | *then on an orthographic drawing*[1] |
|---|---|
| (1) a straight line is parallel to the horizontal plane of projection, | the vertical projection of this line is parallel to the coordinate axis and its horizontal projection forms an angle with the coordinate axis; |

[1] See foot-note on page 8.

(2) a straight line is parallel to the vertical plane of projection, the horizontal projection of this line is parallel to the coordinate axis and its vertical projection forms an angle with the coordinate axis;

(3) a straight line is parallel to the coordinate axis, its horizontal and vertical projections are parallel to the coordinate axis;

(4) a straight line lies in a plane perpendicular to the coordinate axis (a profile line), its horizontal and vertical projections lie on a common perpendicular to the coordinate axis;

(5) a straight line is perpendicular to the horizontal plane of projection (a horizontal projecting line), the horizontal projection of the line is a point and its vertical projection is a straight line perpendicular to the coordinate axis;

(6) a straight line is perpendicular to the vertical plane of projection (a vertical projecting line), the vertical projection of the line is a point and its horizontal projection is a straight line perpendicular to the coordinate axis.

## PROBLEMS

8. Draw the projections of the straight line *AB* and construct an orthographic drawing, given that the line is:

(1) parallel to the horizontal plane of projection (Fig. 23);

(2) parallel to the vertical plane of projection (Fig. 24);

(3) parallel to the coordinate axis (Fig. 25);

(4) perpendicular to the horizontal plane of projection (Fig. 26);

(5) perpendicular to the vertical plane of projection (Fig. 27).

9. Read the following orthographic drawings (fill in the blanks):

(1) The line *AB* is in the (?) quadrant and its end *A* rests against the (?) half-plane of the (?) plane of projection (Fig. 28).

(2) The line *AB* is in the (?) quadrant and its end *A* rests against the (?) half-plane of the (?) plane of projection (Fig. 29).

(3) The line *AB* lies in the (?) quadrant, is parallel to the (?) plane of projection and its end *A* rests against the (?) plane of projection (Fig. 30).

(4) The line *AB* lies in the (?) quadrant, is perpendicular to the (?) plane of projection and its end *B* rests against the (?) half-plane of the (?) plane of projection (Fig. 31).

(5) The line *AB* is in the (?) quadrant and its end *A* rests against the (?) half-plane of the (?) plane of projection (Fig. 32).

(6) The line *AB* lies in the (?) half-plane of the (?) plane of projection (Fig. 33).

(7) The line *AB* lies in the (?) quadrant, is parallel to the (?) plane of projection and its end *A* rests against the (?) half-plane of the (?) plane of projection (Fig. 34).

(8) The line *AB* lies in the (?) quadrant, is perpendicular to the (?) plane of projection and its end *A* rests against the (?) half-plane of the (?) plane of projection (Fig. 35).

(9) The line *AB* lies in the (?) quadrant and is parallel to (?) (Fig. 36).

(10) The line *AB* lies in the (?) half-plane of the (?) plane of projection (Fig. 37).

(11) The line *AB* lies in the (?) quadrant and its end *A* rests against the (?) half-plane of the (?) plane of projection and its end *B*, against the (?) half-plane of the (?) plane of projection (Fig. 38).

(12) The line *AB* lies in the (?) half-plane of the (?) plane of projection and is parallel to (?) (Fig. 39).

(13) The line *AB* lies in the (?) half-plane of the (?) plane of projection and is parallel to (?) (Fig. 40).

10. Construct an orthographic drawing of a straight line, given that:

(1) it is oblique, lies in the second quadrant, and its end *A* rests against the vertical plane of projection;

(2) it lies in the first quadrant, is parallel to the vertical plane of projection, and its end $A$ rests against the horizontal plane of projection;

(3) it lies arbitrarily in the front half-plane of the horizontal plane of projection;

(4) it lies in the fourth quadrant, is perpendicular to the horizontal plane of projection, and its end $A$ is equidistant from the planes of projection;

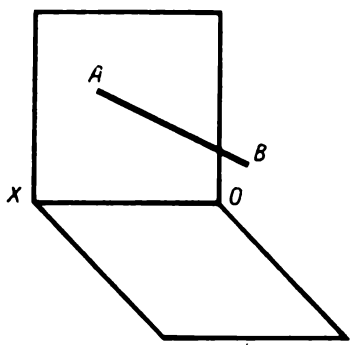



FIG. 23.

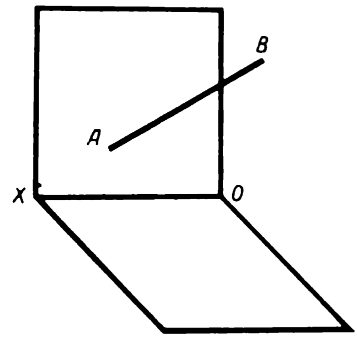



FIG. 24.

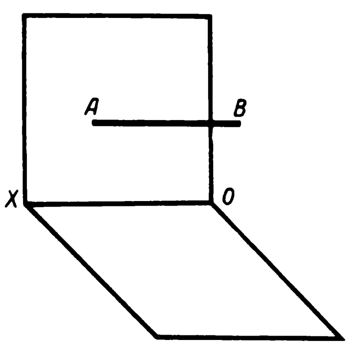



FIG. 25.

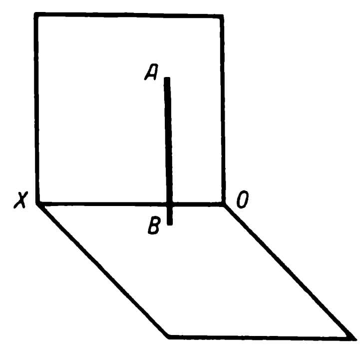



FIG. 26.

(5) it lies in the third quadrant, is parallel to the horizontal plane of projection, and its end $A$ rests against the plane of projection;

(6) it lies in the bisector plane of the first quadrant, and is parallel to the coordinate axis;

(7) it lies in the fourth quadrant, is parallel to the vertical plane of projection, and its end $A$ is equidistant from the planes of projection;

(8) it is oblique, lies in the third quadrant, its end $A$ rests against the horizontal plane of projection, and its end $B$ is equidistant from the planes of projection;

(9) it lies in the upper half-plane of the vertical plane of projection and is parallel to the coordinate axis;

(10) it lies in the second quadrant, its end $A$ rests against the coordinate axis, and its end $B$ is equidistant from the planes of projection;

(11) it lies in the third quadrant, is perpendicular to the vertical plane of projection, and its end $B$ rests against the plane of projection.

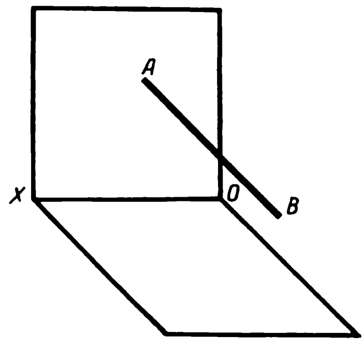



FIG. 27.

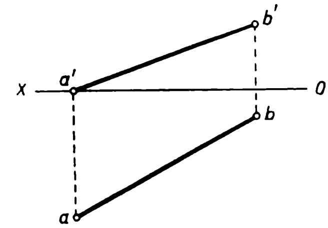



FIG. 28.

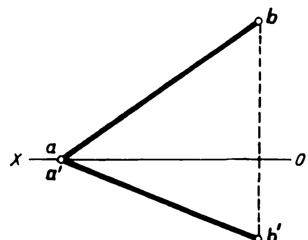



FIG. 29.

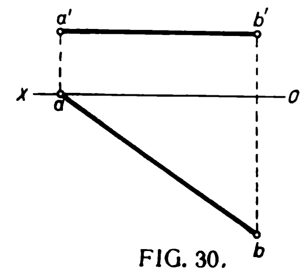



FIG. 30.

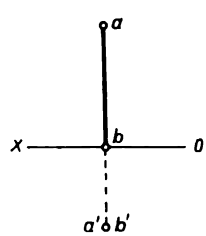



FIG. 31.

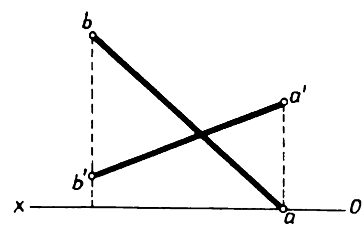



FIG. 32.

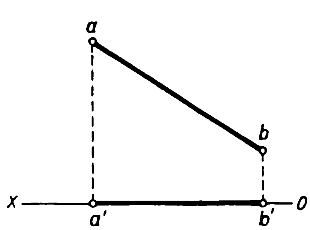



FIG. 33.

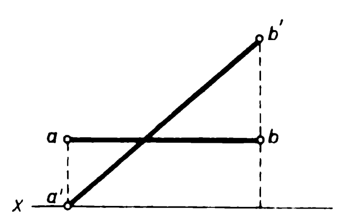



FIG. 34.

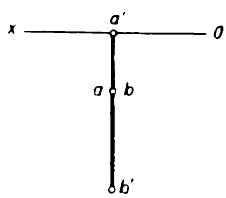



FIG. 35.

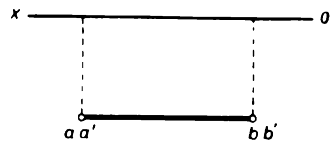



FIG. 36.

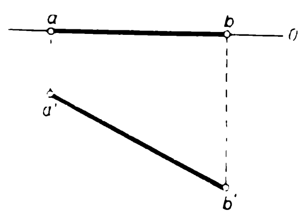



FIG. 37.

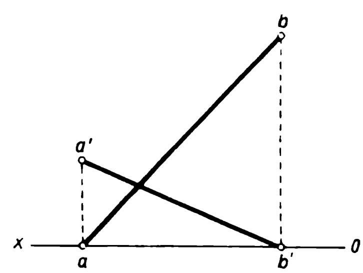



FIG. 38.

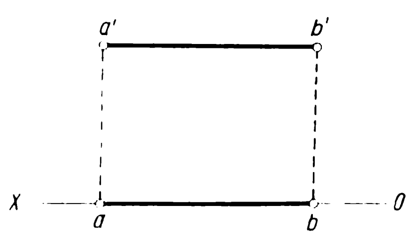



FIG. 39.

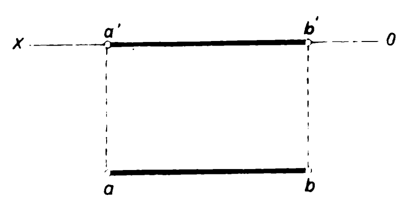



FIG. 40.

CHAPTER IV

# AN ORTHOGONAL SYSTEM OF THREE PLANES OF PROJECTION

Make a drawing of three projection planes; write down the names and designations of the planes, coordinate axes, half-planes, and octants (Fig. 41).

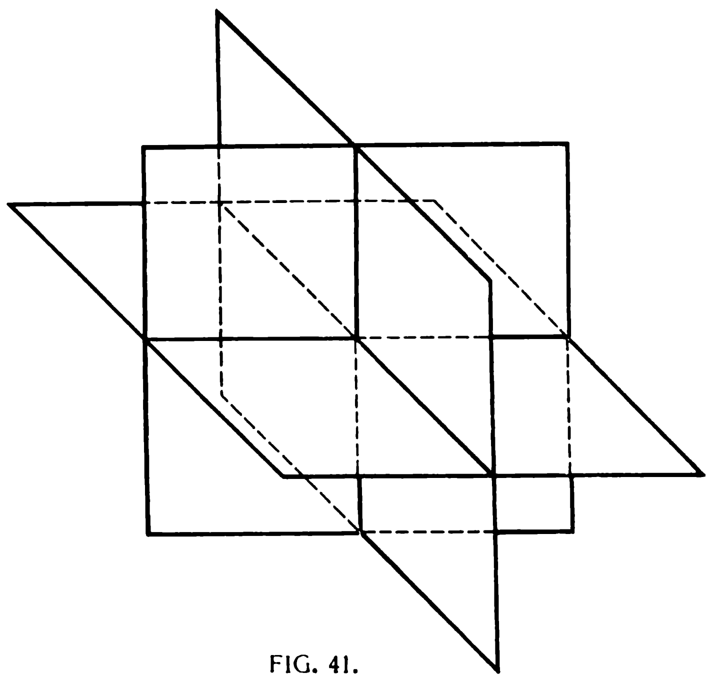


FIG. 41.

**Check-Up Questions**

Give full answers to the following questions:

1. Along what axis do the following pairs of projection planes intersect: horizontal and vertical; horizontal and profile; vertical and profile?
2. How are the following projection planes designated: horizontal, vertical, profile?
3. What half-planes form the first, second, third, fourth, fifth, sixth, seventh, eighth octant?
4. What serves as a boundary between the following pairs of octants: I and IV, II and III, V and VIII, VI and VII; I and II, III and IV, V and VI, VII and VIII; I and V, II and VI, III and VII, IV and VIII?
5. Enumerate the octants found above the horizontal plane of projection, below the horizontal plane, in front of the vertical plane, behind the vertical plane, to the left of the profile plane, to the right of the profile plane of projection.
6. What position relative to the projection planes is occupied by a point located in the first, second, third, fourth, fifth, sixth, seventh, eighth octant?
7. To what octant does a point belong if it lies on the positive axis $OX$, $OY$, $OZ$; on the negative axis $OX$, $OY$, $OZ$?

8. In what half-planes are the projections of a point if the latter lies in the first, second, third, fourth, fifth, sixth, seventh, eighth octant?

9. Where are the projections of a point situated if the latter lies in the following projection planes and axes: $H$ (or $H_1$, $H_2$, $H_3$); $V$ (or $V_1$, $V_2$, $V_3$); $W$ (or $W_1$, $W_2$, $W_3$); $\pm OX$; $\pm OY$; $\pm OZ$?

10. What is common to all points lying in the horizontal, vertical, profile planes of projection, respectively?

11. Where is a point found in space if its horizontal projection lies in $H$ (or $H_1$, $H_2$, $H_3$); if its vertical projection lies in $V$ (or $V_1$, $V_2$, $V_3$); if its profile projection lies in $W$ (or $W_1$, $W_2$, $W_3$)?

12. What half-planes, when brought to coincidence, are found above the $x$-axis, below the $x$-axis; to the right of the $z$-axis, to the left of the $z$-axis?

13. Draw (with the planes brought to coincidence) the half-planes forming the first, second, third, fourth, fifth, sixth, seventh, eighth octant.

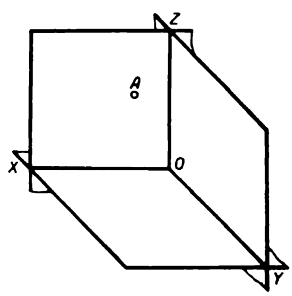



FIG. 42.

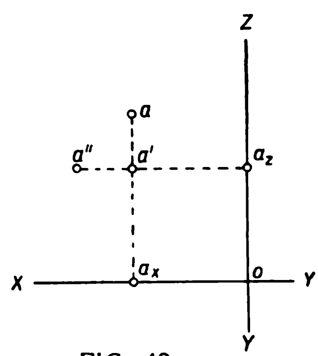



FIG. 43.

14. At what position of a point in space will its horizontal orthographic projection be found above the $x$-axis, below the $x$-axis; to the left of the $z$-axis, to the right of the $z$-axis?

15. At what position of a point in space will its vertical orthographic projection be found above the $x$-axis, below the $x$-axis; to the left of the $z$-axis, to the right of the $z$-axis?

16. At what position of a point in space will its profile orthographic projection be found above the $x$-axis, below the $x$-axis; to the left of the $z$-axis, to the right of the $z$-axis?

17. At what position of a point lying outside the coordinate axis can its vertical and horizontal projections (vertical and profile projections, all three projections) coincide?

Before giving answers to the following questions (18 through 28) construct the three projections of a point $A$ in the first octant of a projection drawing and indicate the coordinates of the point and its projections (Fig. 42).

18. What coordinates define the horizontal projection of a point, the vertical projection of a point, the profile projection of a point?

19. What projections of a point are found on the common perpendicular to the $x$-axis ($z$-axis) after the projection planes are brought to coincidence?

20. What is the procedure of finding the third projection of a point, given the other two projections? For instance, how to find the profile projection of a point, given its horizontal and vertical projections, and so on?

21. What determines the distance from a point in space to the profile plane on an orthographic drawing?

22. What is the position of a point if any two of its coordinates equal zero? (For instance, $x = 0$, $z = 0$.)

23. What is the position of a point if any one of its coordinates equals zero?

24. In what octants is the $x$-coordinate of a point positive, negative?

25. In what octants is the $y$-coordinate of a point positive, negative?

26. In what octants is the $z$-coordinate of a point positive, negative?

27. What signs have the $x$-, $y$-, $z$-coordinates of a point located in the first, second, third, fourth, fifth, sixth, seventh, eighth octant?

28. How does one lay off on an orthographic drawing a length determining the $x$-coordinate if it is positive, negative; a length determining the $y$-coordinate when constructing the profile projection of a point, if this coordinate is positive, negative?

EXAMPLES

**Example 6**

Make an orthographic drawing of the point $A$ (15, —24, 15) (Fig. 43).

*Solution.* Mark off a segment $Oa_x = 15$ mm (the abscissa) on the positive $x$-axis, draw through the point $a_x$ a straight line perpendicular to this axis and lay off on it upward segments $a_xa = 24$ mm (the ordinate) and $a_xa' = 15$ mm (the $z$-coordinate). To determine the profile projection ($a''$) of the point, draw through the point $a'$ a straight line perpendicular to the $z$-axis and lay off on it leftward a segment $a_za'' = 24$ mm (the ordinate).

**Example 7**

Given: the point $A$ (—15, —24, —15). Make an orthographic drawing of a point $B$, which is symmetrical to the point $A$ relative to the horizontal (Fig. 44), vertical (Fig. 45), and profile (Fig. 46) projection planes, respectively.

*Solution.* Point $A$ is situated to the right of the profile plane of projection, behind the vertical plane and below the horizontal plane, that is in the seventh octant. Make its orthographic drawing: mark off a segment $Oa_x = 15$ mm (the abscissa) on the negative $x$-axis, draw through the point $a_x$ a straight line perpendicular to the $x$-axis and lay off on it a length $a_xa = 24$ mm (the ordinate) upward and $a_xa' = 15$ mm (the $z$-coordinate) downward, then draw through the point $a'$ a straight line perpendicular to the $z$-axis and lay off on it leftward a length $a_za'' = 24$ mm (the ordinate).

1. The point $B$, which is symmetrical to the given point relative to the horizontal plane of projection, lies in the sixth octant, i.e. $B$ (—15, —24, 15). Mark off upward lengths $a_xb = 24$ mm (the ordinate) and $a_xb' = 15$ mm (the $z$-coordinate) on the common perpendicular and determine the profile projection ($b''$) of the point $B$.

2. The point $B$, which is symmetrical to the given point relative to the vertical plane of projection, lies in the eighth octant, i.e. $B$ (—15, 24, —15). Construct an orthographic drawing of the point $A$ and lay off downward on the common perpendicular lengths $a_xb = 24$ mm (the ordinate) and $a_xb' = 15$ mm (the $z$-coordinate). Then draw a straight line through the point $b'$ and perpendicular to the $z$-axis and lay off on it to the right a length $a_zb'' = 24$ mm (the ordinate).

3. The point $B$, which is symmetrical to the given point relative to the profile plane of projection, lies in the third octant, i.e. $B$ (15, —24, —15). Construct an orthographic drawing of the point $A$ and lay off a length $Ob_x = 15$ mm (the abscissa) on the positive $x$-axis, draw through the point $b_x$ a straight line perpendicular to the $x$-axis, and lay off on it a length $b_xb = 24$ mm (the ordinate) upward and $b_xb' = 15$ mm (the $z$-coordinate) downward. Then find the profile projection ($b''$) of the point $B$.

**Example 8**

Given: the point $A$ (15, 24, —15). Make an orthographic drawing of a point $B$, which is symmetrical to the point $A$ relative to the $x$-axis (Fig. 47), $y$-axis (Fig. 48), $z$-axis (Fig. 49).

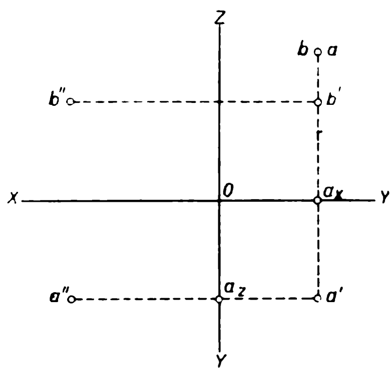



FIG. 44.

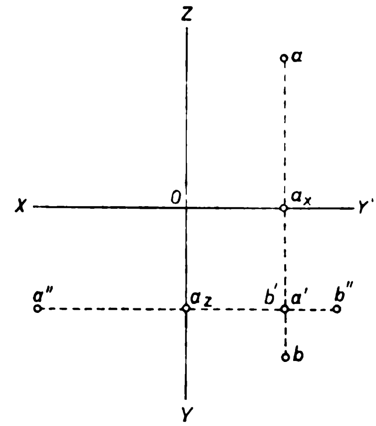



FIG. 45.

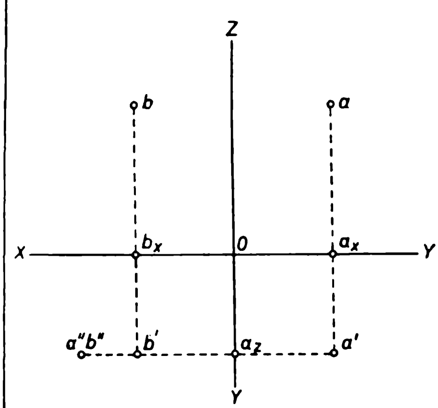



FIG. 46.

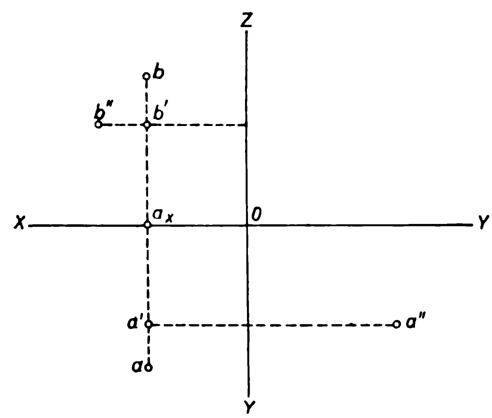



FIG. 47.

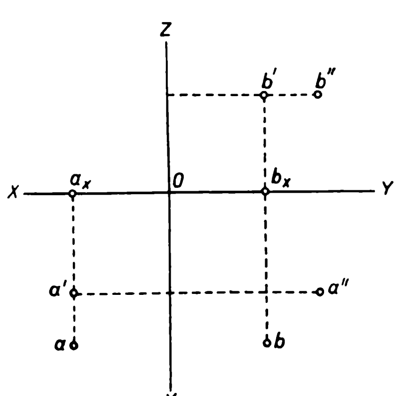



FIG. 48.

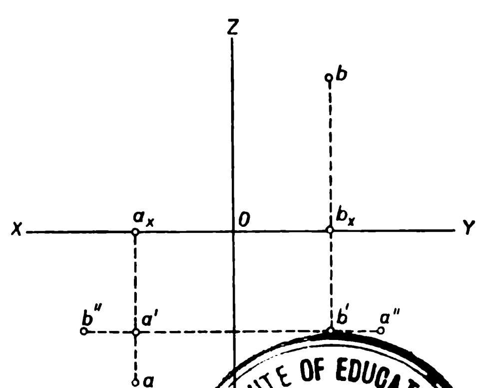



FIG. 49.

*Solution.* The point $A$ is located to the left of the profile plane of projection, in front of the vertical plane, and below the horizontal plane of projection, i.e. in the fourth octant. Make its orthographic drawing: mark off a length $Oa_x = 15$ mm (the abscissa) on the positive $x$-axis, draw a straight line through the point $a_x$ and perpendicular to the $x$-axis and lay off on it downward lengths $a_xa = 24$ mm (the ordinate) and $a_xa' = 15$ mm (the $z$-coordinate). Then find the profile projection ($a''$) of the point $A$.

1. The point $B$, which is symmetrical to the given point relative to the $x$-axis, lies in the second octant, i.e. its coordinates will be 15, —24, 15. Mark off lengths $a_xb = 24$ mm (the ordinate) and $a_xb' = 15$ mm (the $z$-coordinate) on the common perpendicular upward. Then find the profile projection ($b''$) of the point $B$.

2. The point $B$, being symmetrical to the given point relative to the $y$-axis and thus lying in the fifth octant, is defined by the coordinates (—15, 24, 15). Make an orthographic drawing of the point $A$ and lay off a length $Ob_x = 15$ mm (the abscissa) on the negative $x$-axis, draw a straight line through the point $b_x$ and perpendicular to the $x$-axis and lay off on it a length $b_xb = 24$ mm (the ordinate) downward and $b_xb' = 15$ mm (the $z$-coordinate) upward. Then determine the profile projection ($b''$) of the point $B$.

3. The point $B$, being symmetrical to the given point relative to the $z$-axis and thus lying in the seventh octant, is determined by the coordinates (—15, —24, —15). Make an orthographic drawing of the point $A$ and lay off a length $Ob_x = 15$ mm (the abscissa) on the negative $x$-axis, draw through the point $b_x$ a straight line perpendicular to the $x$-axis and lay off on it a length $b_xb = 24$ mm (the ordinate) upward and $b_xb' = 15$ mm (the $z$-coordinate) downward. Then find the profile projection ($b''$) of the point $B$.

**Example 9**

Given: the horizontal projection ($a$) of the point $A$ situated in the third quadrant. Construct its vertical and profile projections, observing the condition $z = x + 12$ mm (Fig. 50).

*Solution.* The vertical projection ($a'$) of the point must lie below the $x$-axis on a perpendicular to it passing through the horizontal projection of the point. The coordinate $x$ defines the distance from the horizontal projection ($a$) of the point to the $z$-axis.

The coordinate $z$ determines the distance from the vertical projection ($a'$) of the point to the $x$-axis. Hence, draw a straight line through the point $a$ perpendicular to the $x$-axis to intersect it at point $a_x$. On this perpendicular lay off a length $a_xa' = aa_y + 12$ mm from the point $a_x$ downward. The end point of this line will be the vertical projection ($a'$) of the point.

Then find the profile projection ($a''$) of the point $A$ from the two projections ($a$ and $a'$).

**Example 10**

Construct the $z$-axis and the horizontal projection of the point $A$ lying in the second quadrant, given the projections $a$ and $a''$, and the ratio $\frac{x}{y} = 3$ (Fig. 51).

*Solution.* The given projections of the point $A$ must lie to the left of the $z$-axis. Since the distance from the vertical projection $a'$ of the point to the $z$-axis is $x$, and the distance from the profile projection $a''$ of the point to the $z$-axis is $y$, the length $a'a''$ is equal to the difference $(x—y)$. Observing the given condition $\frac{x}{y} = 3$, we may write $\frac{x-y}{y} = \frac{3-1}{1} = 2$, whence $\frac{x-y}{2} = y$.

Now bisect the segment $a'a''$ and draw the $z$-axis at right angles to the $x$-axis at a distance of $\frac{a'a}{2}$ to the right of the profile projection $a''$ of the point. Then find the horizontal projection $a$ of the point $A$.

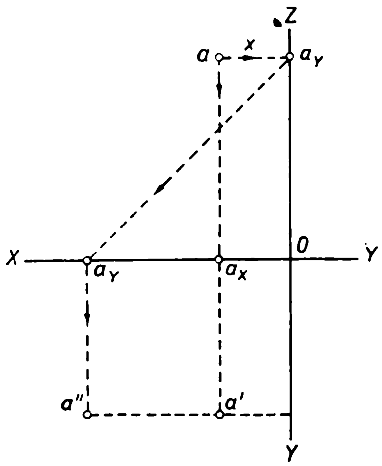



FIG. 50.

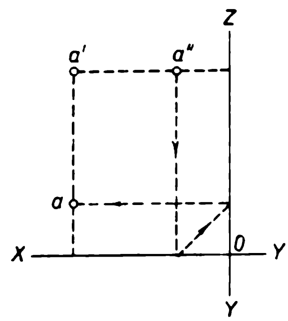



FIG. 51.

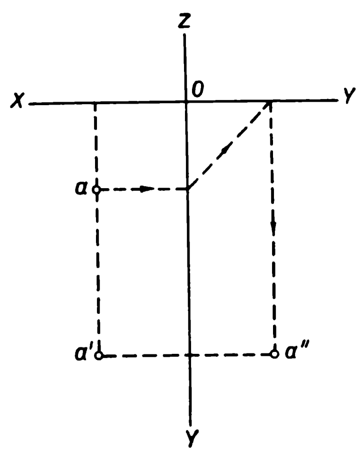



FIG. 52.

**Example 11**

Construct the axis $OX$ and the profile projection of the point $A$ lying in the fourth quadrant, given the projections $a$ and $a'$, and the ratio $\frac{y}{z}=\frac{1}{3}$ (Fig. 52).

*Solution.* The given projections of the point $A$ must lie below the axis $OX$. Since the distance from the horizontal projection of the point to the axis $OX$ is $y$ and the distance from the vertical projection of the point to the axis $OX$ is $z$, the segment $a'a$ between the projections of the point is equal to $(z-y)$. Observing the given condition $\frac{y}{z}=\frac{1}{3}$, we may write $\frac{z}{y}=\frac{3}{1}$, whence $\frac{z-y}{y}=\frac{3-1}{1}=2$, or $\frac{z-y}{} = y.$

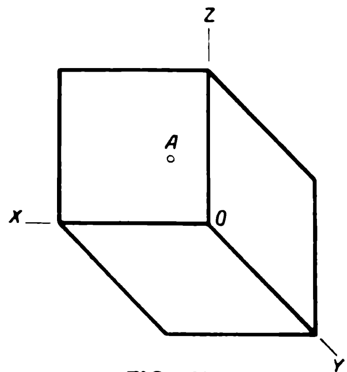


FIG. 53.

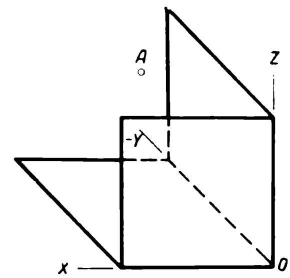


FIG. 54

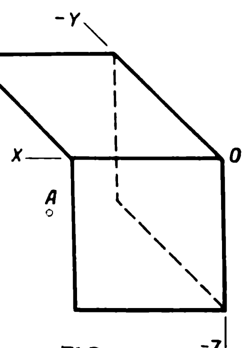


FIG. 55.

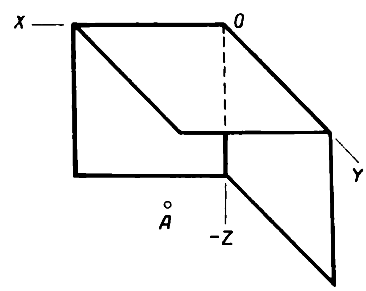


FIG. 56.

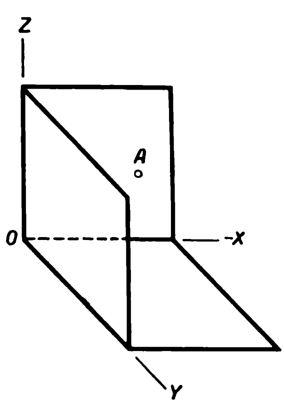


FIG. 57.

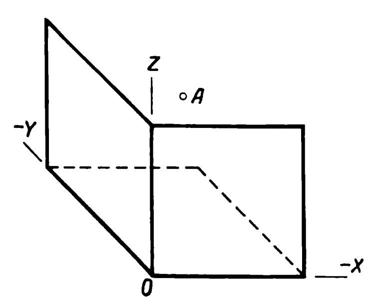


FIG. 58.

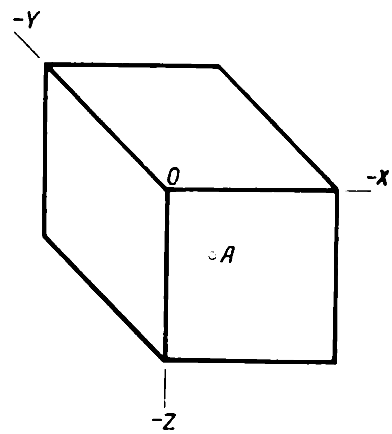



FIG. 59.

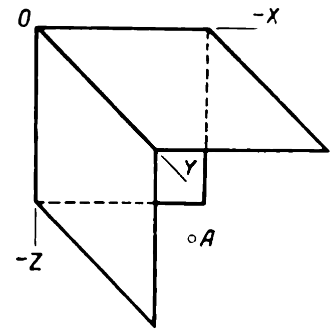



FIG. 60.

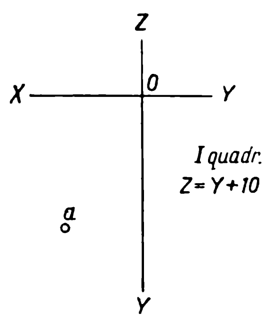



FIG 61.

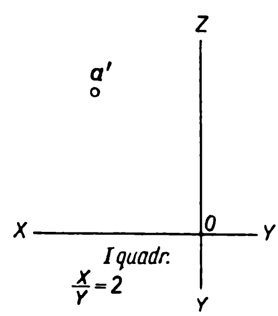



FIG. 62.

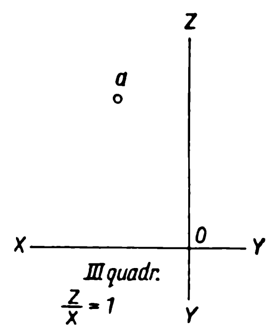



FIG. 63.

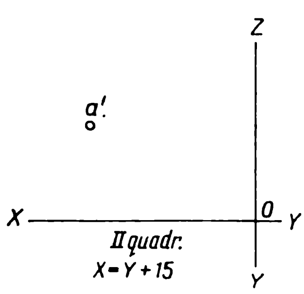



FIG. 64.

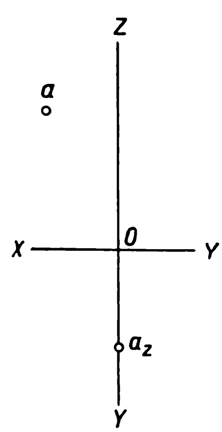



FIG. 65.

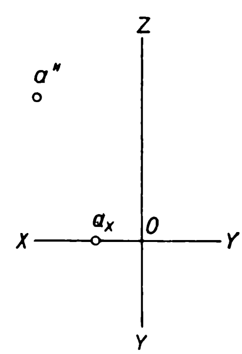



FIG. 66.

Now divide $a'a$ in half and construct the axis $OX$ above the point $a$ at a distance of $\frac{a'a}{2}$. Then find the profile projection $a''$ of the point $A$.

PROBLEMS

11. Construct the three projections of a point $A$ located in the first (second, third, fourth, fifth, sixth, seventh, eighth) octant and make its orthographic drawing (Figs. 53 to 60).

12. Construct the three projections of a point $A$, given that $z_A = 0$ (Figs. 53, 55, 58, 60); $y_A = 0$ (Figs. 54, 56, 57, 59); $x_A = 0$ (Figs. 54, 56, 57, 59); and make its orthographic drawing.

13. Make an orthographic drawing of a point $A$ from the coordinates tabulated below:

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $x$ | 20 | 15 | 15 | 20 | −15 | −20 | −20 | −15 | 25 | −25 |
| $y$ | 15 | −25 | −25 | 20 | 25 | −15 | −15 | 25 | −25 | 25 |
| $z$ | 25 | 35 | −20 | −30 | 20 | 25 | −25 | −35 | 25 | −25 |
| $x$ | 15 | −15 | 20 | −15 | 0 | 0 | 0 | 0 | 0 | −20 |
| $y$ | −25 | 25 | 0 | 0 | −20 | 25 | −20 | −20 | 0 | 0 |
| $z$ | 0 | 0 | −30 | 30 | 30 | −35 | 20 | 0 | 20 | 0 |

14. Find the lacking projections of the point $A$, given one of its projections, the coordinate ratio (or other data), and the quadrant it lies in (Figs. 61 to 66).

15. Draw the lacking coordinate axis and determine the projection of the point $A$, given the coordinate ratio (Figs. 67 to 72).

16. Construct the third projection of the line $AB$, triangle $ABC$ (Figs. 73 to 78).

17. Construct the projections of the triangle $ABC$, given the coordinates of its vertices: $A$ (20, 0, 0), $B$ (0, 30, 0), and $C$ (0, 0, 25).

18. Construct the projections of the triangle $ABC$, given the coordinates of its vertices: $A$ (20, 30, 0), $B$ (20, 0, 30), and $C$ (0, 20, 30).

19. Given: the point $A$ (20, 30, 20). Make an orthographic drawing of a point $B$, which is symmetrical to the point $A$ relative to the horizontal plane of projection, vertical plane, profile plane of projection (make three drawings).

20. Given: the point $A$ (20, 30, 20). Make an orthographic drawing of the point $B$, which is symmetrical to the point $A$ relative to the $x$-, $y$-, $z$-axis (make three drawings).

21. Construct the projections of a cube with the base $ABCD$ lying in the vertical plane of projection, given the diagonal $AC$ of the base, and show the projections of each face, of each edge (Fig. 79).

22. Construct the projections of a right regular trihedral prism 50 mm high, given the side $AB$ of its base lying in the horizontal plane of projection; show the projections of each face, of each edge (Fig. 80).

23. Construct the projections of a right regular trihedral pyramid 60 mm high, given the side $AB$ of its base lying in the vertical plane of projection, and show the projections of each face, of each edge (Fig. 81).

24. Construct the projections of a right circular cylinder with the base lying in the vertical plane of projection and with centre $C$ (30, 0, 30), given the altitude = = 60 mm and the radius of the base = 20 mm.

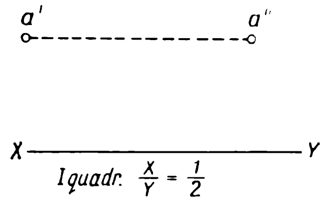



FIG. 67.

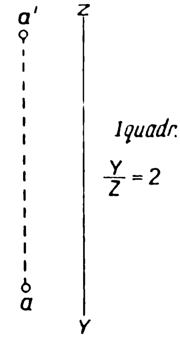



FIG. 68.

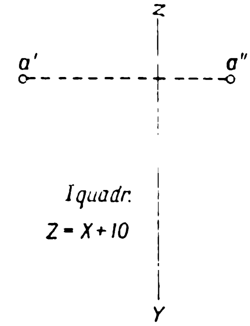



FIG. 69.

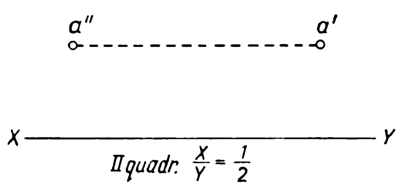



FIG. 70.

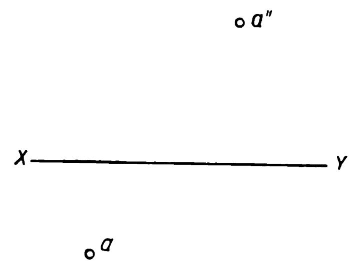



FIG. 72.

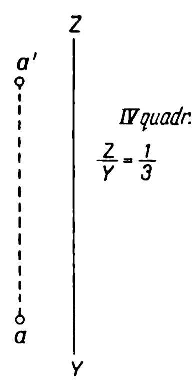



FIG. 71.

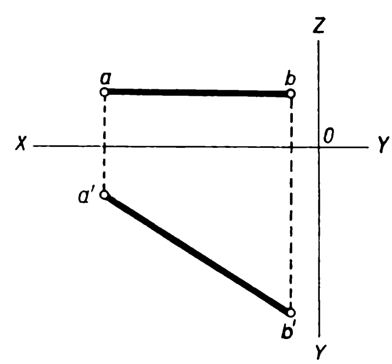



FIG. 73.

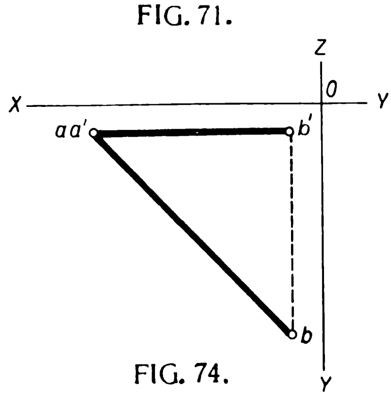



FIG. 74.

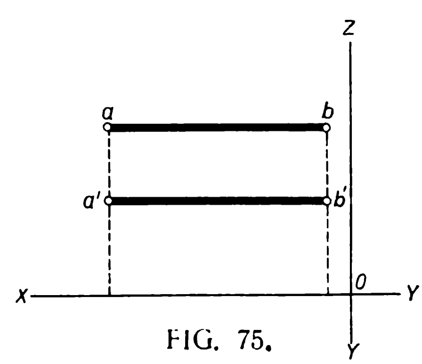



FIG. 75.

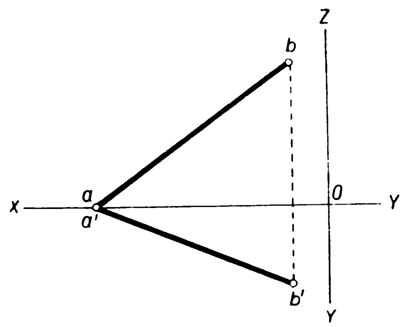



FIG. 76.

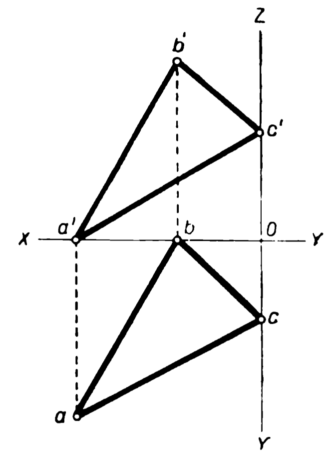



FIG. 77.

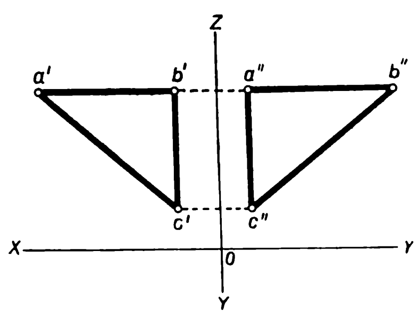



FIG. 78.

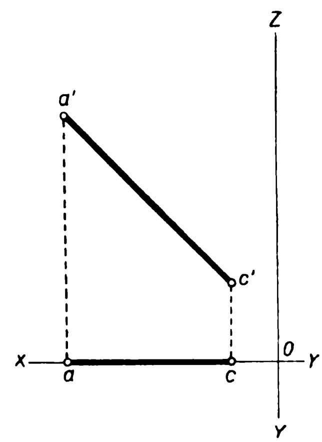



FIG. 79.

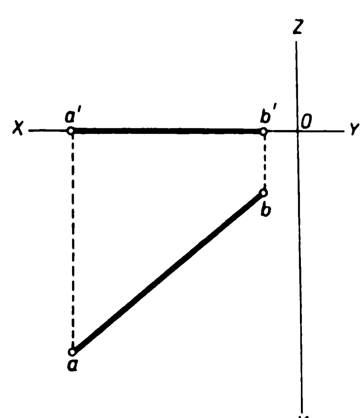



FIG. 80.

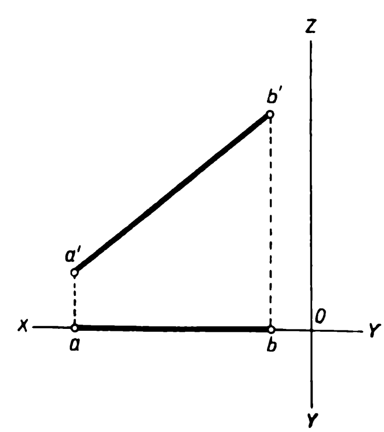



FIG. 81.

25. Construct the locus of straight lines in space passing through a point $S$ (30, 30, 50) and forming an angle of 70° with the horizontal plane of projection.

26. Construct the locus of points in space situated at a distance of 20 mm from a point $C$ (30, 30, 35).

CHAPTER V

# MUTUAL POSITIONS OF A POINT AND A LINE-SEGMENT

| *If in space* | *then on an orthographic drawing* |
|---|---|
| a point lies on a line-segment | the projections of the point must lie on like projections of the line-segment. |

*Note.* In the case of a profile line the converse theorem holds only in the system $H$, $V$, $W$. It is necessary to have two projections to check whether the profile projection of the point lies on the profile projection of the line-segment.

Conclusion:

| *If in space* | *then on an orthographic drawing* |
|---|---|
| a line-segment passes through a point | the projections of the line-segment pass through the like projections of the point. |

EXAMPLES

**Example 12**

Do the points $A$, $B$, $C$, $D$ lie on the line-segment $MN$ (Fig. 82)?

*Solution.* The projections ($a$, $a'$) of the point $A$ lie on the like projections ($mn$, $m'n'$) of the line-segment; hence, the point $A$ lies on the line-segment $MN$.

The projections ($b$, $b'$) of the point $B$ lie on unlike projections of the line-segment ($mn$, $m'n'$), hence the point $B$ does not lie on the line-segment $MN$.

The points $C$ and $D$ do not lie on the line-segment either (why?).

**Example 13**

Does the point $C$ lie on the profile line-segment $AB$ (Fig. 83)?

*Solution.* Since the given line is a profile line, it is necessary to check additionally the mutual position of the profile projection of the point and that of the line-segment. Constructing the profile projection of the line-segment and that of the point, we see that the profile projection of the point does not lie on the profile projection of the line-segment; hence, the point $C$ does not lie on the line-segment $AB$.

**Example 14**

Given: the profile line-segment $AB$ and the vertical projection ($c'$) of a point $C$ lying on the line-segment. Find the horizontal projection ($c$) of the point (Fig. 84).

*Solution.* To find the horizontal projection ($c$) of the point, one should know its profile projection ($c''$), which must lie on the profile projection ($a''b''$) of the line-segment and on a line-segment perpendicular to the axis $OZ$ and drawn through the point $c'$. Thus we find the profile projection ($a''b''$) of the line-segment. The point of intersection of the extension of this line and the perpendicular to the axis $OZ$ will be the profile projection ($c''$) of the point. Then find the horizontal projection ($c$) of the point.

**Example 15**

Given: the profile line-segment $AB$ and the horizontal projection ($c$) of the point $C$ lying on the line-segment. Find the vertical projection ($c'$) of the point (Fig. 85).

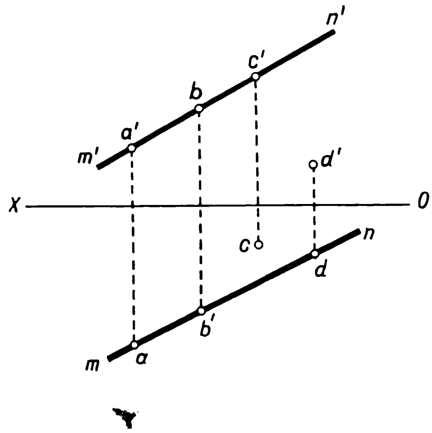



FIG. 82.

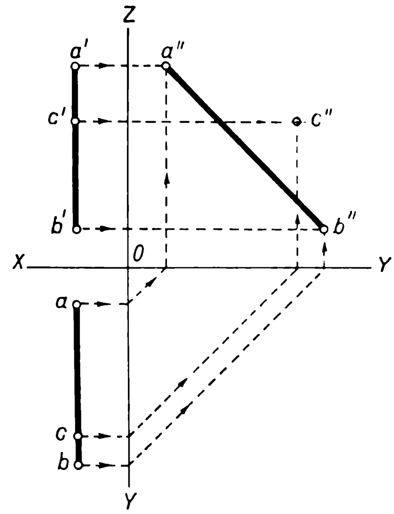



FIG. 83.

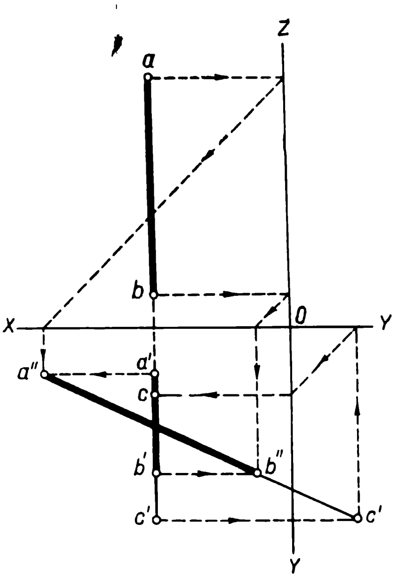



FIG. 84.

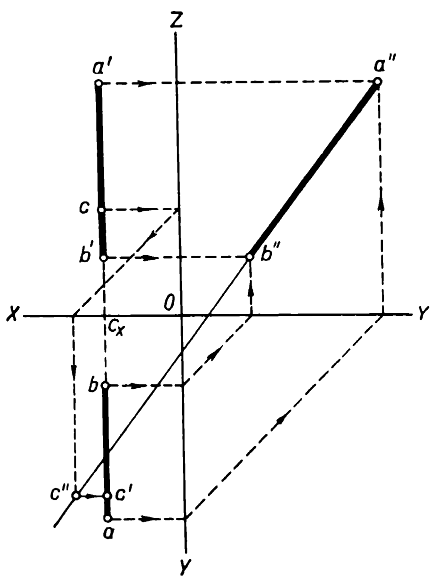



FIG. 85.

*Solution.* To find the vertical projection ($c'$) of the point one should know its profile projection ($c''$). The latter must lie on the profile projection ($a''b''$) of the line-segment at a distance $y$ to the left (why?) of the axis $OZ$. Thus we find the profile projection ($a''b''$) of the line-segment. The point of intersection of this line and a straight line drawn parallel to the axis $OZ$ at a distance of $c_x c$ (i.e. $y$) to the left of it will be the profile projection ($c''$) of the point. Then find $c'$.

**Example 16**

Draw a straight line through the points $A$ and $B$ (Fig. 86).

*Solution.* Draw the two projections of the required line: the horizontal one through the points $a$ and $b$ and the vertical one through the points $a'$ and $b'$.

**Example 17**

Given: the point $A$ and unlike projections of points $B$ and $C$. Find the lacking projections of the points $B$ and $C$, knowing that all of them should lie on the straight line $MN$ (Fig. 87).

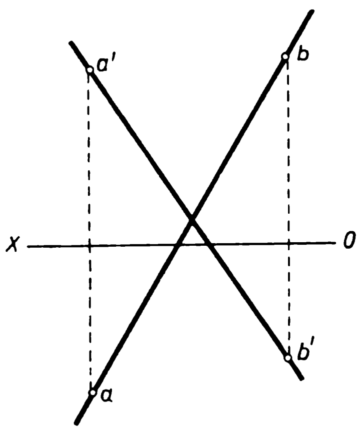



FIG. 86.

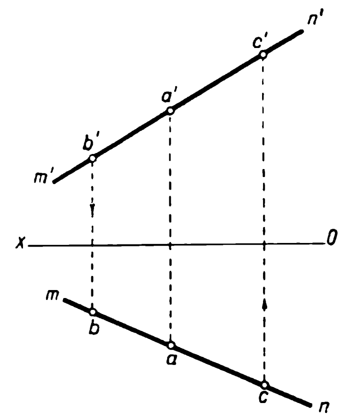



FIG. 87.

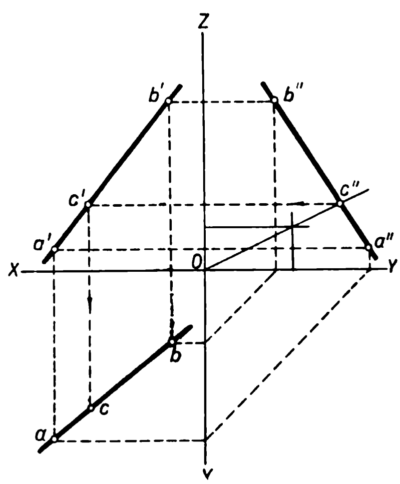



FIG. 88.

*Solution.* Draw the projections of the line $MN$: the horizontal one ($mn$) through the points $a$ and $c$, and the vertical one through the points $a'$ and $b'$. Then find the horizontal projection ($b$) of the point $B$ on the line $mn$ and the vertical projection ($c'$) of the point $C$ on the line $m'n'$.

**Example 18**

Find a point $C$ on the straight line $AB$, given its coordinate ratio $\frac{z}{y} = \frac{1}{2}$ (Fig. 88).

*Solution.* Coordinates $z$ and $y$ define the profile projection ($c''$) of the point $C$. The locus of the points of a plane in the $ZOY$ system with a coordinate ratio $\frac{z}{y} = \frac{1}{2}$ is a straight line defined by the equation $y = 2z$. The profile projection ($c''$) of the point $C$ must lie on the profile projection ($a''b''$) of the given line $AB$

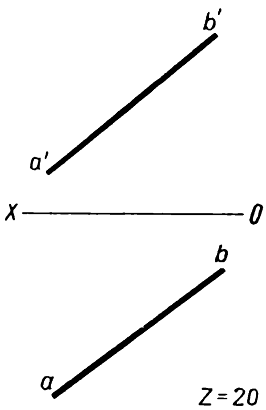



FIG. 89.

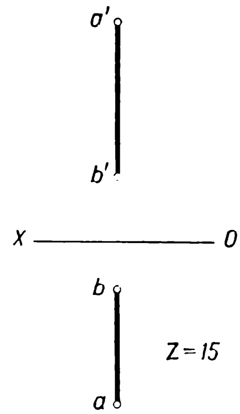



FIG. 90.

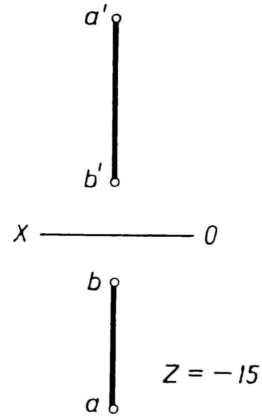



FIG. 91.

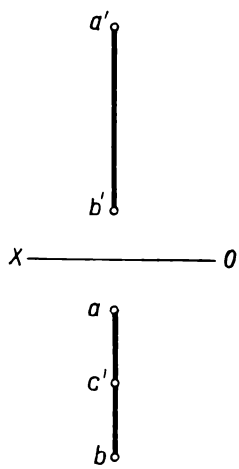



FIG. 92.

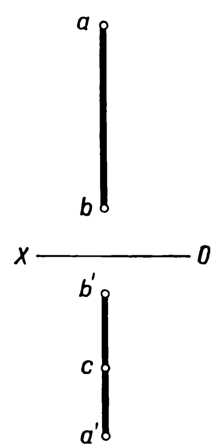



FIG. 93.

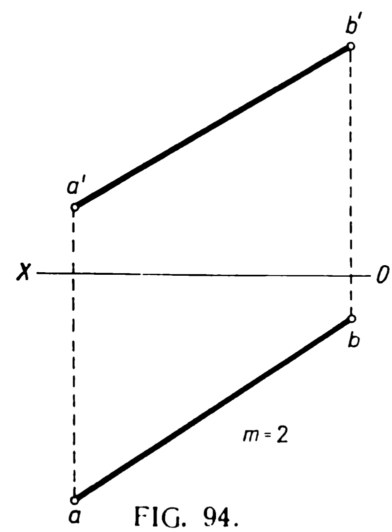



FIG. 94.

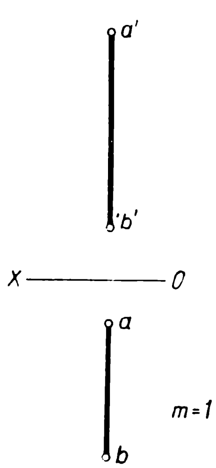



FIG. 95.

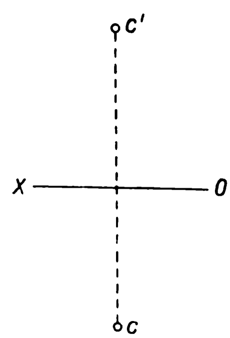



FIG. 96.

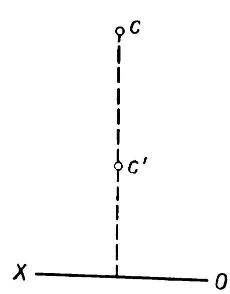



FIG. 97.

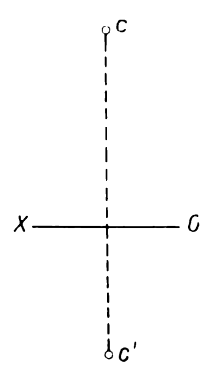



FIG. 98.

and at the same time on the line $y = 2z$, i.e. at the point of their intersection. Hence, find the profile projection ($a''b''$) of the line-segment $AB$ from the given projections, draw a straight line $y = 2z$ in the $ZOY$ system to obtain the profile projection ($c''$) of the point $C$ at their intersection. Then find from $c''$ the horizontal and vertical projections of the required point on the like projections of the line $AB$.

### PROBLEMS

27. Find a point on the line $AB$, given its distance from some plane of projection (Figs. 89 to 91).

28. Find a point $C$ on the line-segment $AB$, given one of the projections of the point (Figs. 92, 93).

29. What is common to all points lying, respectively, on a horizontal projecting line, on a vertical projecting line, on a profile projecting line on an orthographic drawing?

30. Given one of the projections of a point lying on a line, is it always possible to find its lacking projection (s) in the orthogonal system of two (three) planes of projection?

31. On the line-segment $AB$ find a point $C$ specified by a coordinate ratio $\frac{z}{y} = m$ (Figs. 94, 95).

32. Through the point $C$ draw a straight line $AB$ parallel to the horizontal plane of projection (Fig. 96), to the vertical plane of projection (Fig. 97), to the profile plane of projection (Fig. 98).

## CHAPTER VI

# TRACES OF A LINE

The *trace of a line* is the point at which the line intersects a projection plane.

The *horizontal trace of a line* is the point of intersection of the line with the horizontal plane of projection.

The *vertical trace of a line* is the point of intersection of the line with the vertical plane of projection.

The horizontal trace of a line is designated as $H$, and its projections as ($h$, $h'$).

The vertical trace of a line is designated as $V$, and its projections as ($v$, $v'$).

*Rule.* To find the horizontal trace of a line proceed as follows: prolong the vertical projection of the line to intersect the $x$-axis at the point $h'$. At this point erect a perpendicular to the coordinate axis to intersect the horizontal projection of the line at the point $h$ ($H$). The point thus obtained is the horizontal trace of the line.

The vertical trace of a line is found in much the same manner: prolong the horizontal projection of the line to intersect the $x$-axis at the point $v$. At this point erect a perpendicular to the coordinate axis to intersect the vertical projection of the line at the point $v'$ ($V$). This point is the vertical trace of the line.

*Note.* Following the above rule we can find only the points $h'$ and $v$ for a profile line. To determine $h$ and $v'$ it is necessary first to find $h''$ and $v''$ at the intersection of $a''b''$ and the $y$- and $z$-axis, respectively, and then $h$ ($H$) and $v'$ ($V$) from $h'$, $h''$ and $v$, $v''$, respectively.

### EXAMPLES

**Example 19**

Construct the projections of a line, given its traces (Fig. 99).

*Solution.* The desired line passes through the traces—points $H$ and $V$. Hence, the projections of the line must pass through the like projections of these points.

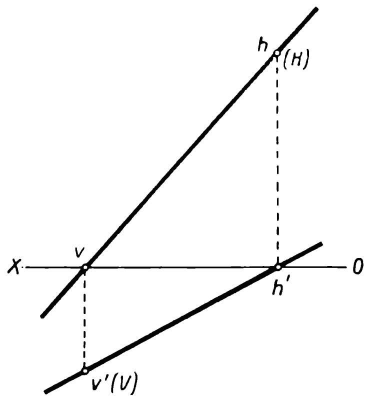



FIG. 99.

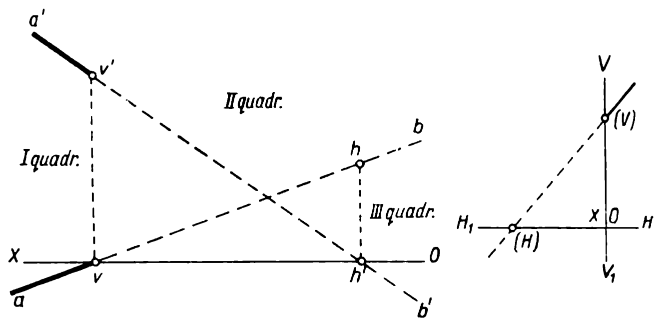



FIG. 100.

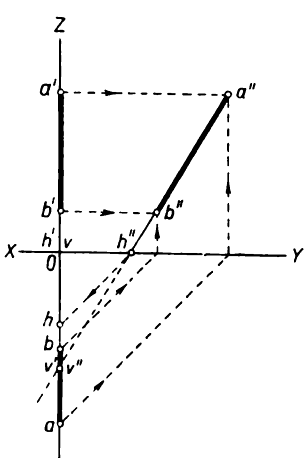



FIG. 101.

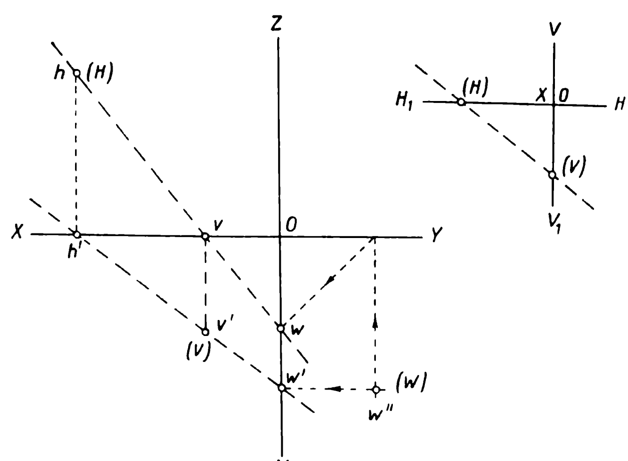



FIG. 102.

Find the projections ($h$, $h'$) and ($v$, $v'$) of the points and draw the horizontal projection of the line through the points $h$ and $v$, and the vertical projection of the line through the points $v'$ and $h'$.

**Example 20**

Determine the traces of the line-segment $AB$ and indicate its visible and invisible parts (Fig. 100).

*Solution.* Prolong the horizontal projection ($ab$) of the line-segment to intersect the $x$-axis at point $v$. At this point erect a perpendicular to the coordinate axis to intersect the vertical projection of the line-segment at the point $v'$ ($V$). The point of intersection yields the vertical trace of the line-segment. Then prolong the vertical projection ($a'b'$) of the line-segment to intersect the coordinate axis at the point $h'$. At this point erect a perpendicular to the coordinate axis to intersect the horizontal projection of the line-segment at the point $h$ ($H$). Thus we obtain the horizontal trace of the line-segment.

Hence, the horizontal trace of the desired line-segment lies in the rear half-plane of the horizontal plane of projection, the vertical trace lying in the upper half-plane of the vertical plane of projection. Such a line passes through the first, second, and third quadrants; its part between the traces is invisible. The line is visible beyond its vertical trace and is invisible beyond the horizontal trace.

**Example 21**

Given: the profile line-segment $AB$; required: to find its traces (Fig. 101).

*Solution.* Prolong the given projections to intersect the coordinate axis at the points $h'$ and $v$. Construct the profile projection of the given line-segment and prolong it to intersect the $y$- and $z$-axis at the points $h''$ and $v''$, respectively. Then determine $h$ ($H$) and $v'$ ($V$) from $h'$, $h''$, and $v$, $v''$, respectively.

**Example 22**

Construct the projections of a line-segment, given its traces (points $H$ and $W$). Find the vertical trace of the line-segment and indicate the visible and invisible parts of the line (Fig. 102).

*Solution.* Find the projections of the given traces (points $H$ and $W$), then draw the projections of the desired line-segment through the like projections of the points $H$ and $W$.

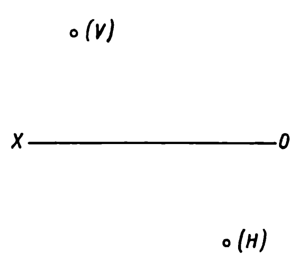



FIG. 103.

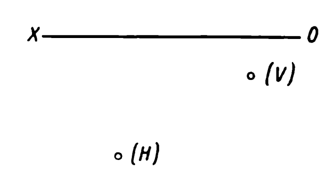



FIG. 104.

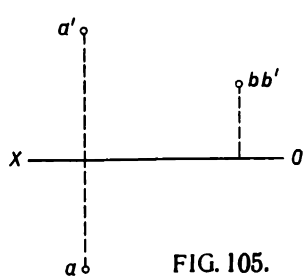



FIG. 105.

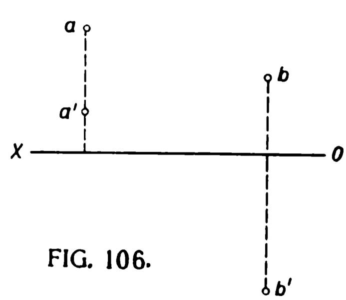



FIG. 106.

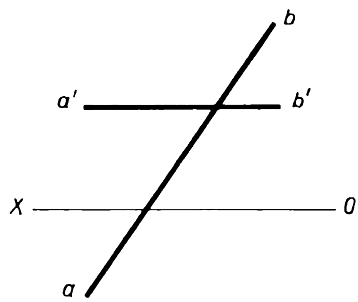



FIG. 107.

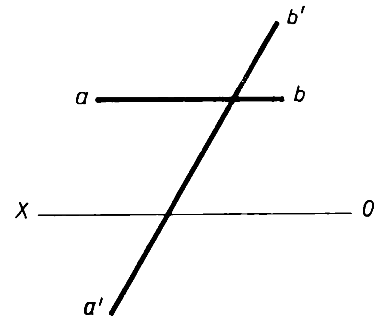



FIG. 108.

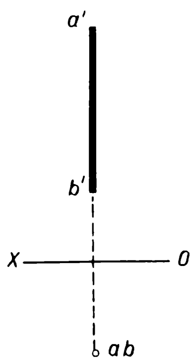



FIG. 109.

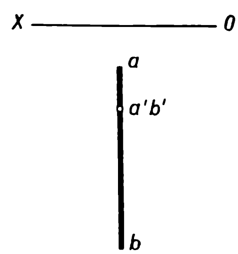



FIG. 110.

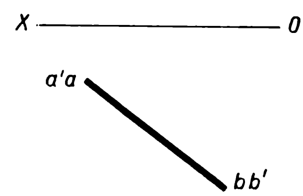



FIG. 111.

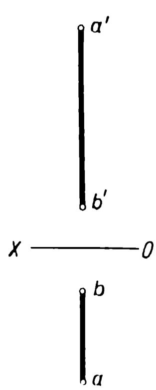



FIG. 112.

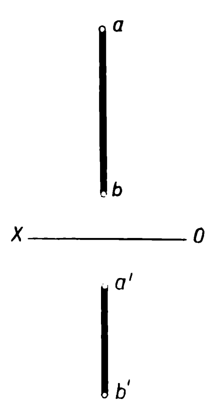



FIG. 113.

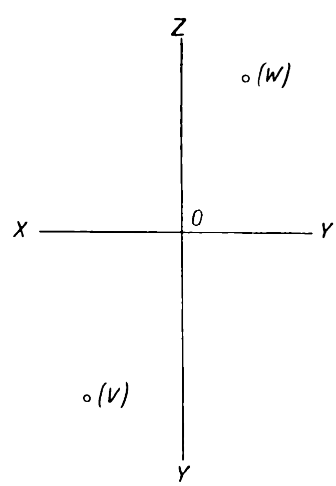



FIG. 114.

To obtain the vertical trace of the line-segment find the point $v$ at the intersection of the horizontal projection of the line-segment and the $x$-axis; at this point erect a perpendicular to the axis to intersect the vertical projection of the line-segment at $v'$, which is the vertical trace of the line-segment. The line is invisible.

PROBLEMS

33. Enumerate the lines having only one trace in the $H$-$V$ system; name it.
34. Enumerate the lines having in the $H$-$V$-$W$ system:
(1) only one trace (name it);
(2) two traces (name them).
35. What is common to the traces of a profile line on an orthographic drawing?
36. When can the traces of a profile line coincide on an orthographic drawing?

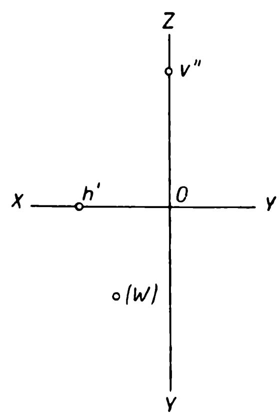



FIG. 115.

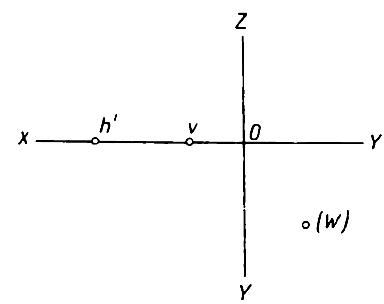



FIG. 116.

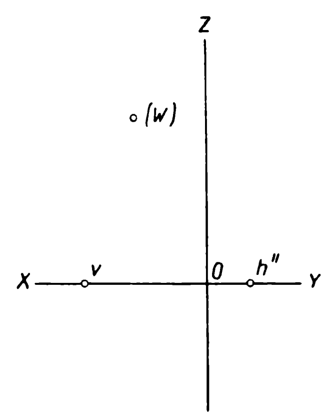



FIG. 117.

37. Give an example where a line-segment is not defined by the horizontal and vertical traces; indicate additional conditions necessary for conclusive determination of this line-segment.

38. Enumerate the cases of coincidence of the trace of a line with one of its projections.

39. Construct the projections of a line-segment, given its traces (Figs. 103, 104).

40. Find the traces of a line-segment passing through the points $A$ and $B$ (Figs. 105, 106).

41. Find the traces of the line-segment $AB$ (Figs. 107 to 113).

42. Construct the three projections of the desired line-segment, given its traces (or a trace and one of the projections of each of the other two traces) (Figs. 114 to 117).

CHAPTER VII

# THE RELATIVE POSITIONS OF TWO STRAIGHT LINES

| *If in space* | *then on an orthographic drawing*[1] |
|---|---|
| (1) line-segments intersect, | their like projections intersect and the points of intersection lie on a single perpendicular to the coordinate axis; |
| (2) line-segments are parallel, | their like projections are also parallel. |

In a particular case intersecting lines may be mutually perpendicular.

The theorem concerning the projections of two mutually perpendicular lines is given in Chapter X.

| | |
|---|---|
| (3) line-segments do not intersect and are not parallel to each other (skew lines), | the points of intersection of their like projections do not lie on a single perpendicular to the axis. |

*Note.* The relative positions of two lines, one of which is a profile line, are determined from the third projection. For instance, two profile line-segments are parallel if their profile projections are parallel to each other.

EXAMPLES

**Example 23**

Determine the relative positions of the line-segments $AB$ and $CD$ in space (Fig. 118).

*Solution.* The points of intersection of the like projections of the given line-segments lie on a common perpendicular to the coordinate axis. Let us designate the point of intersection of the horizontal projections of the line-segments as $k$ and that of their vertical projections as $k'$. The point $(k, k')$ lies on the line-segments $AB$ and $CD$, i.e. it is their common point; hence, the line-segments $AB$ and $CD$ *intersect* in space.

**Example 24**

Determine the relative positions of the line-segments $AB$ and $CD$ (Fig. 119).

*Solution.* The points of intersection of the like projections of the line-segments lie on a common perpendicular to the coordinate axis. Designate the point of intersection of the horizontal projections as $k$, and that of their vertical projections as $k'$. The point $(k, k')$ lies on the line-segment $(ab, a'b')$. To find the position of the point $(k, k')$ relative to the profile line $CD$, construct the profile projections of the line-segment $(cd, c'd')$ and those of the point $(k, k')$. The profile projection $(k'')$ of the point does not lie on the profile projection $(c''d'')$ of the line-segment $CD$, that is, the point $(k, k')$ does not lie on the profile line $(cd, c'd')$ and is *not common* to the given line-segments $AB$ and $CD$. Hence, the latter are *skew lines*.

**Example 25**

Determine the relative positions of the line-segments $AB$ and $CD$ (Fig. 120).

*Solution.* The horizontal and vertical projections of the given line-segments $AB$ and $CD$ are parallel to each other; hence, the line-segments $AB$ and $CD$ are parallel.

**Example 26**

Determine the relative positions of the line-segments $AB$ and $CD$ (Fig. 121).

*Solution.* The horizontal and vertical projections of two profile lines lying in different planes are always parallel to each other, and that is why in order to find

[1] See foot-note on page 8.

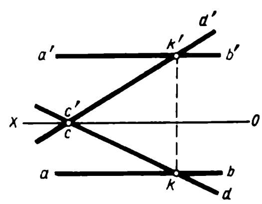



FIG. 118.

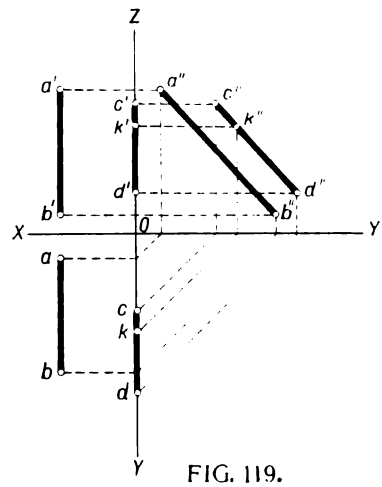



FIG. 119.

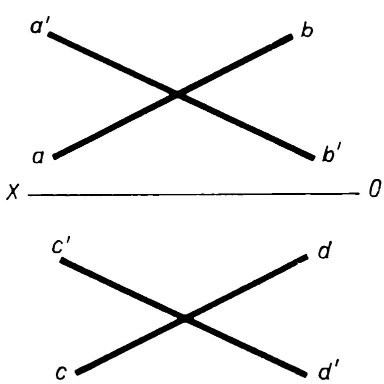



FIG. 120.

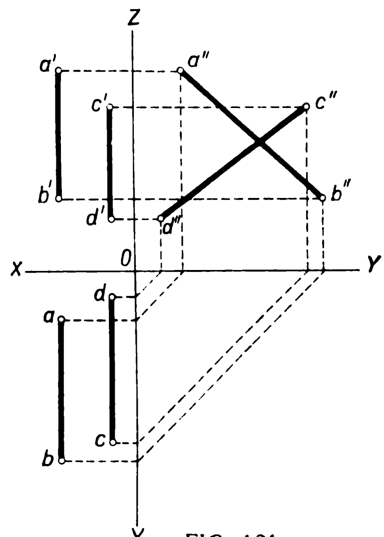



FIG. 121.

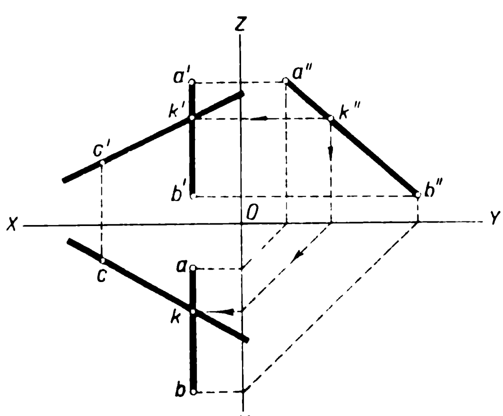



FIG. 122.

their relative positions in space it is necessary to construct their profile projections ($a''b''$) and ($c''d''$). The profile projections of the line-segments ($ab$, $a'b'$) and ($cd$, $c'd'$) intersect; hence, $AB$ and $CD$ are *skew lines.*

**Example 27**

Given: the straight line $AB$ and the point $C$. Draw through the point $C$ an arbitrary line intersecting the given line $AB$ (Fig. 122).

*Solution.* Take a random point $K$ on the line-segment $AB$. Since the given line $AB$ is a profile line, construct its profile projection and mark on it arbitrarily the profile projection ($k''$) of the point $K$. From this projection find the horizontal and vertical projections ($k$, $k'$) of the point on the like projections of the line-segment $AB$. Then draw the respective projections of the desired line: the horizontal one through the points $c$ and $k$, and the vertical one through $c'$ and $k'$.

**Example 28**

Given: the line-segment $AB$ and the point $C$. Through the point $C$ draw a line intersecting the given line-segment $AB$ and parallel to the vertical plane of projection (Fig. 123).

*Solution.* The projections of the desired line must pass through the like projections of the point $C$. Since the line must be parallel to the vertical plane of projection, its horizontal projection will be parallel to the coordinate axis. Hence, draw the horizontal projection of the desired line through the point $c$ and parallel to the coordinate axis to intersect the line-segment $ab$ at the point $k$. From the latter point find the point $k'$ and draw the vertical projection of the line through the points $k'$ and $c'$.

**Example 29**

Given: the line-segment $AB$ and the point $K$. Draw through the point $K$ a line parallel to the given line-segment $AB$ (Fig. 124).

*Solution.* The projections of the desired line must pass through the like projections of the point $K$, and the like projections of the given and desired lines must be parallel to each other. Hence, draw the respective projections of the desired line: the horizontal one ($km$) through the point $k$ and parallel to the line $ab$, and the vertical one ($k'm'$) through the point $k'$ and parallel to the line $a'b'$.

**Example 30**

Given: the line-segment $AB$ and the point $K$. Draw through the point $K$ a straight line parallel to the given line-segment $AB$ (Fig. 125).

*Solution.* The required line is also a profile line. As is known, the condition of parallelism of two profile lines is parallelism of their profile projections. Hence, find the profile projection ($a''b''$) of the line-segment $AB$ and the profile projection ($k''$) of the point $K$. Draw the profile projection of the desired line through the point $k''$ and parallel to the line $a''b''$; limit it by an arbitrary line-segment $c''d''$, and then construct its horizontal ($cd$) and vertical ($c'd'$) projections from its profile projection.

**Example 31**

Intersect the line-segments $AB$ and $CD$ with an arbitrary line (Fig. 126).

*Solution.* Take an arbitrary point on each of the given line-segments. The line drawn through these points will be the desired line. Hence, assume an arbitrary point ($m$, $m'$) on the line-segment $AB$ and a point ($n$, $n'$) on the line-segment $CD$. Then draw the horizontal projection of the desired line through the points $m$ and $n$, and its vertical projection through the points $m'$ and $n'$.

This problem can be solved in a different way: intersect the vertical projections of the arbitrary line and mark the points of intersection ($m'$ and $n'$). Find the point $m$ on the horizontal projection of $AB$ from the point $m'$, and find the point $n$ on the horizontal projection of $CD$ from the point $n'$. Draw the horizontal

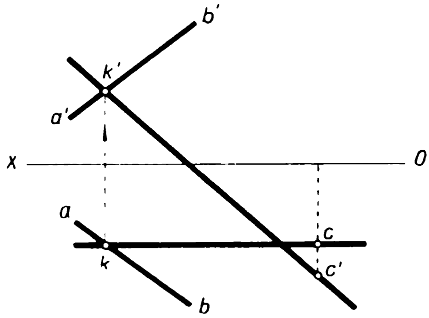


FIG. 123.

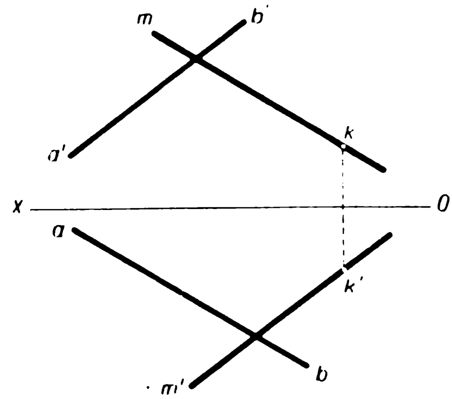


FIG. 124.

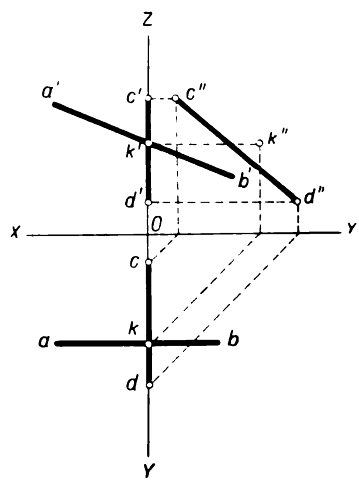


FIG. 125.

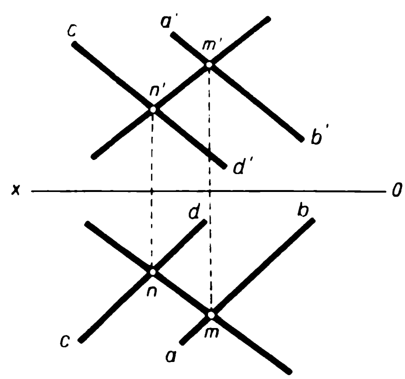


FIG. 126.

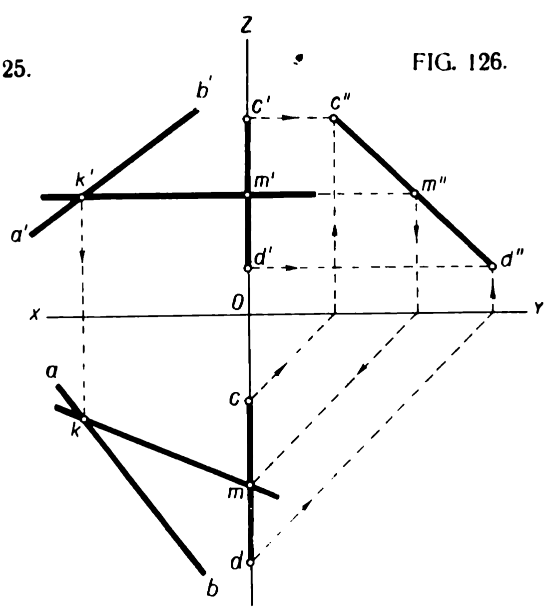


FIG. 127.

projection of the desired line through the obtained points $m$ and $n$. We may also begin to solve this problem by drawing arbitrarily the horizontal projection of the desired line and then proceeding as above.

**Example 32**

Intersect the line-segments $AB$ and $CD$ with a line parallel to the horizontal plane of projection (Fig. 127).

*Solution.* The desired line must be parallel to the horizontal plane of projection and, consequently, its vertical projection must be parallel to the $x$-axis. Hence, draw arbitrarily the vertical projection of the desired line parallel to the coordinate axis. Designate its points of intersection with the line-segments $a'b'$ and $c'd'$ as $k'$ and $m'$. From the points $k'$ and $m'$ find the points $k$ and $m$ on the line-segments $ab$ and $cd$. Draw the horizontal projection ($km$) of the desired line through the points $k$ and $m$.

PROBLEMS

43. Determine the relative positions of the line-segments $AB$ and $CD$ (Figs. 128 to 137).

44. Intersect the line-segment $AB$ with the line $MN$ passing through the point $C$ and parallel to: the horizontal plane of projection (Fig. 138), the vertical plane of projection (Fig. 139).

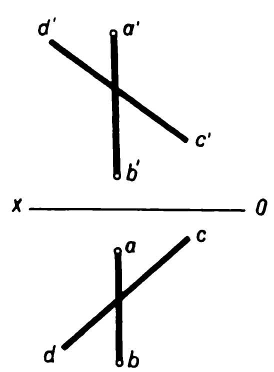


FIG. 128.

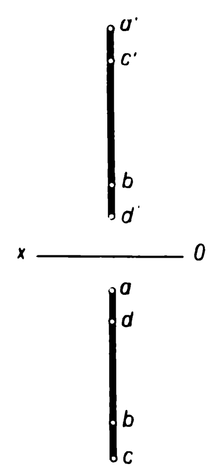


FIG. 129.

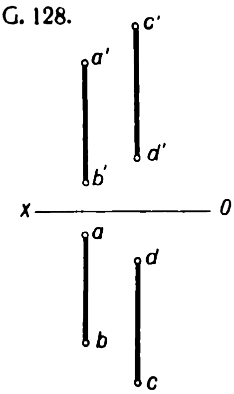


FIG. 130.

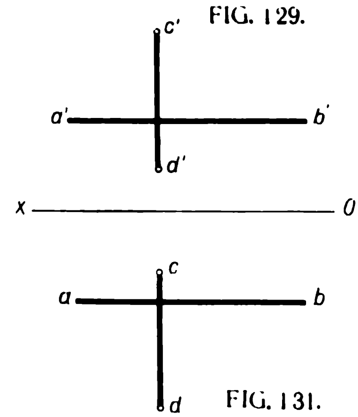


FIG. 131.

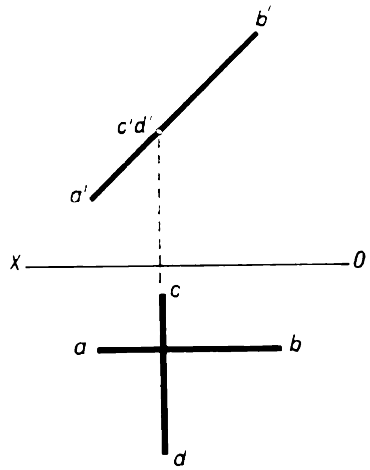



FIG. 132.

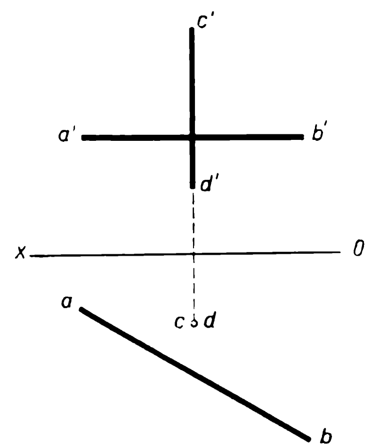



FIG. 133.

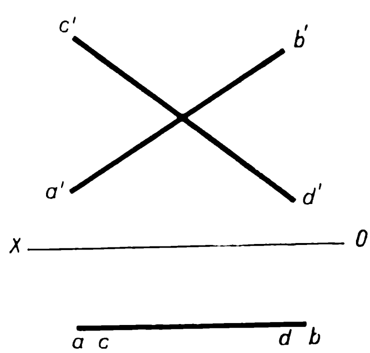



FIG. 134.

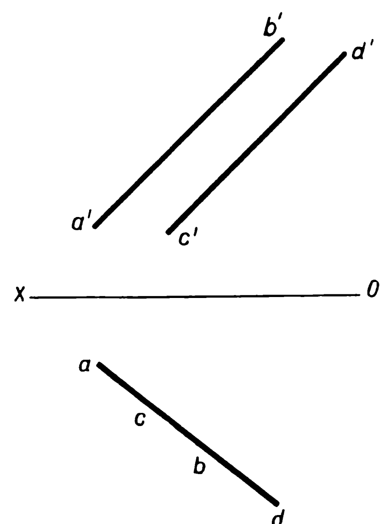



FIG. 135.

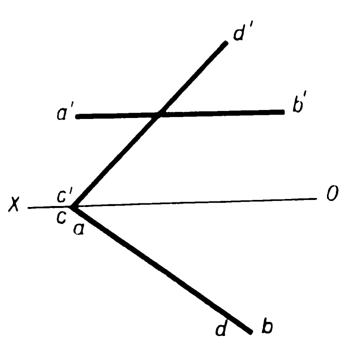



FIG. 136.

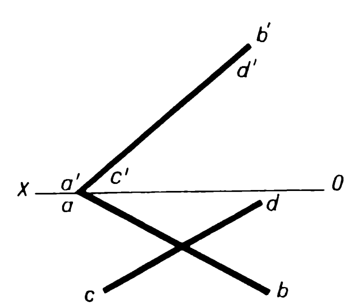



FIG. 137.

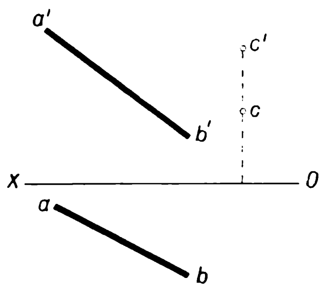



FIG. 138.

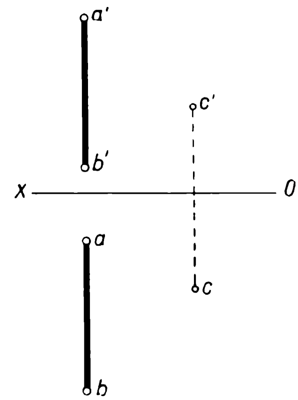



FIG. 139.

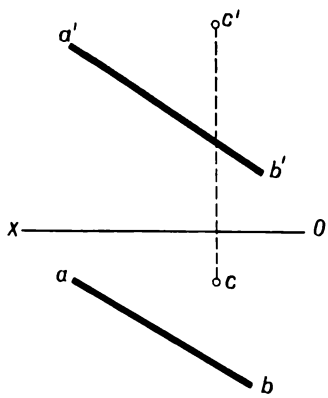



FIG. 140.

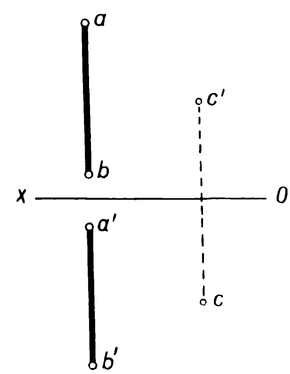



FIG. 141.

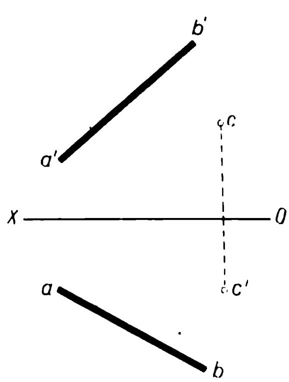



FIG. 142.

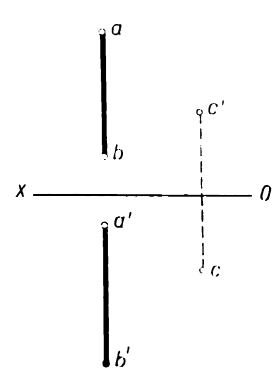



FIG. 143.

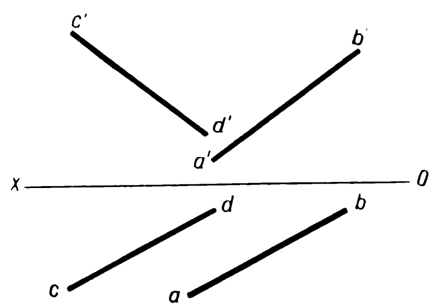



FIG. 144.

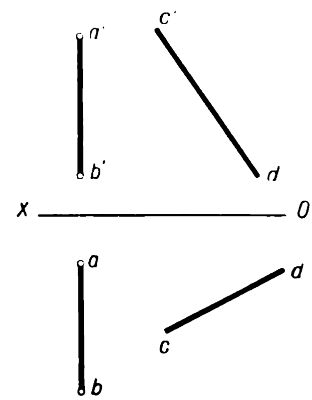



FIG. 145.

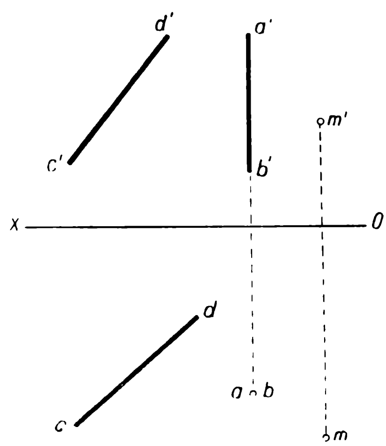



FIG. 146

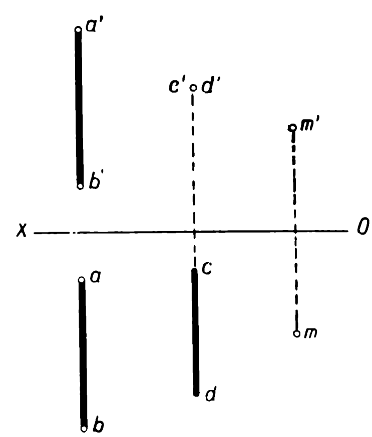



FIG. 147.

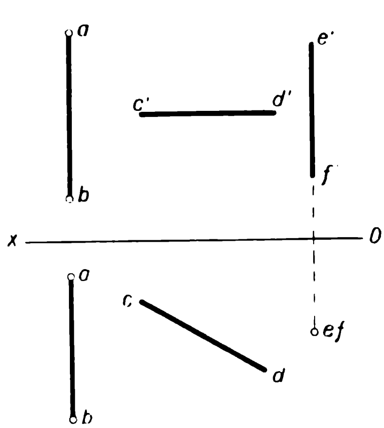



FIG. 148.

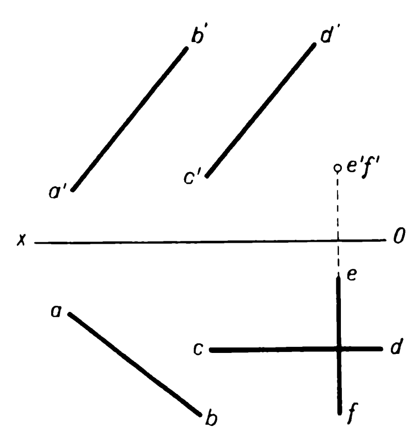



FIG. 149.

45. Through the point $C$ draw a line intersecting the line-segment $AB$ and the coordinate axis (Figs. 140, 141).

46. Through the point $C$ draw a line parallel to the line-segment $AB$ (Figs. 142, 143).

47. Intersect the line-segments $AB$ and $CD$ with the line $MN$, which is parallel to: the horizontal plane of projection (Fig. 144), the vertical plane of projection (Fig. 145), the coordinate axis.

48. Intersect the line-segments $AB$ and $CD$ with the line $EF$ passing through the point $M$ (Figs. 146, 147).

49. Intersect the line-segments $AB$, $CD$, and $EF$ with an arbitrary line $MN$ (Figs. 148, 149).

CHAPTER VIII

# DETERMINING THE LENGTH OF A LINE-SEGMENT AND THE ANGLES OF INCLINATION OF A LINE TO THE PLANES OF PROJECTION

A line-segment parallel to any projection plane is projected on this plane in true length (i.e. without foreshortening).

If a line-segment is parallel to the horizontal plane of projection, the angle at which the horizontal projection of this line-segment is inclined to the coordinate axis is equal to the angle at which the line-segment is inclined to the vertical plane of projection.

If a line-segment is parallel to the vertical plane of projection, the angle at which the vertical projection of this line-segment is inclined to the coordinate axis is equal to the angle at which the line-segment is inclined to the horizontal plane of projection.

The true length of a line-segment is determined from its projections as the hypotenuse of a right triangle one leg of which is one of the projections of the given line-segment, the second leg being equal to the absolute value of the algebraic difference of the distances from the ends of the other projection of the line-segment to the coordinate axis.

The angle formed by one leg of the triangle (say, the horizontal projection of the line-segment) and its hypotenuse (i.e. its true length) is equal to the angle at which the line-segment is inclined to the horizontal plane of projection.

The angle formed by the other leg of the triangle (i.e. the vertical projection of the line-segment) and its hypotenuse (i.e. its true length) is equal to the angle at which the line-segment is inclined to the vertical plane of projection.

## EXAMPLES

**Example 33**

Determine the true length of the line-segment $AB$ (Fig. 150).

*Solution.* Construct a right triangle given its two legs: the horizontal projection ($ab$) of the line-segment and a line-segment of length $|z + z_1|$. The hypotenuse of this triangle will yield the true length of $AB$.

The same result may be obtained by constructing a right triangle whose legs are the vertical projection ($a'b'$) of the line-segment and a line-segment of length $|y_1 - y|$. The hypotenuse of this triangle will yield the true length of the given line-segment.

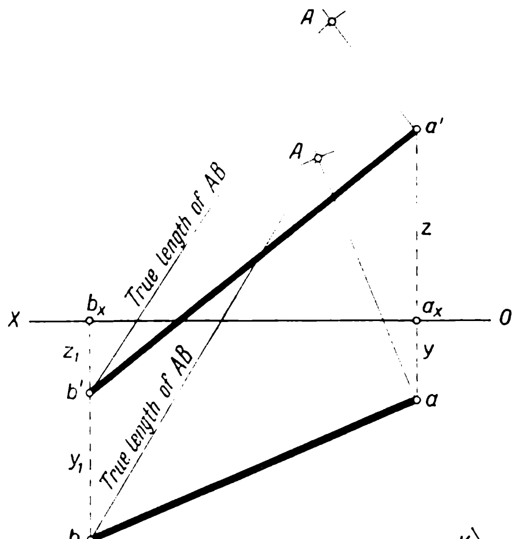



FIG. 150.

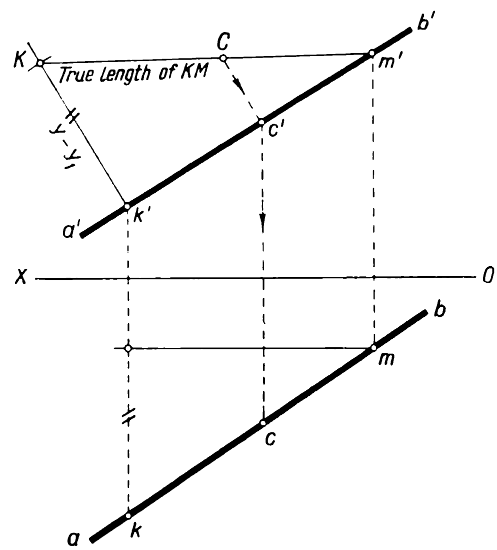



FIG. 151.

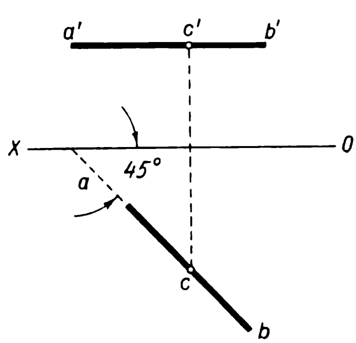



FIG. 152.

**Example 34**

Lay off a length of 28 mm on the line-segment $AB$ from the point $K$ in the direction of the point $B$ (Fig. 151).

*Solution.* Mark an arbitrary line-segment $KM$ on $AB$ and determine its true length. For this purpose construct a right triangle, given two legs ($k'm'$) and $|y - y_1|$. Mark off a length $KC$ equal to 28 mm on the hypotenuse of the obtained triangle and drop from the point $C$ a perpendicular onto $a'b'$ to intersect it at the point $c'$. From the latter find the point $c$ on $ab$. The projections of the desired line-segment are $kc$ and $k'c'$.

**Example 35**

Through the point $C$ draw a straight line parallel to the horizontal plane of projection and inclined at an angle of 45° to the vertical plane of projection (Fig. 152).

*Solution.* Since the required line-segment $AB$ is parallel to the horizontal plane of projection, its vertical projection must be parallel to the coordinate axis. For the line-segment to be inclined to the vertical plane of projection at an angle of 45°, its horizontal projection must be inclined at the same angle to the coordinate axis. Thus, through the point $c'$ draw the vertical projection ($a'b'$) of the line-segment parallel to the coordinate axis, and through the point $c$ the horizontal projection ($ab$) at an angle of 45° to the coordinate axis. There are two such lines, only one being shown in the drawing.

**Example 36**

Determine the angles of inclination of the line-segment $AB$ to the projection planes (Fig. 153).

*Solution.* Construct right triangles $abB$ and $a'b'A$ as above. The angle $\alpha$ is the angle of inclination of the given line-segment to the horizontal plane of projection, $\beta$ being the angle at which the same line-segment is inclined to the vertical plane of projection.

**Example 37**

Through the point $C$ draw a straight line inclined at an angle $\alpha$ to the horizontal plane of projection, and at an angle $\beta$ to the vertical plane of projection $[\alpha + \beta < 90°]$ (Fig. 154).

*Solution.* First draw an auxiliary line inclined to the projection planes at the given angles, for which purpose take an arbitrary point $A$ lying in the vertical plane of projection and through its vertical projection ($a'$) draw the line-segment $a'b_1$ inclined at an angle $\alpha$ to the coordinate axis. Using this line-segment as the hypotenuse, construct a right triangle with an angle $\beta$ at the vertex $a'$. To do this, bisect the line-segment $a'b_1$ and describe a semicircle of radius $\frac{a'b_1}{2}$ from the mid-point as centre. Then, through the point $a'$ draw a leg at an angle $\beta$ to the line-segment $a'b_1$ to intersect the arc at the point $K$, and join this point to the point $b_1$. The leg $Ka'$ is equal in length to the vertical projection of the auxiliary line. To determine its position, from $a'$ as centre strike an arc of radius $a'K$ to intersect the coordinate axis at the point $b'$; the line-segment $a'b'$ is the vertical projection of the auxiliary line.

The leg $Kb_1$ defines the difference of the distances from the end points of the horizontal projection of the line to the coordinate axis. To determine the position of the horizontal projection of the line-segment erect at the point $b'$ a perpendicular to the coordinate axis and lay off on it a line-segment $bb'$ equal to $Kb_1$. Now join $a$ to $b$ to obtain the horizontal projection ($ab$) of the auxiliary line. Then, through the projections ($c$, $c'$) of the point $C$ draw the projections ($mn$, $m'n'$) of the desired line-segment parallel to the projections of the previously constructed line-segment ($ab$, $a'b'$).

*Note.* The point $A$ may lie in the horizontal plane of projection. It is advisable to consider this alternative solution.

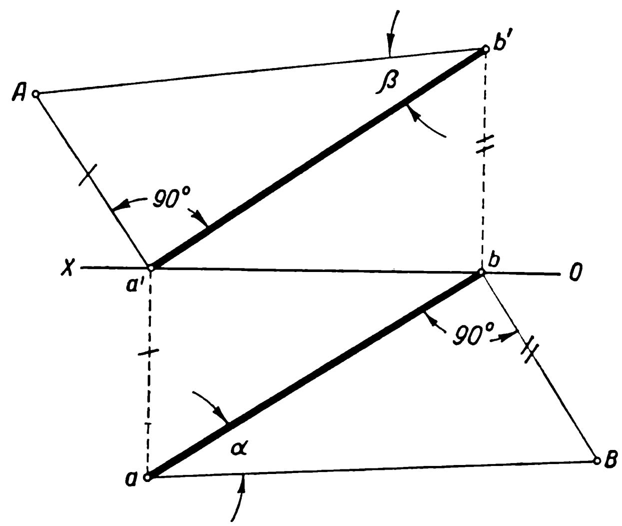



FIG. 153.

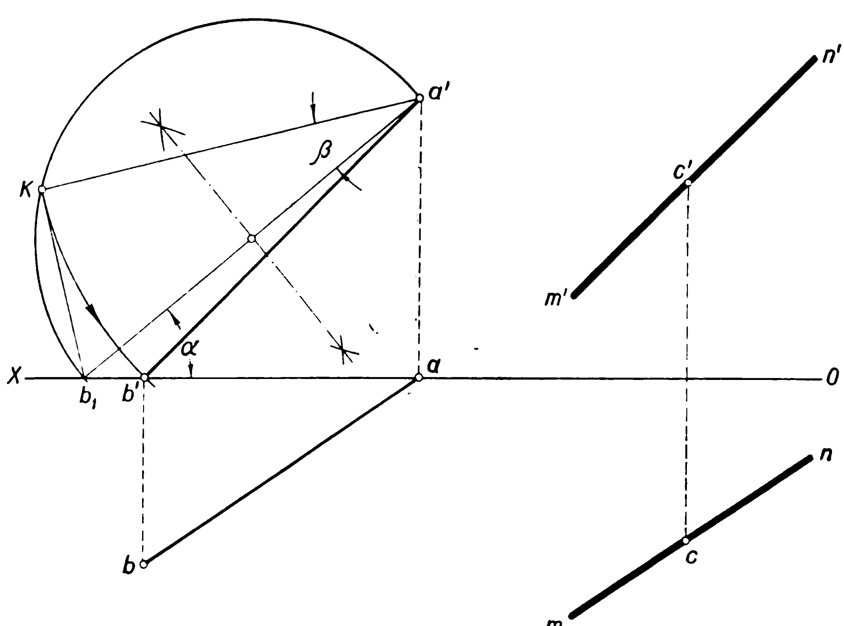



FIG.154.

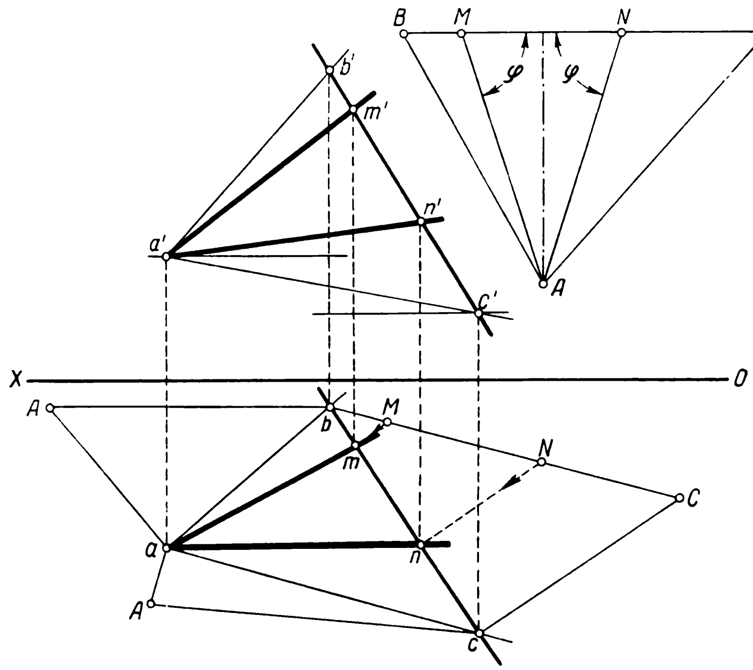



FIG. 155.

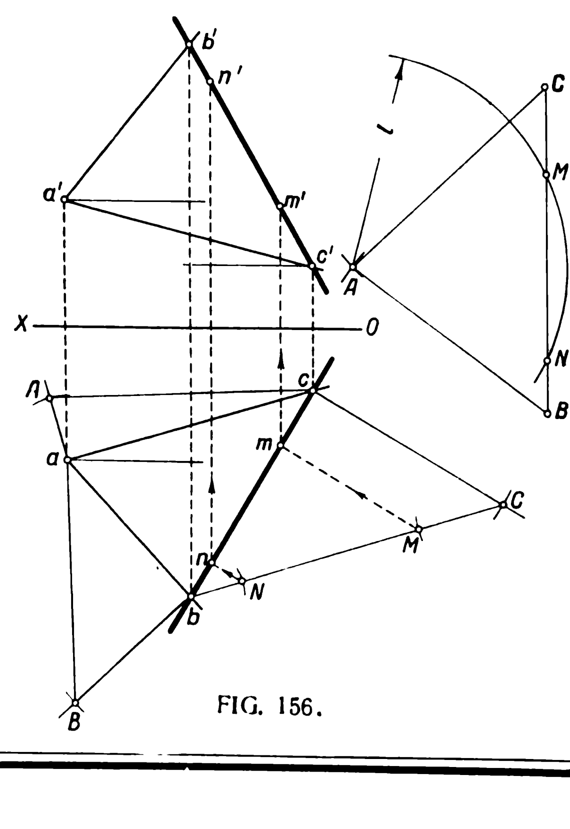



FIG. 156.

**Example 38**

Given: the line-segment $BC$ and the point $A$. Through $A$ draw a straight line intersecting $BC$ at a given angle $\varphi$ (Fig. 155).

*Solution.* Join $b$ ($b'$), $c$ ($c'$), and $a$ ($a'$) and determine the true size of the triangle ($abc$, $a'b'c'$) thus obtained. Construct an auxiliary triangle $ABC$ and, through the point $A$, draw straight lines $AM$ and $AN$ at the given angle $\varphi$ to $BC$. Then lay off on $bC$ line-segments $bM$ and $bN$ equal to $BM$ and $BN$, respectively, and drop from the points $M$ and $N$ perpendiculars onto $bc$ to obtain points $m$ and $n$. From the latter points find the points $m'$, $n'$. The desired lines are ($am$, $a'm'$) and ($an$, $a'n'$).

**Example 39**

Given: the line-segment $BC$ and the point $A$. Find on $BC$ a point lying at a distance of $l$ mm from the given point $A$ (Fig. 156).

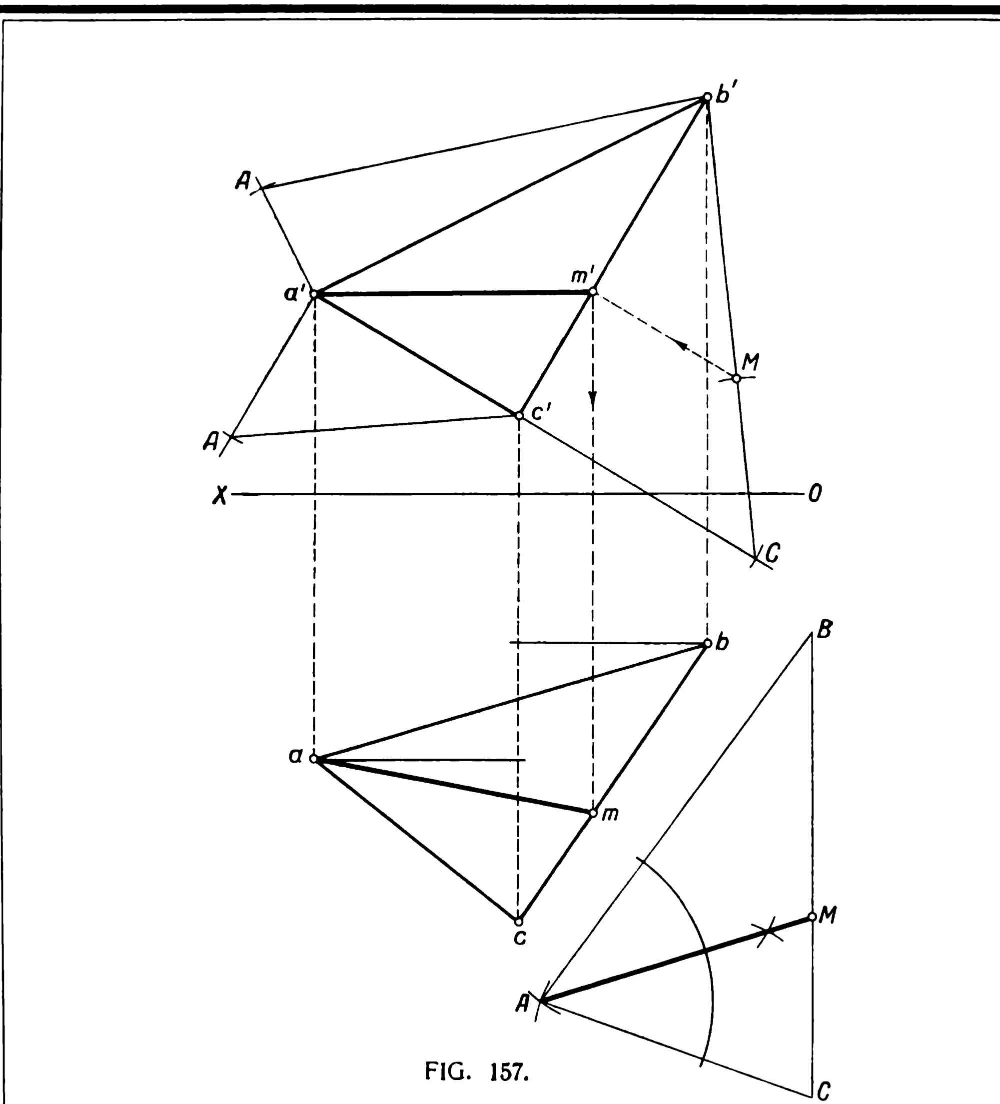



FIG. 157.

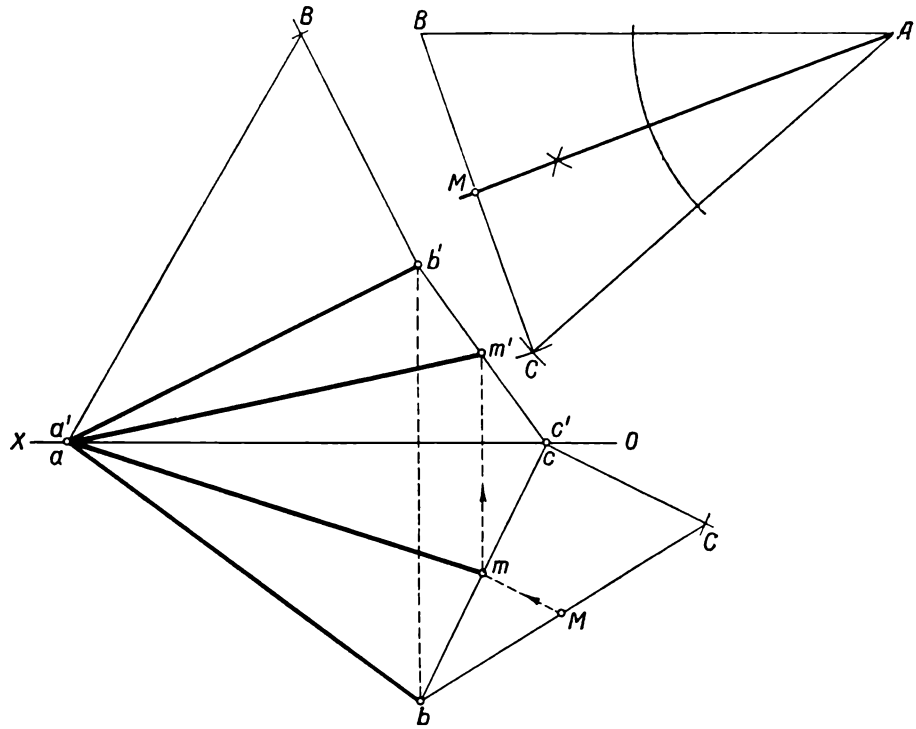



FIG. 158.

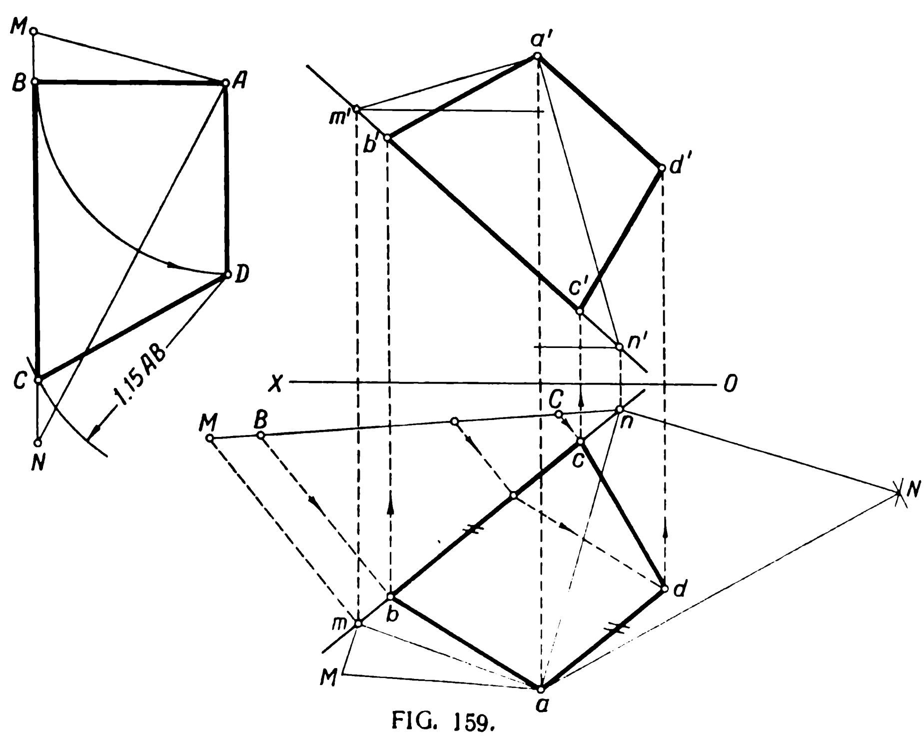



FIG. 159.

*Solution*. Join $b\,(b')$, $c\,(c')$, and $a\,(a')$ and determine the true size of the triangle $(abc, a'b'c')$ thus obtained. Then construct an auxiliary triangle $ABC$ and from $A$ as centre strike an arc of radius $l$ mm to intersect $BC$ at the points $M$ and $N$. Now lay off on $bC$ line-segments $bM$ and $bN$ equal to $BM$ and $BN$, respectively, and drop from the points $M$ and $N$ perpendiculars onto $bc$ to obtain points $m$ and $n$. From the latter points find points $m'$ and $n'$. The points $(m, m')$ and $(n, n')$ are the desired points.

In a particular case only one point may be obtained (when?) or none at all (when?).

**Example 40**

Given: the triangle $ABC$. Draw the bisector of the angle $A$ (Fig. 157).

*Solution*. Find the true size of the given triangle $(abc, a'b'c')$. Construct an auxiliary triangle $ABC$ and draw the bisector of the angle $A$ to intersect the side $BC$ at the point $M$. Now lay off on $b'C$ a line-segment $b'M$ equal to $BM$ and drop from the point $M$ a perpendicular onto $b'c'$ to obtain the point $m'$. From the latter point find the point $m$. The line-segment $(am, a'm')$ is the desired line.

**Example 41**

Given: the straight line $AB$ intersecting the coordinate axis. Draw the bisector of the angle formed by the given line and the coordinate axis (Fig. 158).

*Solution*. Assume an arbitrary point $(c, c')$ on the coordinate axis and join it with the point $(b, b')$ to obtain a triangle $(abc, a'b'c')$. Then find the true size of the triangle. The rest is clear from the drawing (see Example 40).

**Example 42**

Given: the point $A$ and the line-segment $MN$. Construct a right trapezoid $ABCD$, knowing that the longer base $BC$ lies on the line-segment $MN$, the shorter base $AD$ is equal to $AB$, and the lateral side $DC = 1.15\ AB$ (Fig. 159).

*Solution*. To determine the vertices $B$, $C$, $D$ of the trapezoid, construct an auxiliary triangle $(amn, a'm'n')$ and determine its true size. Make a separate drawing of the triangle $AMN$. Drop a perpendicular from the point $A$ onto $MN$ to find the vertex $B$ of the trapezoid. To determine the vertex $D$ draw a straight line through the point $A$ and parallel to $MN$ and lay off on it a length $AB$. The vertex $C$ is determined by striking an arc of radius 1.15 $AB$ from $D$ as centre. The intersection of the arc and the line-segment $MN$ is the desired point. Now do all this on an orthographic drawing.

The construction is obvious from the drawing.

## PROBLEMS

50. Determine the true length of the line-segment $AB$ and the angles at which it is inclined to the projection planes $H$ and $V$ (Fig. 160).

51. What is the geometrical meaning of parallelism of the projections of an oblique line on an orthographic drawing?

52. Through the point $A$ (20, 35) draw a straight line inclined at equal angles to the projection planes $H$ and $V$ (the problem is indeterminate).

53. Find the true size of the triangle $ABC$ (Fig. 161).

54. Through the point $A$ (20, 35) draw a straight line parallel to the vertical plane of projection and inclined at an angle of 45° to the horizontal plane of projection. How many such lines can be drawn?

55. Through the point $A$ (20, 30) draw a straight line inclined at 30° to the horizontal plane of projection and at 45° to the vertical plane. How many lines satisfying these conditions can be drawn?

56. Lay off a length of 15 mm from the point $A$ on the line-segment $AB$ (Fig. 162).

57. Find the centre of the circle described about the triangle $ABC$ (Fig. 163).

58. Find the centre of the circle inscribed in the triangle $ABC$ (Fig. 163).
59. Construct the bisector of the angle $ABC$ (Figs. 164, 165).
60. Drop a perpendicular from the point $A$ onto the line-segment $BC$ (Fig. 166).
61. Determine the distance from the point $A$ to the line-segment $BC$ (Fig. 166).
62. Determine the distance between the parallel lines $AB$ and $CD$ (Fig. 167).
63. Construct a sphere (with $C$ as centre) tangent to the line-segment $AB$ (Fig. 168). (See Problem 61.)
64. Find on the line-segment $AB$ a point 30 mm distant from the point $C$ (Fig. 168). What alternative constructions are possible?
65. Find the points of intersection of the line-segment $MN$ and the sphere (Fig. 169). What alternative constructions are possible? (See Problem 64.)
66. Construct a sphere (with $C$ as centre) cutting a length of 40 mm off the line-segment $AB$ (Fig. 168).

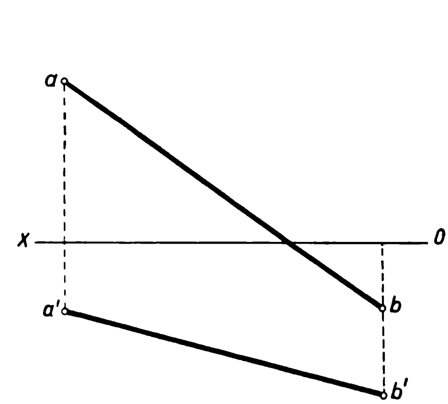



FIG. 160.

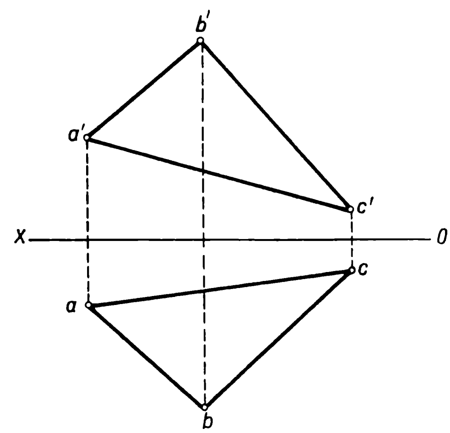



FIG. 161.

67. Construct a right triangle $ABC$ with the right angle $C$ on the line-segment $MN$ (Fig. 170). What alternative constructions are possible?
68. Through the point $C$ draw a straight line intersecting the line-segment $AB$ at an acute angle $\varphi$ equal to 30°, 45°, 60° (Fig. 168). How many such lines are possible?
69. Construct an isosceles triangle $ABC$ with the base $BC$ lying on the line-segment $MN$, given that the length of the lateral side is equal to 1.25 times the altitude (Fig. 171).
70. Construct an isosceles triangle $ABC$ with the base $BC$ lying on the line-segment $MN$, given that the length of the base is 1.5 times the altitude (Fig. 171).
71. Construct an isosceles triangle $ABC$ with the base $BC$ lying on the line-segment $MN$, given that the angle at the base is equal to 30° (Fig. 171).
72. Construct an equilateral triangle $ABC$ with the base $BC$ lying on the line-segment $MN$ (Fig. 171).
73. Construct a right triangle $ABC$ with the leg $BC$ lying on the line-segment $MN$, given that the length of the hypotenuse is equal to $1.25h$ (Fig. 172).
74. Construct a right triangle $ABC$ with the leg $BC$ lying on the line-segment $MN$, given that the acute angle $C$ is equal to 30° (Fig. 172).
75. Construct a right isosceles triangle $ABC$ with the hypotenuse $BC$ lying on the line-segment $MN$ (Fig. 171).
76. Construct a right isosceles triangle $ABC$ with the side $BC$ lying on the line-segment $MN$ (Fig. 172).

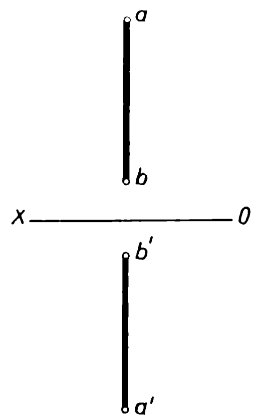



FIG. 162.

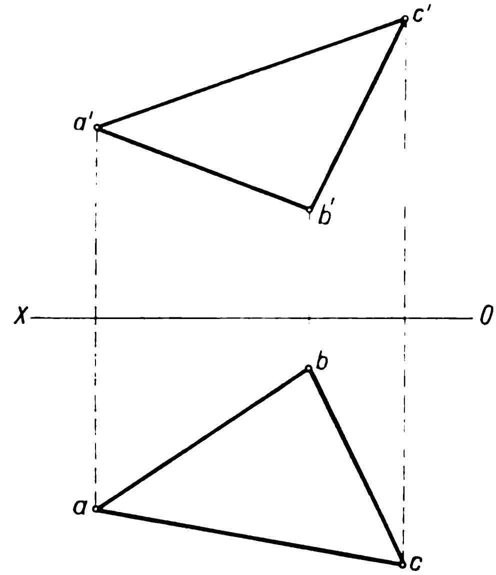



FIG. 163.

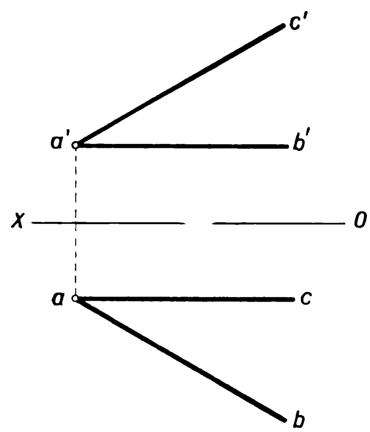



FIG. 164.

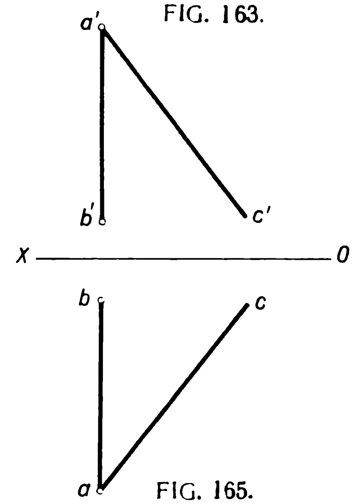



FIG. 165.

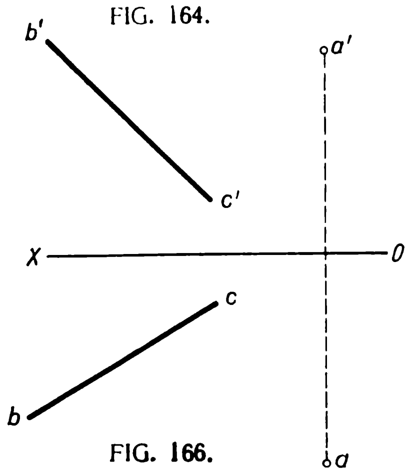



FIG. 166.

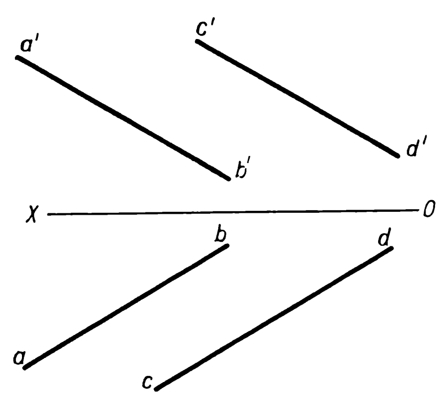



FIG. 167.

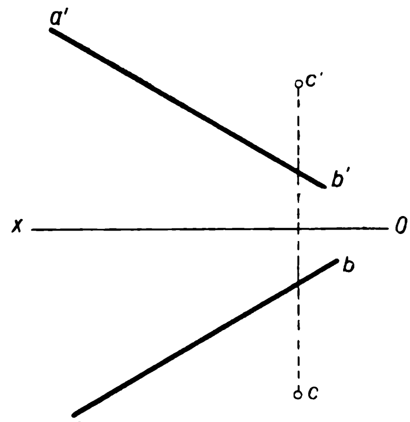



FIG. 168.

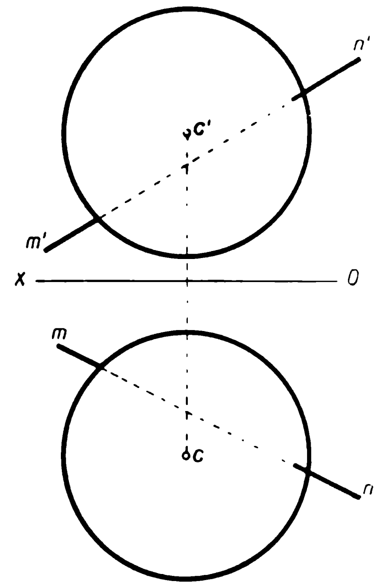



FIG. 169.

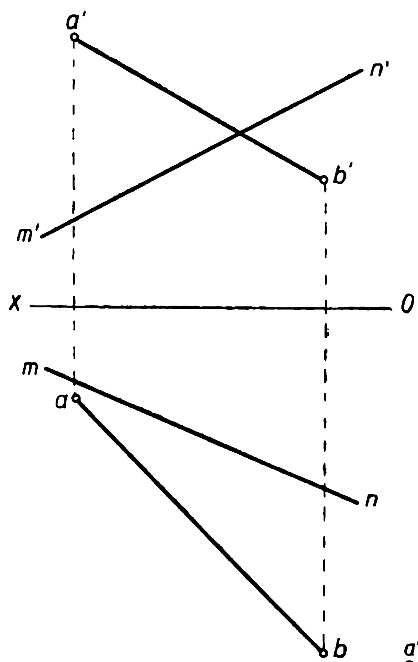



FIG. 170.

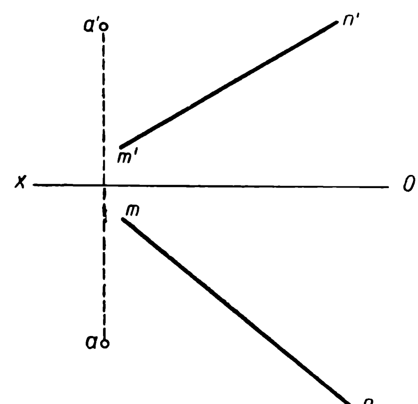



FIG. 171.

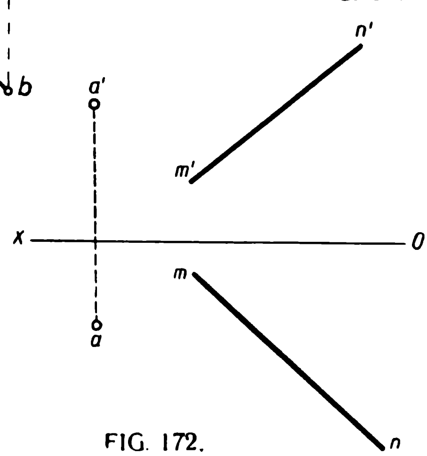



FIG. 172.

77. Construct a rectangle $ABCD$ with the longer base $BC$ lying on the line-segment $MN$, given that the area of the triangle is equal to 1.5 $AB^2$ (Fig. 172).

78. Construct a rectangle $ABCD$ with the longer base $BC$ lying on the line-segment $MN$, given that the length ratio of its sides is 1.5 (Fig. 172).

79. Construct a square $ABCD$ with the side $BC$ lying on the line-segment $MN$ (Fig. 172).

80. Construct a square $ABCD$ with the diagonal $BD$ lying on the line-segment $MN$ (Fig. 171).

81. Construct a rhombus $ABCD$ with the side $BC$ lying on the line-segment $MN$ given that the length of the side is 1.2 times the altitude (Fig. 171).

82. Construct a rhombus $ABCD$ with the side $BC$ lying on the line-segment $MN$, given that the acute angle $B$ equals 60° (Fig. 171).

83. Construct a rhombus $ABCD$ with the longer diagonal $BD$ lying on the line-segment $MN$, given that the length ratio of the diagonals equals 2 (Fig. 171).

84. Construct a parallelogram $ABCD$ with the base $BC$ lying on the line-segment $MN$, given that the acute angle $B$ equals 60° and the diagonal $AC$ is 5 mm longer than the lateral side (Fig. 171).

85. Construct a parallelogram $ABCD$ with the base $BC$ lying on the line-segment $MN$, given that the length of the lateral side is equal to $1.25h$ and the length ratio of its sides equals 2 (Fig. 171).

86. Construct a right-angled trapezoid $ABCD$ with the longer base $BC$ lying on the line-segment $MN$, given that $AD = AB$ and $DC = 1.15\,AB$ (Fig. 172).

87. Construct a right-angled trapezoid $ABCD$ with the longer base $BC$ lying on the line-segment $MN$, given that $AD = AB = \frac{2}{3}BC$ (Fig. 172).

88. Construct a right-angled trapezoid $ABCD$ with the longer base $BC$ lying on the line-segment $MN$, given that $AD = AB$ and the angle $C = 45°$ (Fig. 172).

89. Construct an isosceles trapezium $ABCD$ with the longer base $BC$ lying on the line-segment $MN$, given that $AB = AD = DC = 40$ mm (Fig. 171).

90. Construct an isosceles trapezium $ABCD$ with the longer base $BC$ lying on the line-segment $MN$, given that the acute angle $B = C = 45°$ and the shorter base is equal to the lateral side (Fig. 171).

CHAPTER IX

# DIVIDING A SEGMENT OF ASTRAIGHT LINE IN A GIVEN RATIO

If a point divides a line-segment in the ratio $\frac{m}{n}$, then the projections of this point divide the respective projections of the segment in the same ratio.

Hence, it is not necessary to determine the *true* length of a line-segment dividing it in a given ratio on an orthographic drawing.

EXAMPLES

**Example 43**

Given: the line-segment $MN$ and the point $C$. Draw, through the point $C$, a straight line intersecting the given segment at a point dividing the visible portion of $MN$ in the ratio 2 : 3 in the direction from $H$ to $V$ (Fig. 173).

*Solution.* Find the traces of the given segment ($mn$, $m'n'$) and divide one of its projections, say the horizontal one, with the point $k$ in the ratio 2 : 3. From $k$ find the vertical projection ($k'$) of the point on the vertical projection ($m'n'$) of the line-segment $MN$. Then draw the projections of the desired line through the like projections of the points $K$ and $C$: the horizontal projection of the line through the points $k$ and $c$, and the vertical projection through $k'$ and $c$.

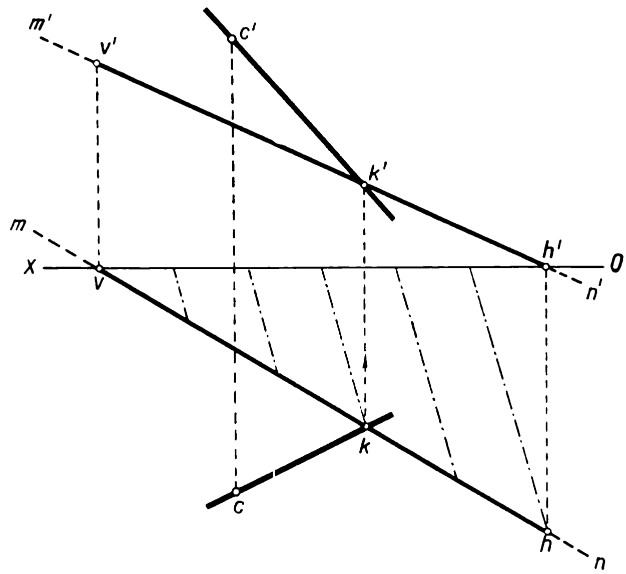



FIG. 173.

PROBLEMS

91. Find the point $C$ dividing the line-segment $AB$ in the ratio: $\frac{m}{n} = \frac{1}{2}$ (Fig. 174); $\frac{m}{n} = \frac{2}{3}$ (Fig. 175).

92. Through the point $C$ draw a straight line intersecting the line-segment $AB$ at a point dividing it in the ratio $\frac{m}{n} = \frac{1}{3}$ (Fig. 176).

93. Find the centre of gravity of the area of the triangle $ABC$ (Fig. 163).

*Note.* The centre of gravity of the area of a triangle lies at the point of intersection of its medians.

94. Find the centre of gravity of the perimeter of the triangle $ABC$ (Fig. 163).

*Note.* The centre of gravity of the perimeter of a triangle lies at the centre of a circle inscribed in a triangle whose vertices lie at the mid-points of the sides of the given triangle.

CHAPTER X

# PROJECTING OF ANGLES

Any angle whose sides are parallel to one of the projection planes is projected on this plane without foreshortening (i.e. in true size).

A *right angle* with at least one side parallel to one of the projection planes is projected on this plane *also* as a right angle.

Hence, if two intersecting lines in space are mutually perpendicular and one of them is parallel to a projection plane, then their projections on this plane are also mutually perpendicular (see Chapter VII).

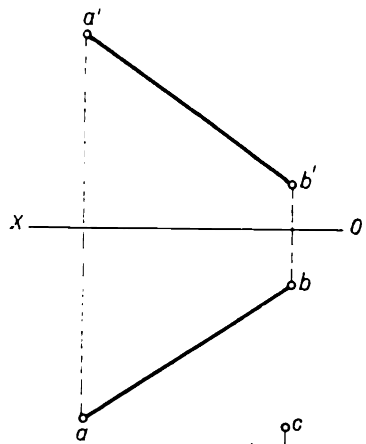



FIG. 174.

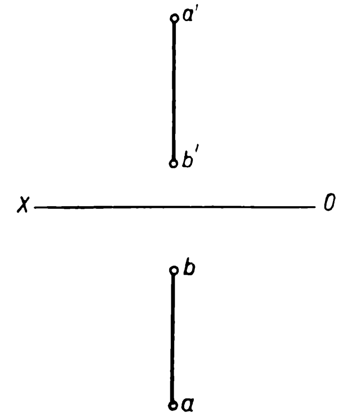



FIG. 175.

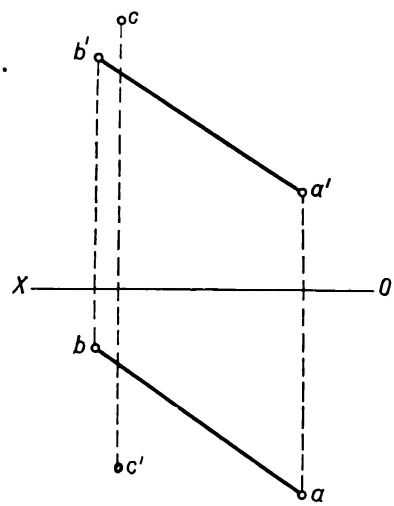



FIG. 176.

EXAMPLES

**Example 44**

Given: the line-segment $AB$ and the point $C$. Draw, through the point $C$, a straight line intersecting $AB$ at right angles (Fig. 177).

*Solution.* The desired line must satisfy three conditions, namely: it has to pass through the point $C$, be perpendicular to the line-segment $AB$ (which is parallel to the vertical plane of projection), and intersect the line $AB$. On an orthographic drawing the projections of the required line must pass through the like projections of the point $C$; the vertical projections of the given and desired lines must be mutually perpendicular, and, finally, the points of intersection of the like projections must lie on a common perpendicular to the coordinate axis.

Hence, draw, through the point $c'$, the vertical projection of the desired line perpendicular to $a'b'$ to intersect it at the point $k'$. From the latter point find the point $k$ on the horizontal projection ($ab$) of $AB$ and draw through it the horizontal projection ($ck$) of the desired line.

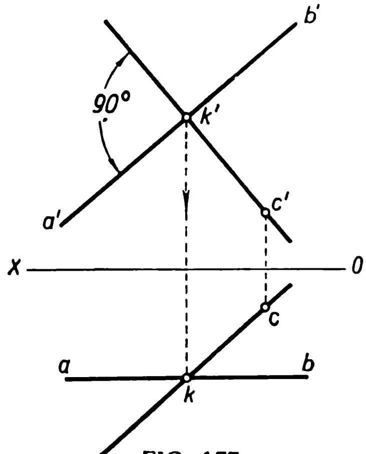



FIG. 177.

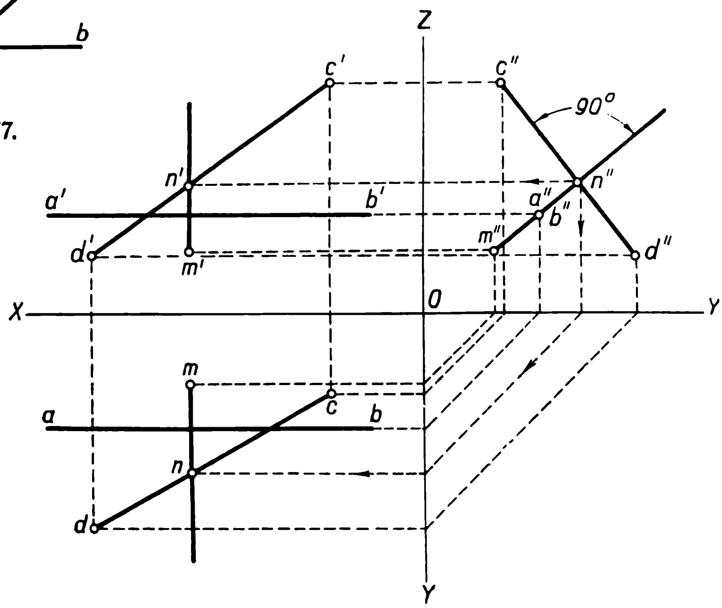



FIG. 178.

**Example 45**

Intersect the line-segments $AB$ and $CD$ with a third line perpendicular to both of them (Fig. 178).

*Solution.* The desired line $MN$ is a profile line, since it must be perpendicular to $AB$, which is parallel to the coordinate axis. For the desired line $MN$ to be perpendicular to $CD$ *as well*, it is necessary that their profile projections ($m''n''$ and $c''d''$) be mutually perpendicular (the theorem on projecting a right angle).

Hence, draw through the point $a''b''$ a line $m''n''$ perpendicular to $c''d''$ until they intersect at the point $n''$. Then find the points $n$ and $n'$ (from $n''$) on the like projections of the line-segment $CD$ and draw the lines $mn$ and $m''n''$.

**Example 46**

Given: the line-segment $AB$ and the point $C$. Determine the distance from $C$ to $AB$ (Fig. 179).

*Solution.* Drop a perpendicular from the given point ($c$, $c'$) onto the line-segment ($ab$, $a'b'$) and find the point of their intersection ($k$, $k'$) by drawing through the point $c$ a straight line perpendicular to $ab$ to obtain the point ($k$) at their intersection. Then find $k'$ and determine the true length of the line-segment ($ck$, $c'k'$).

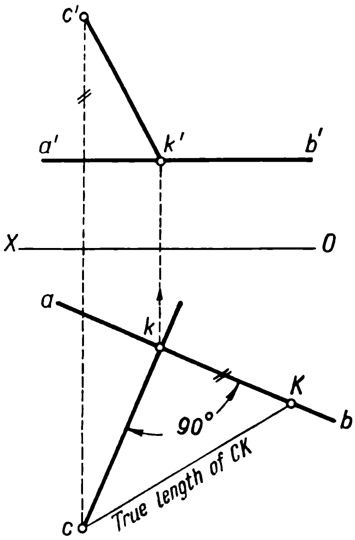



FIG. 179.

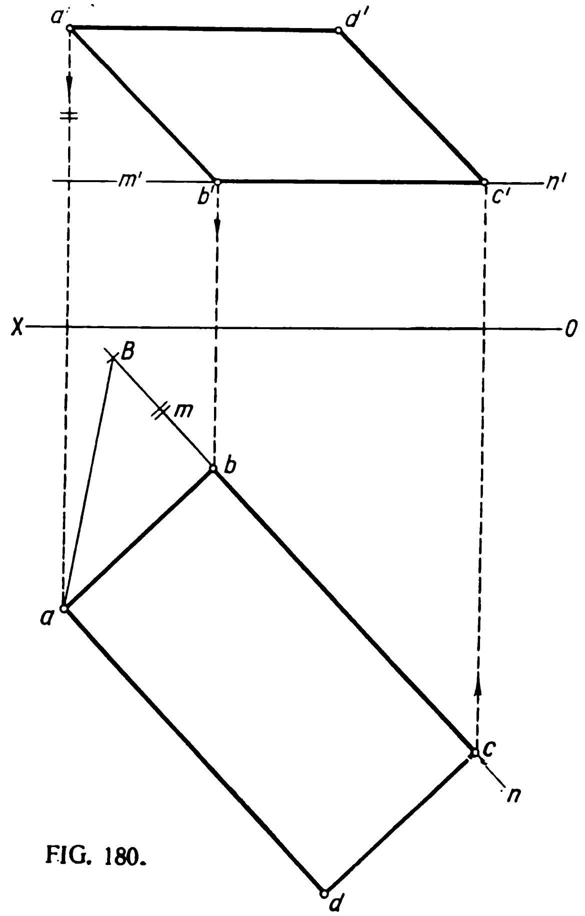



FIG. 180.

**Example 47**

Given: the straight line *MN* (parallel to the horizontal plane of projection) and the vertical projection of the line-segment *AB*, which is perpendicular to *MN*. Construct a rectangle *ABCD* with the base *BC* lying on *MN*, given that $BC = 1.5\,AB$ (Fig. 180).

*Solution.* Determine the point *b* and, by drawing through it a line perpendicular to *mn*, find the horizontal projection (*ab*) of the lateral side *AB*. Find the true length (*aB*) of (*ab*, *a'b'*) and lay off on the given line (*mn*, *m'n'*) from the point (*b*, *b'*) a length of 1.5 *aB* to obtain the point (*c*, *c'*). To complete the construction, draw, through (*c*, *c'*) and (*a*, *a'*), lines parallel to (*ab*, *a'b'*) and (*bc*, *b'c'*).

## PROBLEMS

95. Through the point *C* draw a straight line intersecting the line-segment *AB* at right angles (Figs. 181 to 185).

96. Intersect the line-segments *AB* and *CD* with a straight line perpendicular to them (Figs. 186, 187).

97. Drop a perpendicular from the point *C* onto the line-segment *AB* (Figs. 188, 189).

98. Determine the distance from the point *C* to the line-segment *AB* (Figs. 188, 189).

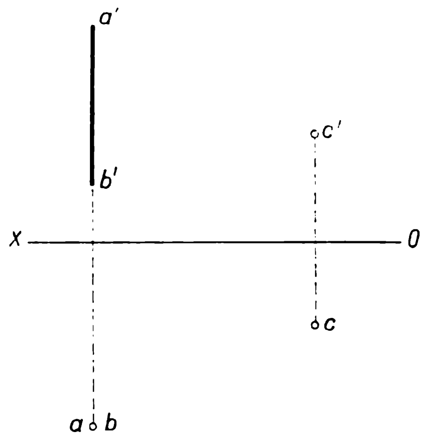



FIG. 181.

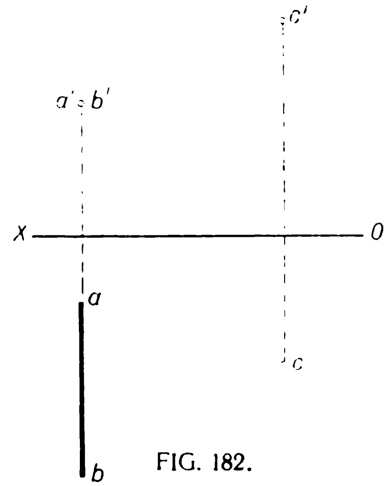



FIG. 182.

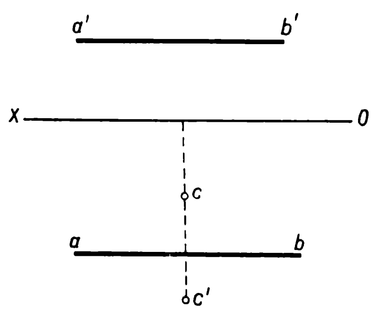



FIG. 183.

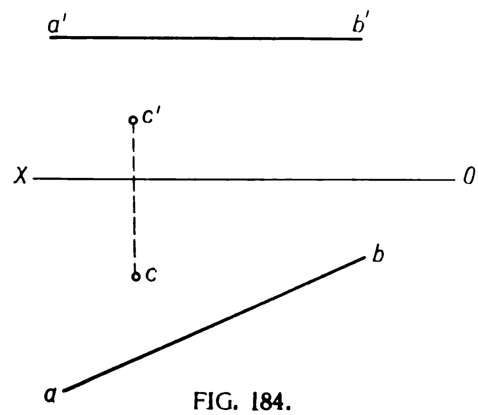



FIG. 184.

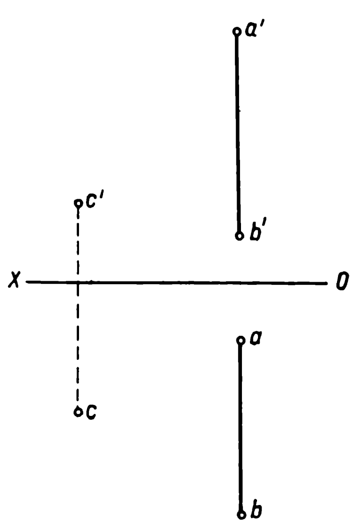



FIG. 185.

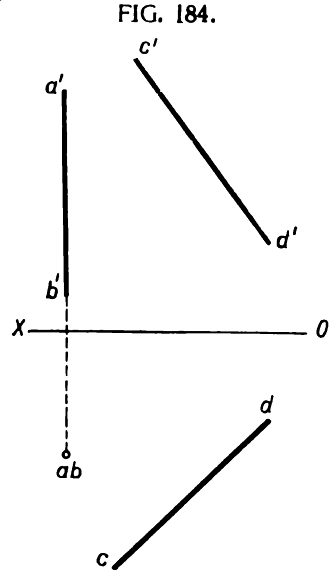



FIG. 186.

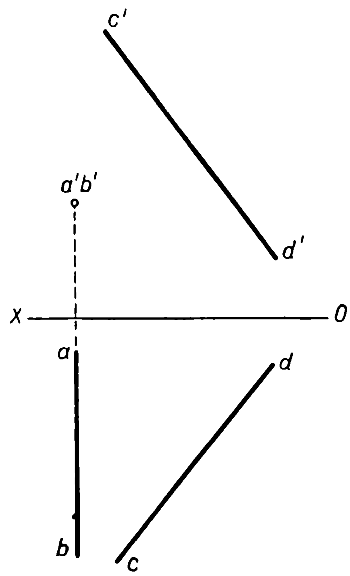



FIG. 187.

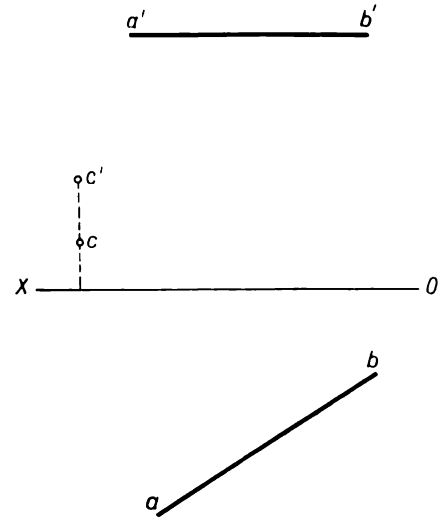



FIG. 188.

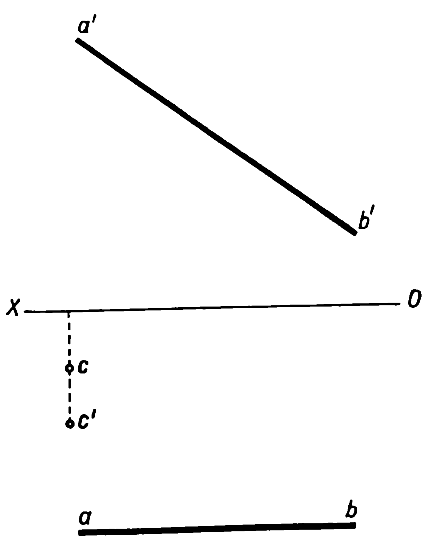



FIG. 189.

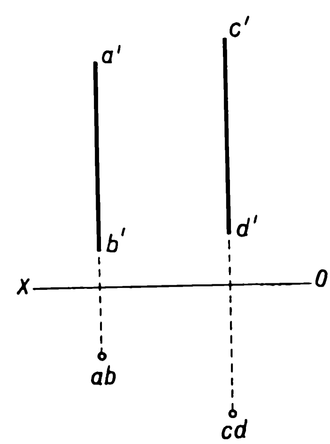



FIG. 190.

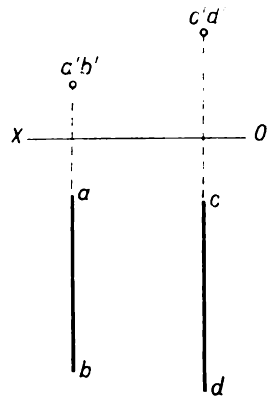



FIG. 191.

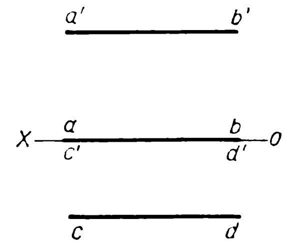



FIG. 192.

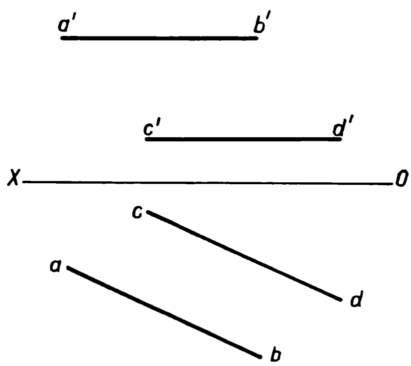



FIG. 193.

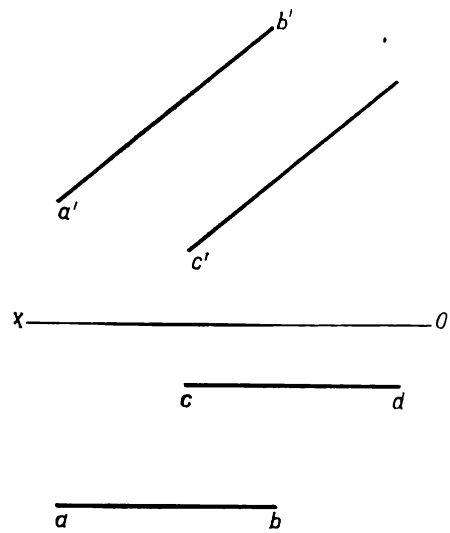



FIG. 194.

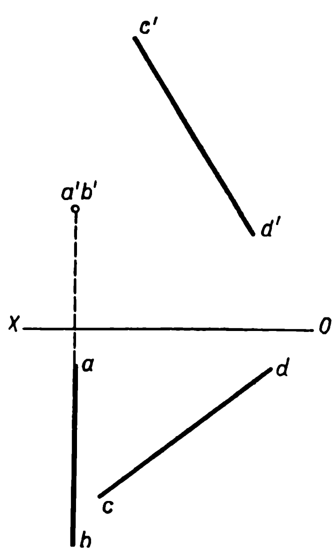



FIG. 195.

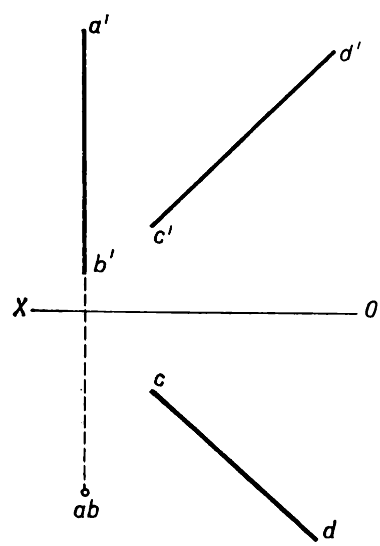



FIG. 196.

99. Determine the distance between the parallel lines $AB$ and $CD$ (Figs. 190 to 194).

100. Determine the distance between the skew lines $AB$ and $CD$ (Figs. 195, 196).

101. Determine the lacking projection of the point $C$, given that the distance $l$ from $C$ to $AB$ is equal to 30 mm (Figs. 197 to 201). What alternative solutions are possible?

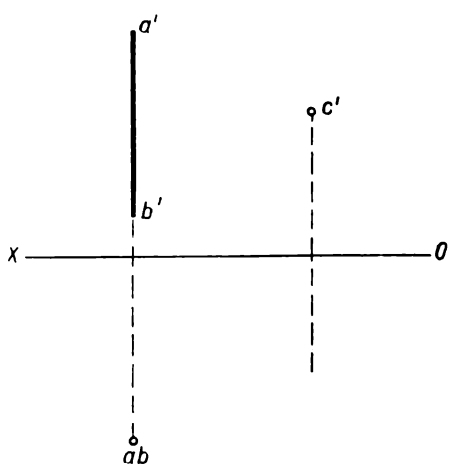



FIG. 197.

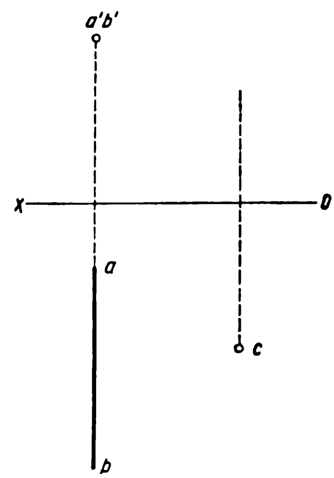



FIG. 198.

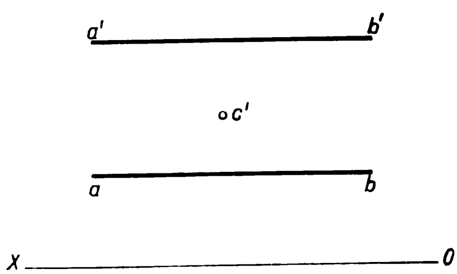



FIG. 199.

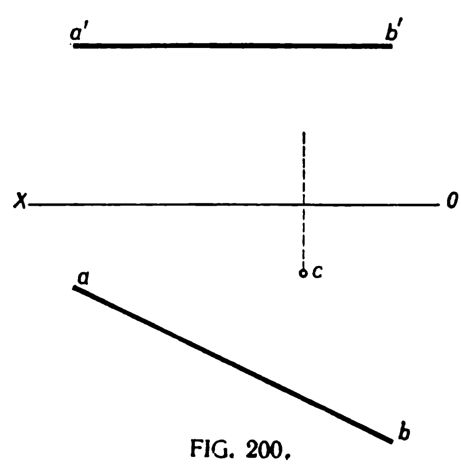



FIG. 200.

102. Determine the lacking projection of the line-segment $CD$, which is parallel to $AB$, given that the distance between them $l = 20$ mm (Figs. 202 to 206). What alternative solutions are possible?

103. Construct a sphere (with $C$ as centre) tangent to the line-segment $AB$ (Figs. 207, 208). (See Problem 98.)

104. Find on the line-segment $AB$ a point 40 mm distant from the point $C$ (Fig. 209). What alternative solutions are possible?

105. Find the point of intersection of the line $AB$ and the sphere (Fig. 210). What alternative solutions are possible? (See Problem 104.)

106. Describe a sphere (with $C$ as centre) cutting a length $l = 40$ mm off the given line $AB$ (Fig. 211) (See Problem 98.)

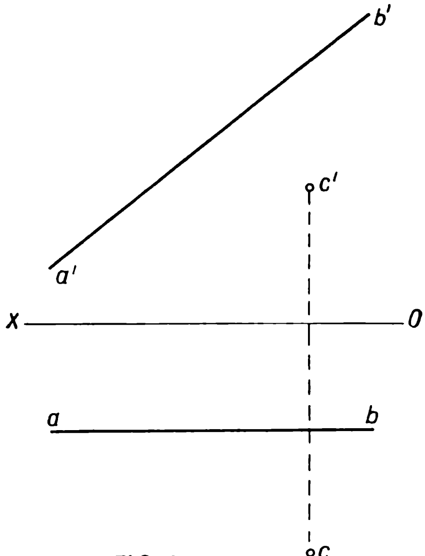



FIG. 201.

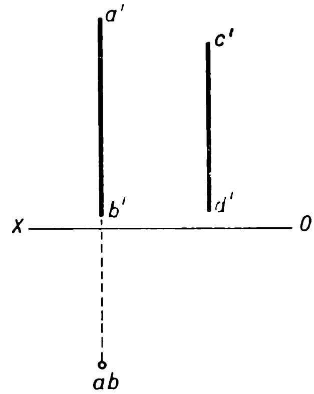



FIG. 202.

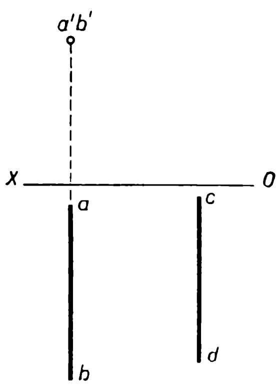



FIG. 203.

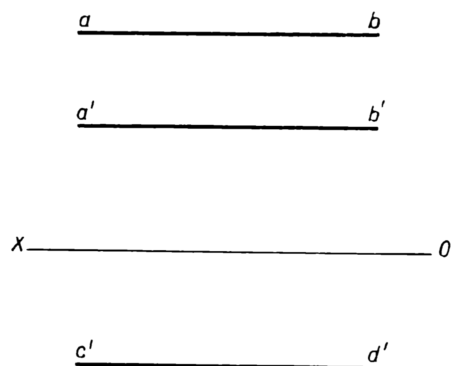



FIG. 204.

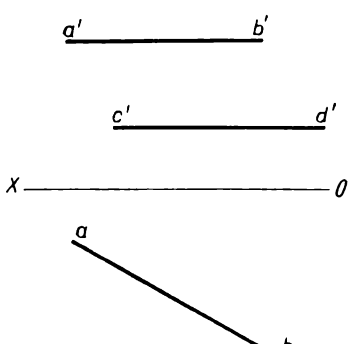



FIG. 205.

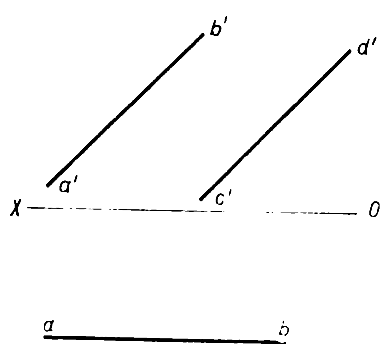
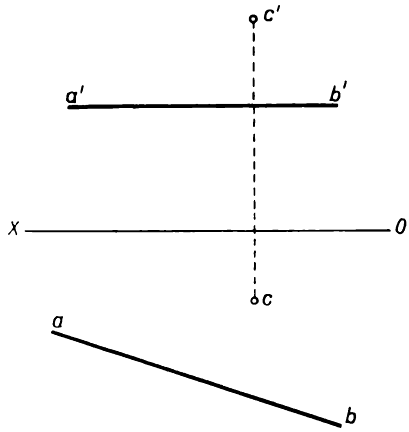



FIG. 206.



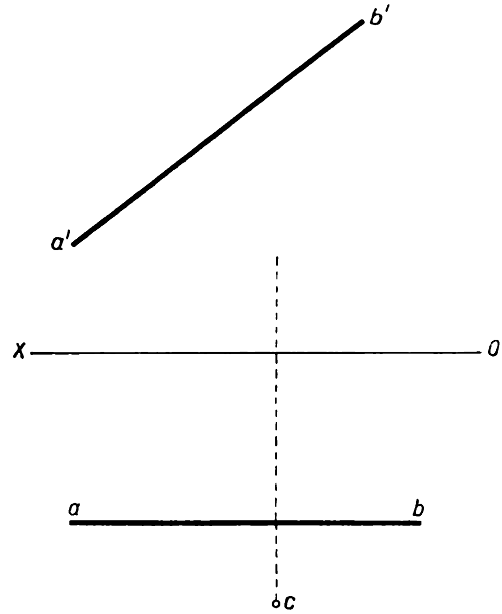
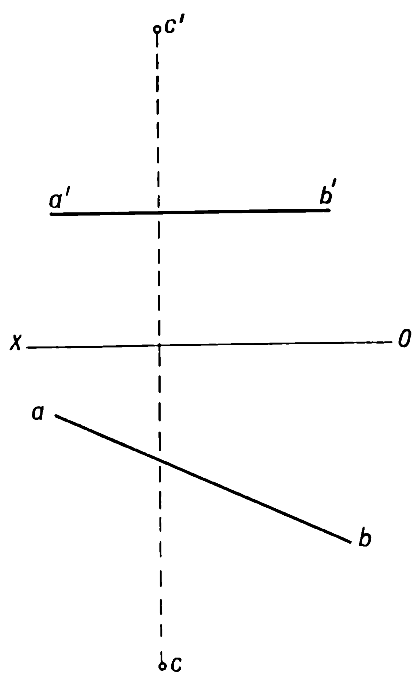




FIG. 207.

FIG. 208.



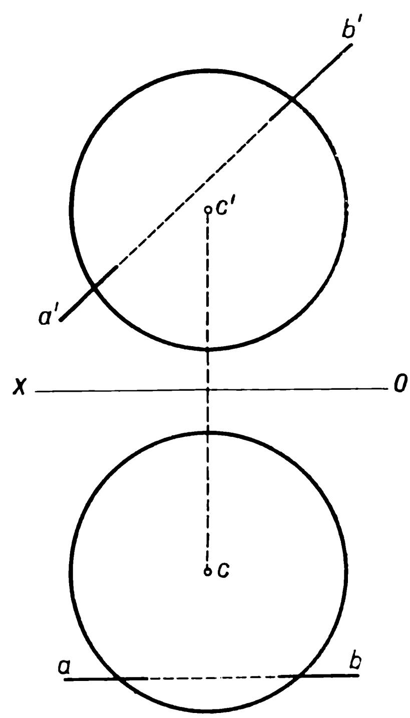




FIG. 209.

FIG. 210.

107. Construct a right triangle $ABC$ with the right angle $B$ on the line-segment $EF$ (Fig. 212). What alternative solutions are possible?

108. Through the point $C$ draw a straight line intersecting the line-segment $MN$ at an acute angle $\varphi$ equal to 30°, 45°, 60° (Fig. 213). How many such lines are possible?

109. Construct an isosceles triangle $ABC$ with the base $BC$ lying on the line-segment $MN$, given that the side is equal to $1.25h$ (Fig. 214).

110. Construct an isosceles triangle $ABC$ with the base $BC$ lying on the line-segment $MN$, given that the length of the base equals $1.5h$ (Fig. 215).

111. Construct an isosceles triangle $ABC$ with the base $BC$ lying on the line-segment $MN$, given that the angle at the base is equal to 30° (Fig. 214).

112. Construct an isosceles triangle $ABC$ with the base $BC$ lying on the line-segment $MN$, given that the side is 10 mm longer than the altitude (Fig. 214).

113. Construct an isosceles triangle $ABC$ with the base $BC$ lying on the line-segment $MN$ and with the vertex $A$ on the line-segment $EF$, given that $K$ is the foot of the altitude $AK$ and the side equals 1.15 $AK$ (Fig. 216).

114. Construct an isosceles triangle $ABC$, given that its base $BC$ (=60 mm) lies on the line-segment $MN$ and the vertex $A$ on the line-segment $EF$ perpendicular to $MN$, the altitude being equal to 40 mm (Fig. 217).

115. Construct an isosceles triangle $ABC$ with the base $BC$ lying on the line-segment $MN$, given that the altitude $AD$ (=40 mm) lies on the line-segment $EF$ and the angle at the base is equal to 30° (Fig. 217).

116. Construct an isosceles triangle $ABC$ with the vertex $A$ lying on the line-segment $EF$ (Fig. 218).

117. Construct an equilateral triangle $ABC$ with the base $BC$ lying on the line-segment $MN$ (Fig. 214).

118. Construct an equilateral triangle $ABC$ with the base $BC$ lying on the line-segment $MN$, given that $K$ is the foot of the altitude (Fig. 219).

119. Construct a right-angled trapezoid $ABCD$ with the longer base $BC$ lying on the line-segment $MN$, given $AD = AB$, $DC = 1.15\ AB$ (Fig. 220).

120. Construct an equilateral triangle $ABC$ with the base $BC$ lying on the line-segment $MN$, given that the altitude $AD$ (=40 mm) lies on the line-segment $EF$ (Fig. 217).

121. Construct an equilateral triangle $ABC$ with the base $BC$ (=50 mm) lying on the line-segment $MN$, and with the vertex $A$ on the line-segment $EF$ perpendicular to $MN$ (Fig. 217).

122. Construct a right-angled triangle $ABC$ with the leg $BC$ lying on the line-segment $MN$, given that the length of the hypotenuse is equal to 1.25 $h$ (Fig. 220).

123. Construct a right-angled triangle $ABC$ with the leg $BC$ lying on the line-segment $MN$, given that the acute angle $C$ is equal to 30° (Fig. 220).

124. Construct a right-angled isosceles triangle $ABC$ with the hypotenuse $AC$ lying on the line-segment $MN$ (Fig. 221).

125. Construct a right-angled isosceles triangle $ABC$ with the leg $BC$ lying on the line-segment $MN$ (Fig. 220).

126. Construct a right-angled triangle $ABC$ with the leg $BC$ lying on the line-segment $MN$, given that the radius of a circle described about the triangle is equal to 0.75 $AB$ (Fig. 220).

127. Construct a right-angled isosceles triangle $ABC$ with the leg $BC$ lying on the line-segment $BM$ and the vertex $A$ on the line-segment $EF$ (Fig. 222).

128. Construct a right-angled triangle $ABC$ with the base $BC$ lying on the line-segment $MN$, given that the leg $AB$ (=30 mm) lies on the line-segment $EF$, and the triangle area is 0.75 $AB^2$ (Fig. 223).

129. Construct a rectangle $ABCD$ with the longer base $BC$ lying on the line-segment $MN$, given that the rectangle area is 1.5 $AB^2$ (Fig. 220).

130. Construct a rectangle $ABCD$ with the longer base $BC$ lying on the line-segment $MN$, given that the ratio of sides is equal to 1.5 (Fig. 220).

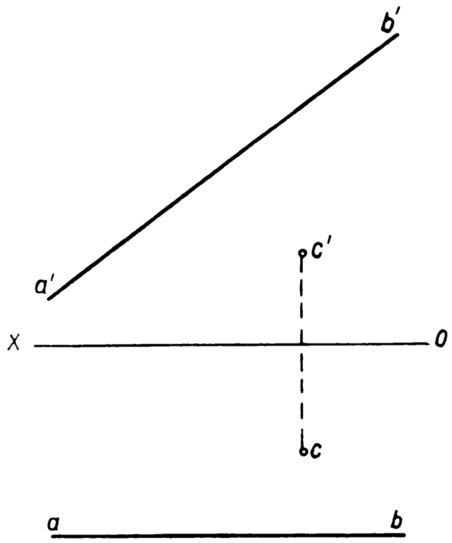



FIG. 211.

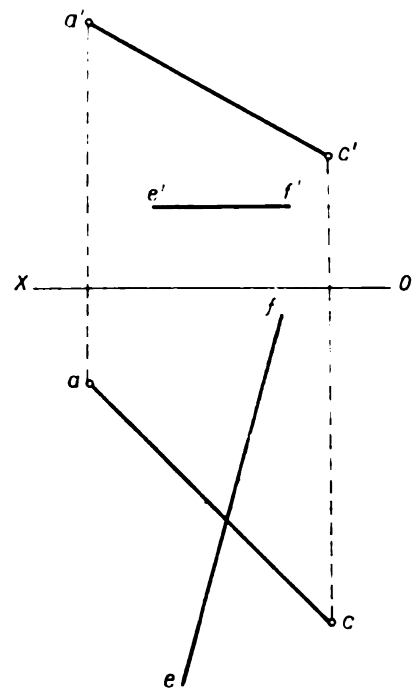



FIG. 212.

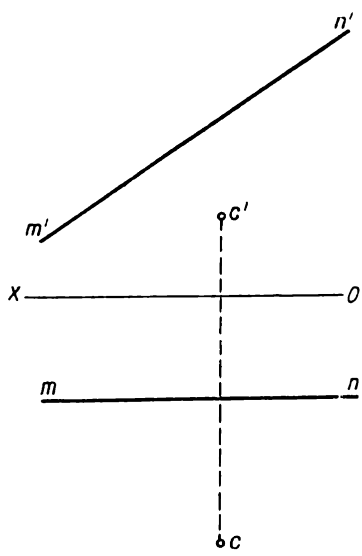



FIG. 213.

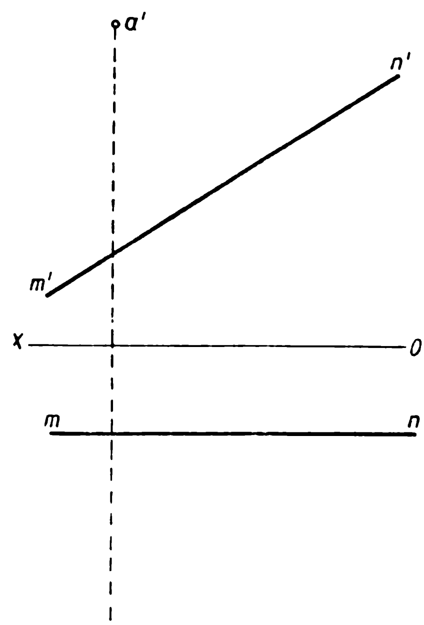



FIG. 214.

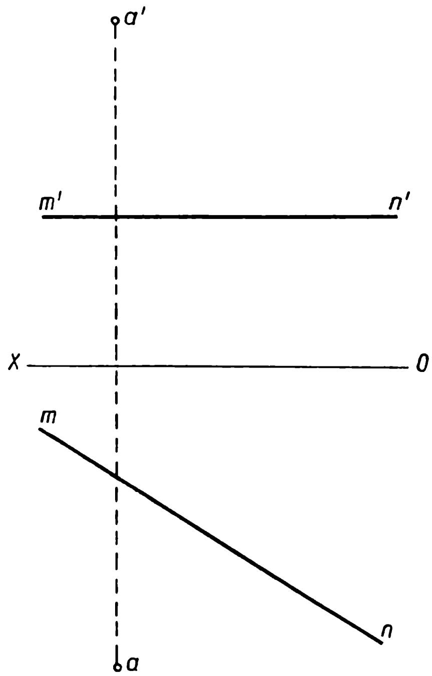



FIG. 215.

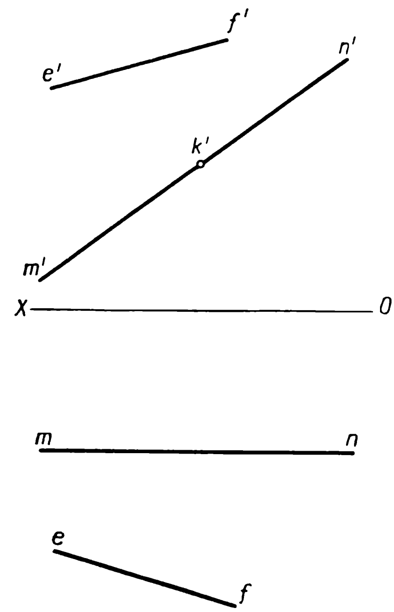



FIG. 216.

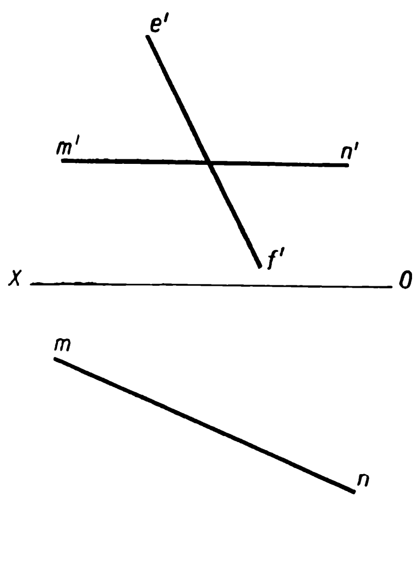



FIG. 217.

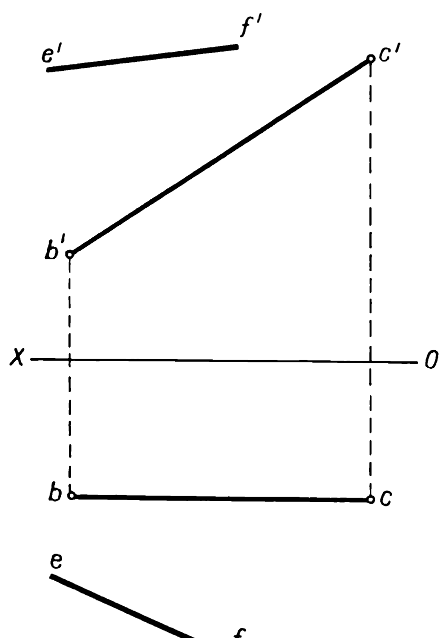



FIG. 218.

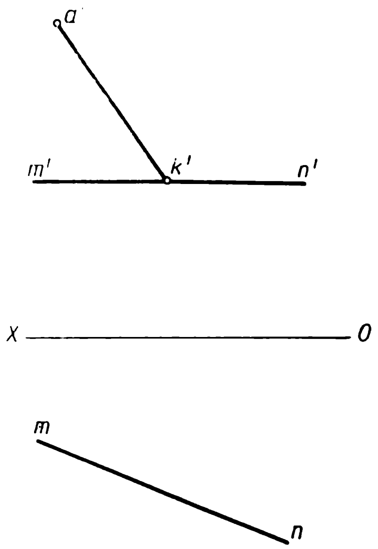



FIG. 219.

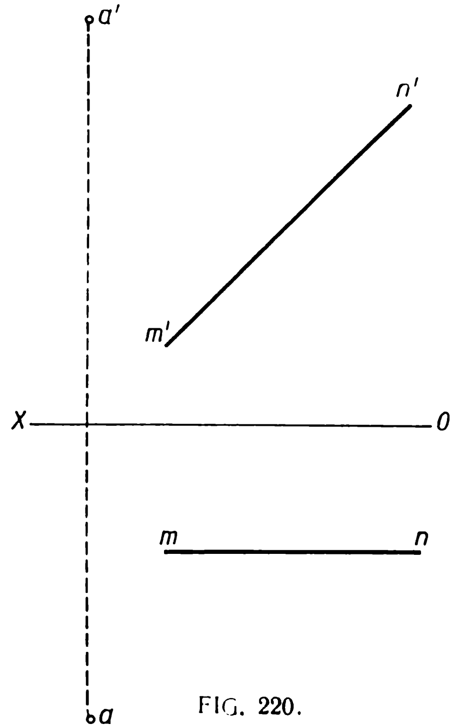



FIG. 220.

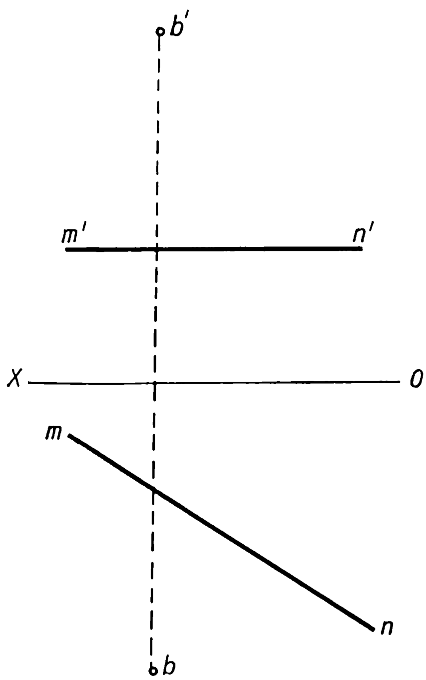



FIG. 221.

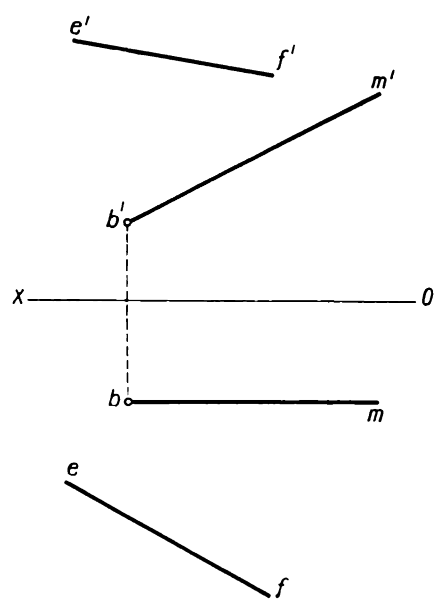



FIG. 222.

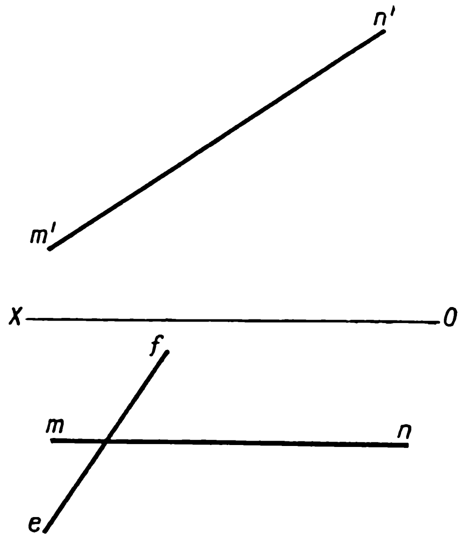



FIG. 223.

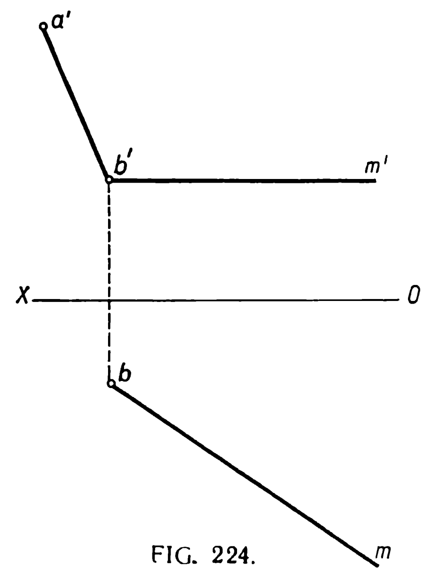



FIG. 224.

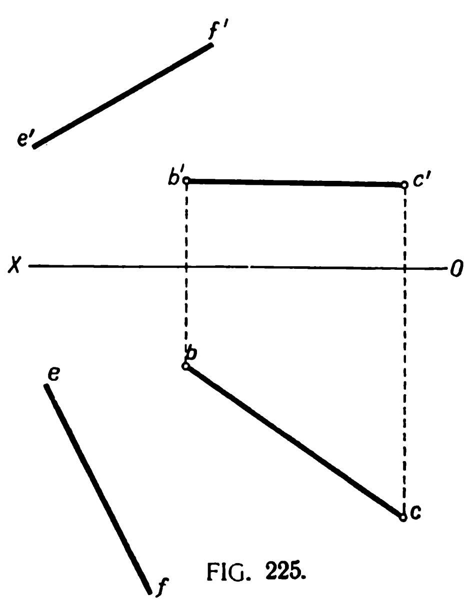



FIG. 225.

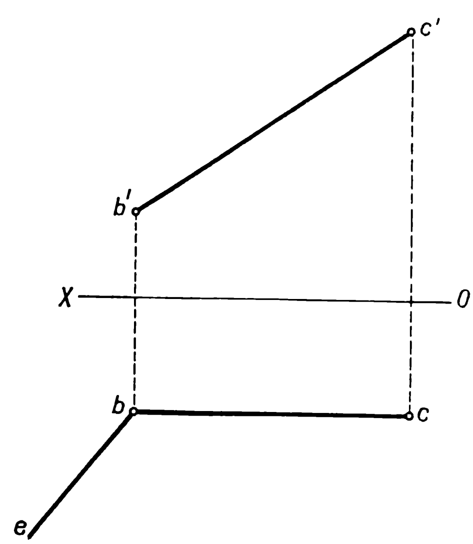



FIG. 226.

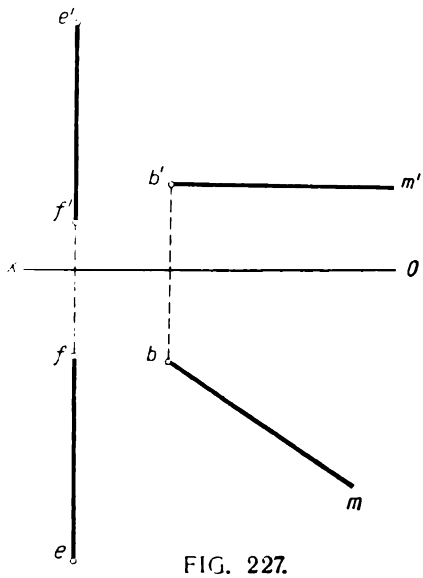



FIG. 227.

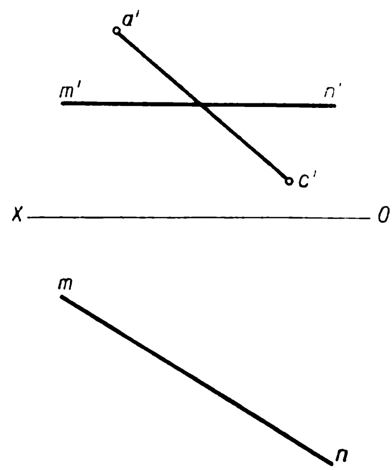



FIG. 228.

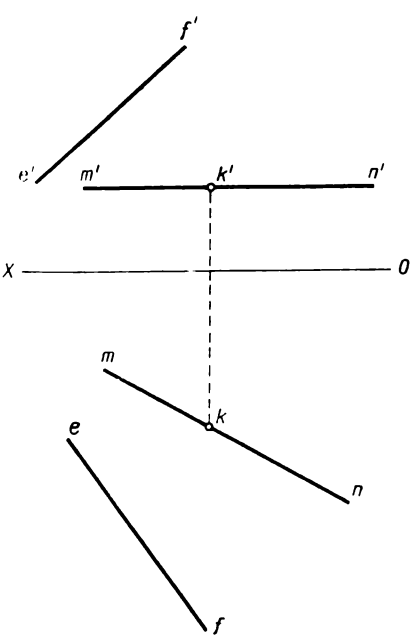



FIG. 229.

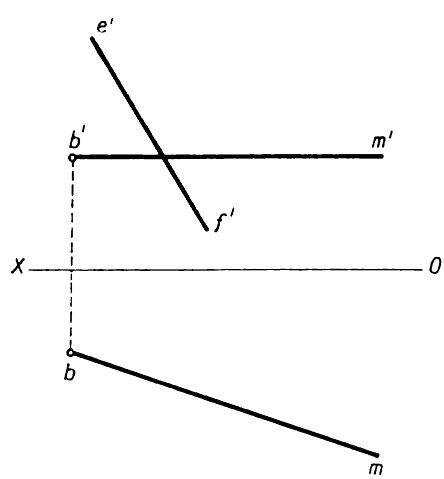



FIG. 230.

131. Construct a rectangle $ABCD$ with the longer side $BC$ lying on the line-segment $BM$, given that the ratio of sides is equal to 2 (Fig. 224).

132. Construct a rectangle $ABCD$ with the longer side $BC$ lying on the line-segment $BM$, and the vertex $A$ on the line-segment $EF$, given that the diagonal equals $2AB$ (Fig. 222).

133. Construct a rectangle $ABCD$ with the longer side $BC$ lying on the line-segment $MN$, given that the side $AB$ (=40 mm) lies on the line-segment $EF$ and the length ratio of the sides equals 1.5 (Fig. 223).

134. Construct a rectangle $ABCD$ with the vertex $A$ lying on the line-segment $EF$ and calculate its area (Fig. 225).

135. Construct a square $ABCD$ with the side $BC$ lying on the line-segment $MN$ (Fig. 220).

136. Construct a square $ABCD$ with the diagonal $BD$ lying on the line-segment $MN$ (Fig. 214).

137. Construct a square $ABCD$ with the side $AB$ lying on the line-segment $BE$ (Fig. 226).

138. Construct a square $ABCD$ with the side $BC$ lying on the line-segment $BM$ (Fig. 224).

139. Construct a square $ABCD$ with the side $BC$ lying on the line-segment $BM$, given that the vertex $A$ lies on the line-segment $EF$ (Fig. 227).

140. Construct a square $ABCD$ with the diagonal $BD$ lying on the line-segment $MN$, given that the vertex $A$ lies on the line-segment $EF$ and $K$ is the point of intersection of the diagonals (Fig. 216).

141. Construct a square $ABCD$ with the diagonal $BD$ lying on the line-segment $MN$ (Fig. 228).

142. Construct a parallelogram $ABCD$ with the base $BC$ lying on the line-segment $MN$, given that the acute angle $B = 60°$, and the diagonal $AC$ is 5 mm longer than the lateral side (Fig. 220).

143. Construct a parallelogram $ABCD$ with the base $BC$ lying on the line-segment $MN$, given that the length of the lateral side is equal to $1.25\,h$, and the length ratio of the sides is 2 (Fig. 220).

144. Construct a right trapezoid $ABCD$ with the longer base $BC$ lying on the line-segment $MN$, given $AD = AB = \frac{2}{3} BC$ (Fig. 220).

145. Construct a parallelogram $ABCD$ with the longer side $BC$ lying on the line-segment $MN$ and with the vertex $A$ on the line-segment $EF$, given that the side $AB$ is 5 mm longer than the altitude $AK$, and the side $BC$ is equal to $1.5\,AK$ (Fig. 229).

146. Construct a parallelogram $ABCD$ with the side $BC = 60$ mm lying on the line-segment $BM$, given that the altitude $AK$ lies on the line-segment $EF$, and the length of the lateral side equals 40 mm (Fig. 230).

147. Construct a rhombus $ABCD$ with the side $BC$ lying on the line-segment $MN$, given that the length of the side is $1.2h$ (Fig. 220).

148. Construct a rhombus $ABCD$ with the side $BC$ lying on the line-segment $MN$, given that the acute angle $B$ is equal to 60° (Fig. 220).

149. Construct a rhombus $ABCD$ with the longer diagonal $BD$ lying on the line-segment $MN$, given that the length ratio of the diagonals is 2 (Fig. 215).

150. Construct a rhombus $ABCD$ with the side $BC$ lying on the line-segment $MN$, given that the length of the side is 1.2 times the altitude $AK$ (Fig. 219).

151. Construct a right trapezoid $ABCD$ with the longer base $BC$ lying on the line-segment $MN$, given that $AD = AB$ and the angle $C$ is equal to 45°, (60°, 30°) (Fig. 220).

152. Construct a rhombus $ABCD$ with the longer diagonal $BD$ lying on the line-segment $MN$, given that the shorter diagonal (= 40 mm) lies on the line-segment $EF$ and the area of the rhombus equals $AC^2$ (Fig. 217).

153. Construct a rhombus $ABCD$ with the vertex $A$ lying on the line-segment $EF$ (Fig. 231).

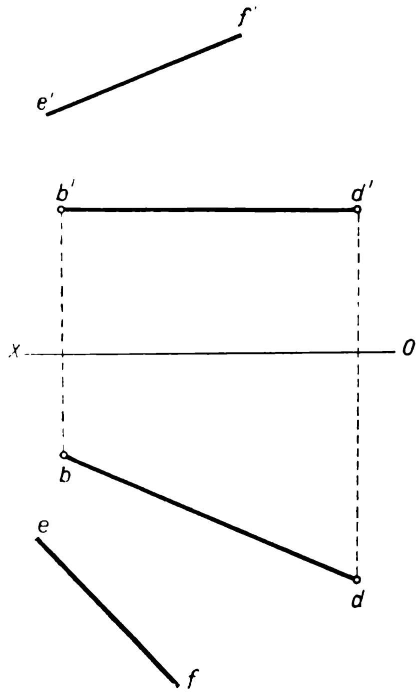



FIG. 231.

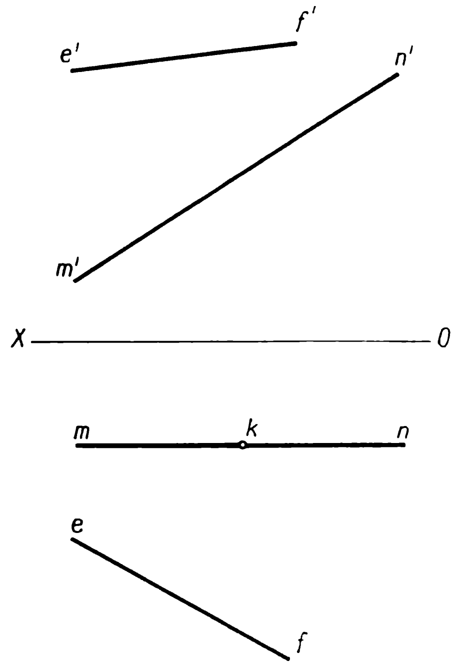



FIG. 232.

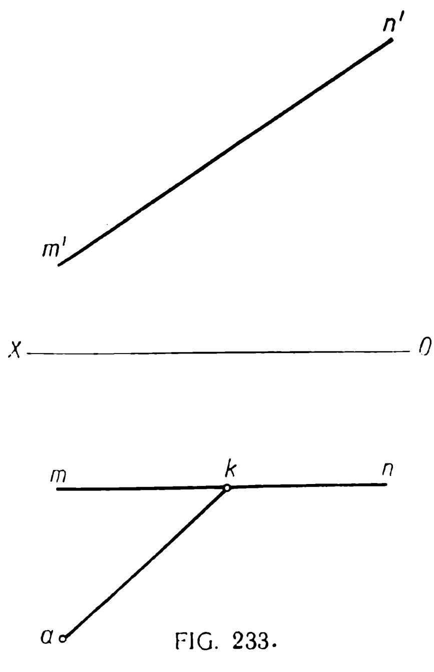



FIG. 233.

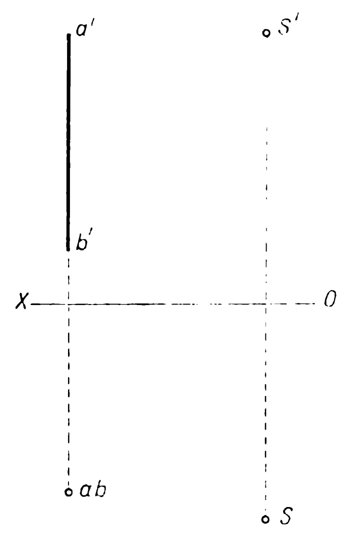



FIG. 234.

154. Construct an equilateral triangle $ABC$ with the base $BC$ lying on the line-segment $MN$, and the vertex $A$ on the line-segment $EF$, given that the point $K$ is the foot of the altitude $AK$ (Fig. 232).

155. Construct a parallelogram $ABCD$ with the longer side $BC$ lying on the line-segment $MN$, given that the point $K$ (the foot of the altitude) divides $BC$ in the ratio 1 : 2 (from $B$ to $C$) and the angle $B$ is 60° (Fig. 233).

156. Construct a rhombus $ABCD$ with the longer diagonal $BD$ lying on the line-segment $MN$ and the vertex $A$ on the line-segment $EF$, given that $K$ is the point of intersection of the diagonals and their length ratio is 2 (Fig. 232).

157. Construct a right trapezoid $ABCD$ with the longer base $BC$ lying on the line-segment $BM$, given $AD = AB$; $CD = 1.2AB$; $B = 90°$ (Fig. 224).

158. Construct a right trapezoid $ABCD$ with the longer base $BC$ lying on the line-segment $BM$, given that the vertex $A$ lies on the line-segment $EF$; $AD = AB$; $B = 90°$; $C = \varphi°$ (Fig. 227).

159. Construct a right trapezoid $ABCD$ with the longer base $BC$ lying on the line-segment $MN$ and the side $AB$ on $EF$, given $B = 90°$; $AD = AB = 40$ mm; $C = 45°$ (Fig. 223).

160. Construct an isosceles trapezoid $ABCD$ with the longer base $BC$ lying on the line-segment $MN$, given $AB = AD = DC = 40$ mm (Fig. 220).

161. Construct an isosceles trapezoid $ABCD$ with the longer base $BC$ lying on the line-segment $MN$, given that the acute angle is equal to 45° and the shorter base is equal to the side (Fig. 220).

162. Through the point $S$ draw a straight line inclined at an angle of 70° to the horizontal plane of projection and 20 mm distant from the line-segment $AB$ (Fig. 234).

# PART TWO

CHAPTER XI

## THE PLANE

The position of a plane in space may be specified in one of the following ways:
(1) by three points not lying on one line;
(2) by a line and a point not lying on the line;
(3) by two intersecting lines;
(4) by two parallel lines.

A plane which is inclined to all projection planes, i.e. is not perpendicular to any of the projection planes, is called an *oblique plane.*

A plane which is perpendicular to the horizontal plane of projection is a *horizontal projecting plane.*

A plane which is perpendicular to the vertical plane of projection is a *vertical projecting plane.*

A plane which is perpendicular to the profile plane of projection is a *profile projecting plane.*

A point taken on lines specifying a plane lies in this plane.

A straight line which has two common points with a plane lies in this plane.

Of the straight lines that may be situated in a given plane, of special importance are the lines of four directions:

*H principal lines* (also called *H parallels* or *horizontal lines*) are lines lying in a given plane and parallel to the horizontal plane of projection;

*V principal lines* (also called *V parallels* or *frontal*, or *vertical lines*) are lines lying in a given plane and parallel to the vertical plane of projection;

*profile lines* are lines lying in a given plane and parallel to the profile plane of projection;

the *steepest lines* are lines drawn in a plane perpendicular to horizontal lines.

A point lies in a plane if the point is situated on a straight line contained in this plane.

Prior to considering the following examples it is necessary to solve Problems 163 through 171.

EXAMPLES

**Example** 48

Draw a straight line in the plane specified by the line-segments $AB$ and $CD$ (Fig. 235).

*Solution.* Assume arbitrary points $(m, m')$ and $(n, n')$ on the lines $(ab, a'b')$ and $(cd, c'd')$ and draw a straight line $(mn, m'n')$ through them.

**Example** 49

Given: the plane specified by the line-segments $AB$ and $CD$. Does the line-segment $MN$ lie in this plane (Fig. 236)?

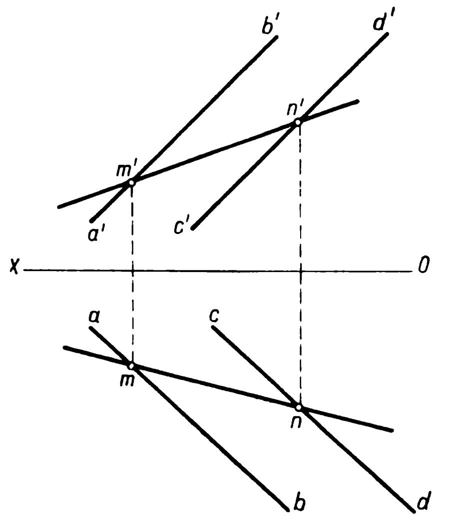



FIG. 235.

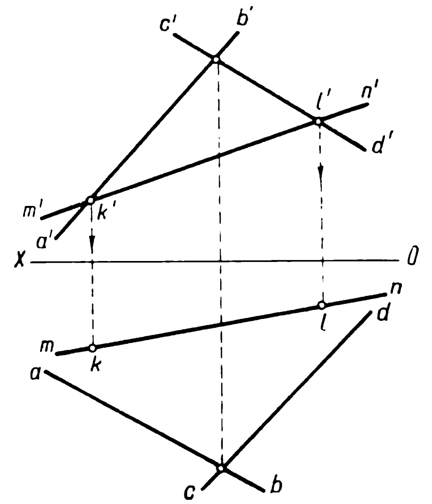



FIG. 236.

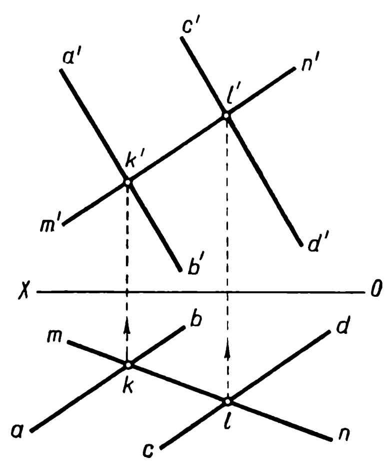



FIG. 237.

*Solution.* Let us designate the point of intersection of the vertical projections of the line-segments $AB$ and $MN$ as $k'$, and that of $CD$ and $MN$ as $l'$. Construct their horizontal projections—points $k$ and $l$—on the horizontal projection ($mn$) of the line-segment $MN$. It is obvious from the construction that the points ($k$, $k'$) and ($l$, $l'$) of the line-segment $MN$ do not belong to the given plane; hence, the line-segment $MN$ *does not lie* in the given plane.

**Example 50**

Given: the plane specified by two parallel lines $AB$ and $CD$, and the horizontal projection of the line-segment $MN$ lying in this plane. Find the vertical projection of the line-segment (Fig. 237).

*Solution.* Designate the points of intersection of the horizontal projections of the line-segments $AB$ and $MN$ as $k$, and $CD$ and $MN$ as $l$. Using the points $k$ and $l$, find $k'$ on $a'b'$ and $l'$ on $c'd'$ and draw through them the desired vertical projection ($m'n'$) of the line-segment $MN$.

**Example 51**

Draw a horizontal line at a distance of 15 mm from the horizontal plane of projection in the plane specified by the line-segment $AB$ and the point $C$ (Fig. 238).

*Solution.* Let us first specify the plane in another way, say, by two intersecting lines. To do this, assume an arbitrary point ($k$, $k'$) on the line ($ab$, $a'b'$) and draw the projections of the line-segment ($ck$, $c'k'$). Then, at a distance of 15 mm from the coordinate axis and parallel to it, draw the vertical projection ($m'n'$) of the horizontal line to intersect the line-segments $a'b'$ and $c'k'$ at the points $d'$ and $e'$. Finally, find points $d$ and $e$ on $ab$ and $ck$ and draw, through them, the horizontal projection ($mn$) of the horizontal line (the drawing gives one alternative solution).

**Example 52**

Draw a frontal line at a distance of 15 mm from the vertical plane of projection in the plane specified by the parallel lines $AB$ and $CD$ (Fig. 239).

*Solution.* At a distance of 15 mm from the coordinate axis and parallel to it, draw the horizontal projection ($mn$) of the frontal line to intersect the line-segments $ab$ and $cd$ at the points $k$ and $l$. Then find points $k'$ and $l'$ on the line-segments $a'b'$ and $c'd'$ and through them draw the vertical projection ($m'n'$) of the frontal line (the drawing gives one alternative solution).

**Example 53**

Draw the steepest line in the plane specified by the intersecting lines $AB$ and $CD$ (Fig. 240).

*Solution.* Draw an arbitrary horizontal line ($mn$, $m'n'$). Since the steepest line must be perpendicular to this horizontal line, draw its horizontal projection, say ($ek$), perpendicular to the horizontal projection ($mn$) of the horizontal line (according to the theorem of projecting a right angle) and then, with the aid of the horizontal projection ($ek$) of the steepest line, find its vertical projection ($e'k'$).

**Example 54**

Take an arbitrary point $K$ on the plane of the triangle $ABC$ (Fig. 241).

*Solution.* Draw an auxiliary line, for instance ($mn$, $m'n'$), in the plane of the given triangle and take an arbitrary point ($k$, $k'$) on it. The latter lies in the plane of the triangle $ABC$.

**Example 55**

Given: the plane specified by a point $C$ and a line $AB$, and the point $K$. Does the point $K$ lie in the given plane (Fig. 242)?

*Solution.* Through the points $c'$ and $k'$ draw the vertical projection of an auxiliary line to intersect the line-segment $a'b'$ at point $m'$; find point $m$ on $ab$ and draw through the points $c$ and $m$ the horizontal projection ($cm$) of the auxiliary line. The line-segment ($cm$, $c'm'$) lies in the given plane and, as is obvious from the drawing, the point ($k$, $k'$) does not lie on this line. Hence, the point ($k$, $k'$) *does not lie* in the given plane.

**Example 56**

Given: the plane specified by the intersecting lines $AB$ and $CD$ and the horizontal projection ($k$) of the point $K$ lying in this plane. Find its vertical projection ($k'$) using a frontal line (Fig. 243).

*Solution.* Draw, through the point $k$ and parallel to the coordinate axis, the horizontal projection ($mn$) of the frontal line which will intersect $ab$ and $cd$ at the points $e$ and $f$. Then find points $e'$ and $f'$ on $a'b'$ and $c'd'$ and draw through them the

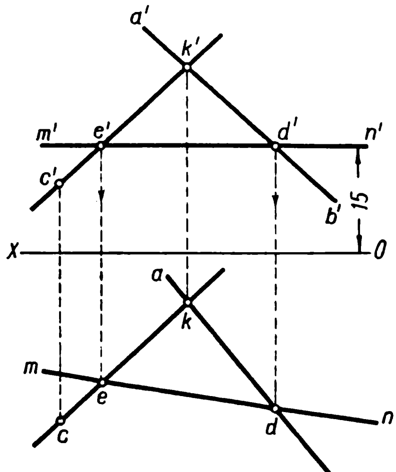



FIG. 238.

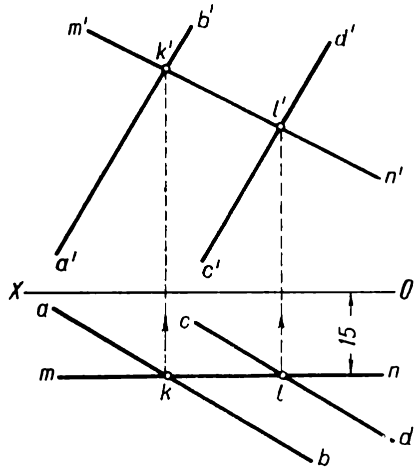



FIG. 239.

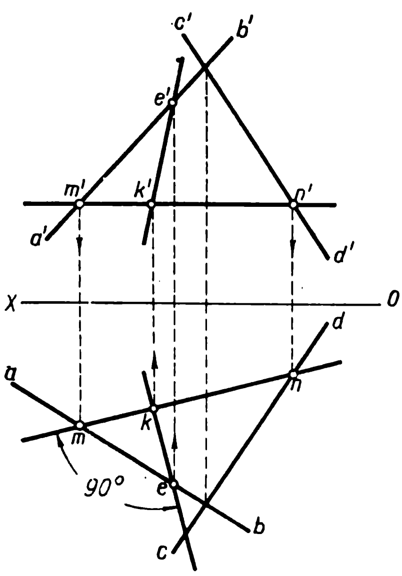



FIG. 240.

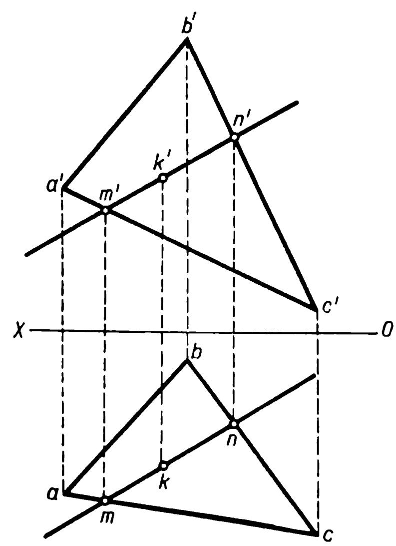



FIG. 241.

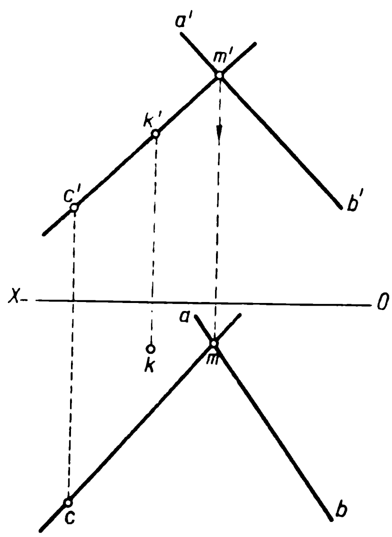



FIG. 242.

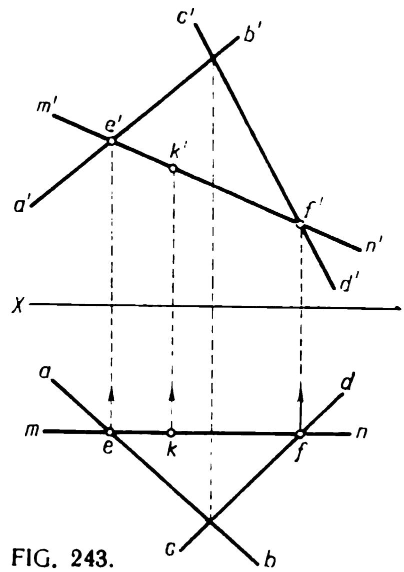



FIG. 243.

vertical projection ($m'n'$) of the frontal line. The latter ($mn$, $m'n'$) lies in the given plane. To ensure that the point ($k$, $k'$) lies in the plane we take the point $k'$ on $m'n'$.

PROBLEMS

163. Specify a horizontal projecting plane by:
(1) two intersecting lines;
(2) a line and a point.
164. Specify a plane parallel to the vertical plane of projection by:
(1) two parallel lines;
(2) three points.

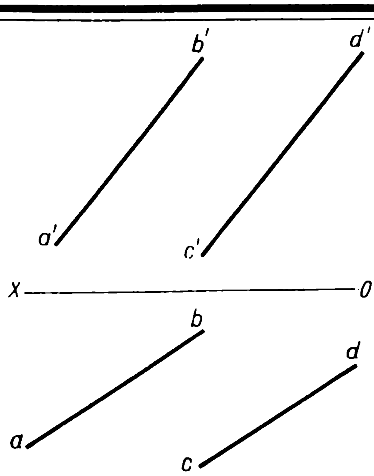



FIG. 244.

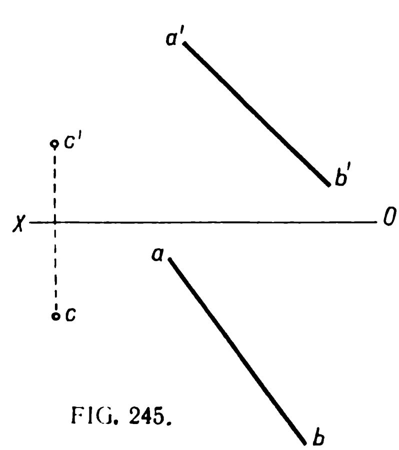



FIG. 245.

165. What is common to the geometrical elements determining a horizontal projecting plane in an orthographic drawing?

166. Specify a vertical projecting plane by:

(1) two parallel lines;

(2) three points.

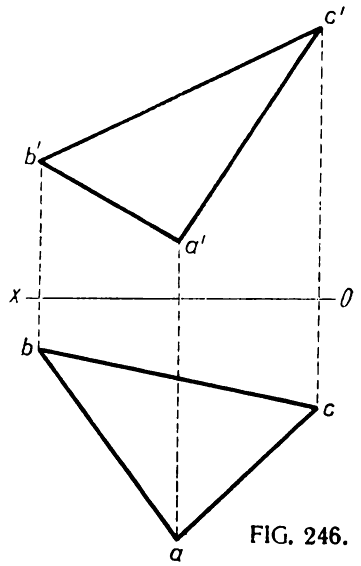



FIG. 246.

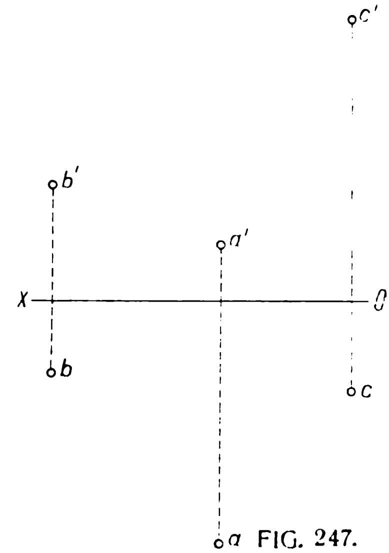



FIG. 247.

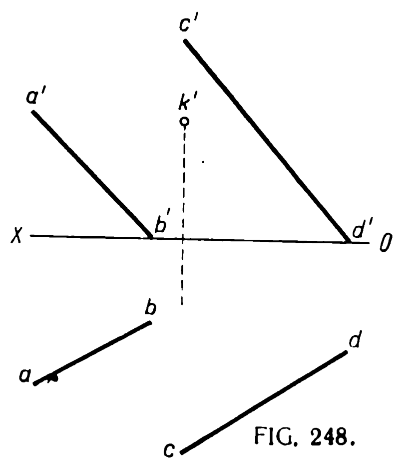



FIG. 248.

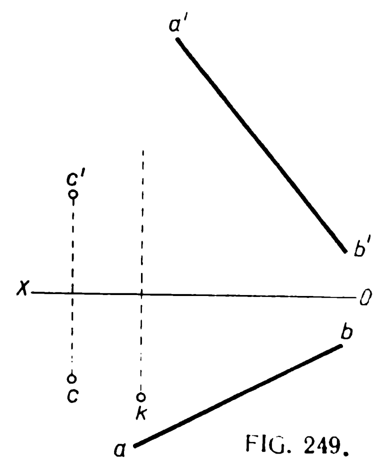



FIG. 249.

167. Specify a plane parallel to the horizontal plane of projection by:

(1) two intersecting lines;

(2) a line and a point.

168. What is common to the geometrical elements determining a vertical projecting plane in an orthographic drawing?

169. Specify a profile projecting plane by:

(1) two intersecting lines;

(2) two parallel lines;

(3) a line and a point;

(4) three points.

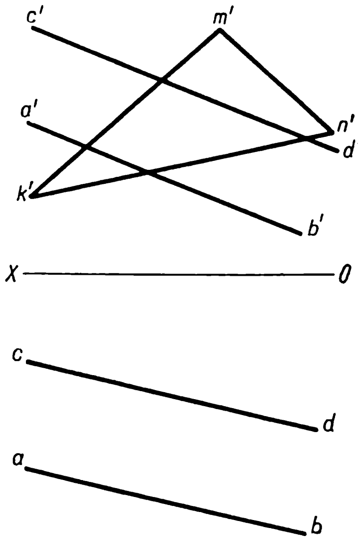



FIG. 250.

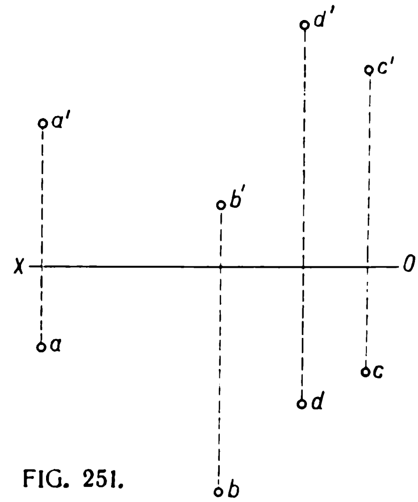



FIG. 251.

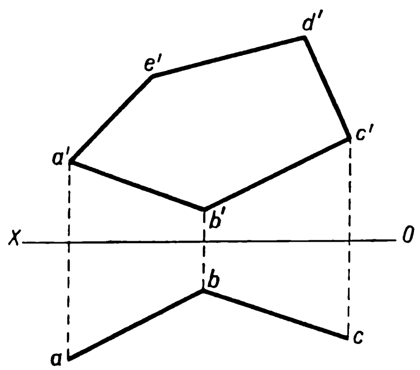



FIG. 252.

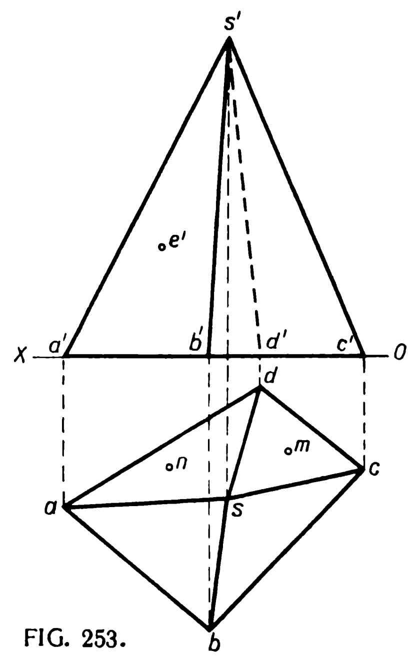



FIG. 253.

170. What is common to the geometrical elements determining a profile projecting plane in an orthographic drawing?

171. Specify an oblique plane by:

(1) two intersecting lines;

(2) two parallel lines;

(3) a line and a point;

(4) three points.

172. Construct in the given plane the geometrical locus of points 15 mm distant from the $H$ plane (Figs. 244 to 247).

173. Construct in the given plane the geometrical locus of points 15 mm distant from the $V$ plane (Figs. 244 to 247).

174. Given: the vertical projection of a point lying in the plane specified by two parallel lines $AB$ and $CD$. Find the horizontal projection (Fig. 248).

175. Given: the horizontal projection of a point lying in the plane specified by the line $AB$ and the point $C$. Find the vertical projection (Fig. 249).

176. Given: the vertical projection of the triangle $KMN$ contained in the plane specified by the parallel lines $AB$ and $CD$. Find the horizontal projection (Fig. 250).

177. Do all the four points $A$, $B$, $C$, $D$ lie in one and the same plane? (Fig. 251.)

178. Determine the horizontal projection of the plane pentagon $ABCDE$, given its vertical projection and the horizontal projection of two adjacent sides (Fig. 252).

179. Given: the pyramid $SABCD$ (Fig. 253).

(1) Take an arbitrary point $K$ on its face $SBC$.

(2) Find the vertical projection ($m'$) of the point $M$ lying on the face $SCD$, given the horizontal projection ($m$).

(3) Find the horizontal projection ($e$) of the point $E$ lying on the face $SAB$, given the vertical projection ($e'$).

(4) Through the point $N$ on the face $SAD$ draw its steepest line.

CHAPTER XII

# CPECIFYING A PLANE BY ITS TRACES. LINES AND POINTS IN A GIVEN PLANE

In a particular case, lines specifying a plane may lie in projection planes. Then such lines are called *traces* of the plane, i.e. lines along which the given plane intersects the planes of projection.

A line lying in the given plane and in the horizontal plane of projection is called the *horizontal trace of the plane*.

A line lying in the given plane and in the vertical plane of projection is referred to as the *vertical trace of the plane*.

An *oblique plane* (not perpendicular to any of the projection planes) has three traces: *horizontal*, *vertical* and *profile*. The traces intersect in pairs on the axes at so-called vanishing points. These points may be regarded as the vertices of trihedral angles formed by the given plane and two of the three projection planes.

Planes specified by their traces are designated by one of the letters $P$, $Q$, $R$, $S$, $T$, etc., and its traces, by the same letters with the subscript $h$ for the horizontal trace ($P_h$, $Q_h$, $R_h$, $S_h$, $T_h$, etc.), the subscript $v$ for the vertical trace ($P_v$, $Q_v$, $R_v$, $S_v$, $T_v$, etc.), and the subscript $w$ for the profile trace ($P_w$, $Q_w$, $R_w$, $S_w$, $T_w$, etc.).

The vanishing points are designated as $P_x$, $Q_x$, $R_x$, $S_x$, $T_x$, etc.

Any point lying on the horizontal or vertical trace of a plane lies in this plane.

A straight line is contained in a plane if its traces lie on like traces of the plane.

A horizontal line in a plane and the horizontal trace of the plane are parallel to each other. Hence, the projections of a horizontal line are parallel to like projections of the horizontal trace of the plane.

A frontal line in a plane and the vertical trace of the plane are parallel to each other. Hence, the projections of a frontal line are parallel to like projections of the horizontal trace of the plane.

The steepest line in a plane and the horizontal trace of the plane are mutually perpendicular. Hence, the horizontal projection of the steepest line in the plane is perpendicular to the horizontal trace of the plane (more precisely, to the horizontal projection of the horizontal trace of the plane).

180. Specify an arbitrary horizontal projecting plane by its traces. Determine orthographically the angle formed by the horizontal trace of this plane and the coordinate axis.

181. Specify by traces a plane parallel to the vertical plane of projection and passing through the first and fourth quadrants; through the second and third quadrants.

182. Specify an arbitrary vertical projecting plane by its traces. Determine orthographically the angle formed by the vertical trace of this plane and the coordinate axis.

183. Specify by traces a plane parallel to the horizontal plane of projection and passing through the first and second quadrants, third and fourth quadrants.

184. Specify by traces a profile projecting plane passing through the following quadrants: II, I and IV; I, II and III; I, IV and III; II, III and IV.

185. What does the coincidence of the traces of a profile projecting plane (in an orthographic drawing) mean?

186. Specify a plane passing through the coordinate axis and through quadrants I and III; II and IV, respectively.

187. Specify the following oblique planes: acute-angled, obtuse-angled, and one with coincident traces.

188. A plane parallel to the horizontal (or vertical) plane of projection is specified by a single trace. But, as is known, one line does not specify a plane in space. Is there a contradiction here?

189. What additional provision is necessary in specifying by traces a plane passing through the coordinate axis?

190. What two lines are the traces of:

(1) an oblique plane;

(2) a horizontal projecting plane;

(3) a vertical projecting plane;

(4) a profile projecting plane?

191. What is common to orthographically represented points, lines, and plane figures lying in:

(1) a horizontal projecting plane;

(2) a vertical projecting plane;

(3) a profile projecting plane?

192. What is the position of the orthographic projections of a horizontal line in an oblique plane specified by its traces? (Why?)

193. What is the position of the orthographic projections of a frontal line in an oblique plane specified by its traces? (Why?)

194. What line is a horizontal line in a horizontal projecting plane?

195. What line is a frontal line in a vertical projecting plane?

196. What line is a horizontal line in a profile projecting plane?

197. What line is a frontal line in a profile projecting plane?

198. What line is the steepest line in a profile projecting plane?

## EXAMPLES

**Example 57**

Construct the traces of the plane specified by the line $AB$ and the point $C$ (Fig. 254).

*Solution.* Let us designate the desired traces of the given plane as $P_h$ and $P_v$. To draw the vertical trace ($P_v$) of the plane it is necessary to have two points belonging to this plane and lying in the vertical plane of projection. One such point ($c$, $c'$) is already given, the second point will be the vertical trace ($v$, $v'$) of the line-segment ($ab$, $a'b'$). Find the point ($v$, $v'$) and draw the projections of the required vertical trace of the plane ($P_v$): the vertical projection through the points $c'$ and $v'$ to intersect the coordinate axis at point $P_x$, and the horizontal projection through the points $c$ and $v$, the latter coinciding with the coordinate axis. One point ($P_x$) of the horizontal trace $P_h$ is known. Find the horizontal trace ($h$, $h'$) of the line-segment ($ab$, $a'b'$), thus obtaining the other point of the horizontal trace of the plane. Now draw the projections of the required horizontal trace ($P_h$) of the given plane: the horizontal projection through the points $h$ and $P_x$, and the vertical projection through the points $P_x$ and $h'$, the latter coinciding with the coordinate axis.

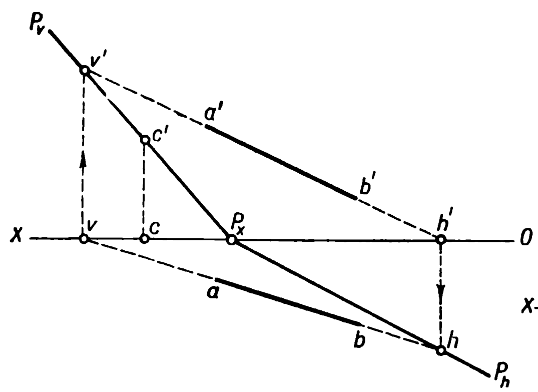



FIG. 254.

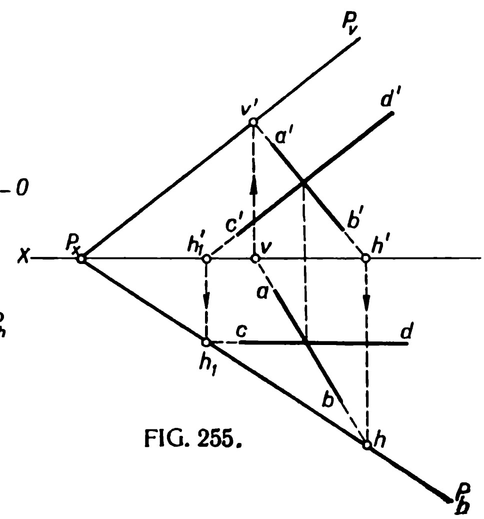



FIG. 255.

*Conclusion.* The horizontal projection of the vertical trace of a plane and the vertical projection of its horizontal trace *always* coincide with the coordinate axis (why?).

**Example 58**

Construct the traces of the plane specified by the intersecting lines $AB$ and $CD$ (Fig. 255).

*Solution.* Find the traces ($h$, $h'$) and ($v$, $v'$) of the line ($ab$, $a'b'$) and the trace ($h_1$, $h_1'$) of the line ($cd$, $c'd'$). Draw the horizontal trace ($P_h$) of the required plane $P$ through the points $h$ and $h_1$ and the vertical trace ($P_v$) through the point $v'$ and parallel to $c'd'$ (why?). The traces $P_h$ and $P_v$ intersect on the coordinate axis at the point $P_x$, which confirms the correctness of the solution.

**Example 59**

Construct the traces of the plane specified by the line $AB$ and the point $C$ (Fig. 256).

*Solution.* The line $AB$ and point $C$ specify an oblique plane (why?).

Designate its traces as $P_h$ and $P_v$. Now let us pass over from specifying the plane by a line and a point to specifying it by two intersecting lines, for which purpose draw a horizontal line through the point ($c$, $c'$) and parallel to the horizontal plane of projection to intersect the line ($ab$, $a'b'$) at point ($k$, $k'$). The point $P_x$ belonging to the line ($ab$, $a'b'$) and lying on the coordinate axis will serve as the vanishing point (why?).

Find the vertical trace ($v$, $v'$) of the horizontal line and draw the vertical trace ($P_v$) of the given plane through the points $P_x$ and $v'$, the horizontal trace ($P_h$) being drawn through the point $P_x$ and parallel to the horizontal projection of the horizontal line.

**Example 60**

Construct the traces of the plane specified by the parallel lines $AB$ and $CD$ (Fig. 257).

*Solution.* The lines $AB$ and $CD$ define a plane parallel to the coordinate axis (why?) with traces $P_h$ and $P_v$ parallel to the coordinate axis. To draw the horizontal trace ($P_h$) we need only one point of the plane lying in the $H$ plane. For the vertical trace $P_v$ we need a point lying in the $V$ plane. The given lines ($ab$, $a'b'$) and ($cd$, $c'd'$) cannot intersect the planes $H$ and $V$; let us intersect them with an arbitrary auxiliary line $MN$ and find its traces. Indeed, the points ($h$, $h'$) and ($v$, $v'$) lie in the given plane,

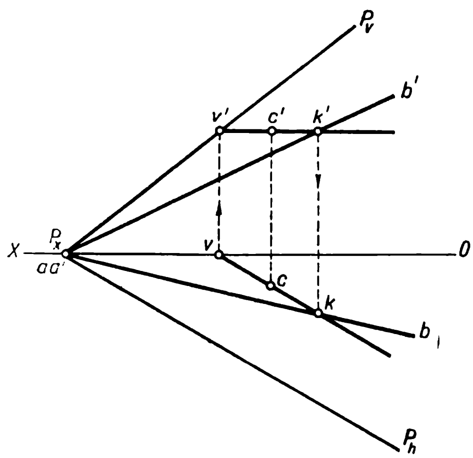


FIG. 256.

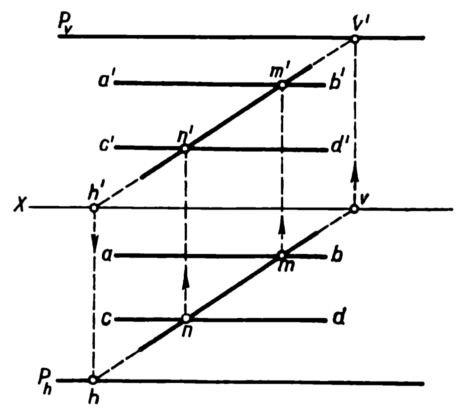


FIG. 257.

since they are located on a straight line contained in the plane. Draw the vertical trace ($P_v$) of the plane through the point $v'$ and parallel to the coordinate axis and the horizontal trace ($P_h$) of the plane through the point $h$ and also parallel to the coordinate axis.

*Conclusion.* To proceed from specifying a plane not by its traces (say, by two intersecting or parallel lines) to specifying it by its traces, one must find the traces of the given lines and draw through them the like traces of the plane. If a plane is specified by a line and a point or by three points not lying on one line, first specify the plane by two intersecting or parallel lines and then proceed as above.

**Example 61**

Find the profile trace of the plane $P$ (Fig. 258).

*Solution.* The profile trace ($P_w$) of the plane must pass through the profile traces of the lines $P_h$ and $P_v$. The line $P_v$ has no profile trace. Find the profile trace ($W$) of the line $P_h$ and draw the profile trace ($P_w$) of the plane through the point $W$ and parallel to the $z$-axis. The point $W$ is sometimes also designated by the letter $P$ with the subscript $y$ ($P_y$).

**Example 62**

Find the profile trace of the plane $P$ (Figs. 259, 260).

*Solution.* Find the profile traces ($W$) and ($W_1$) of the lines $P_h$ and $P_v$ and draw through them the desired profile trace ($P_w$) of the given plane.

The points $W$ and $W_1$ are sometimes also designated by $P_y$ and $P_z$.

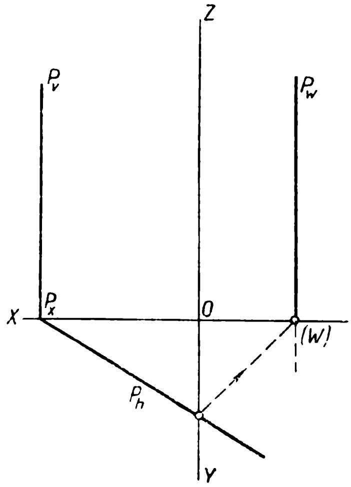



FIG. 258.

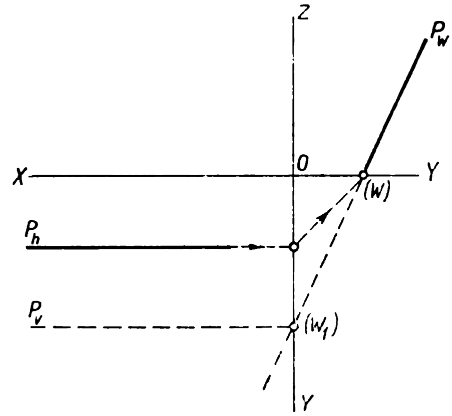



FIG. 259.

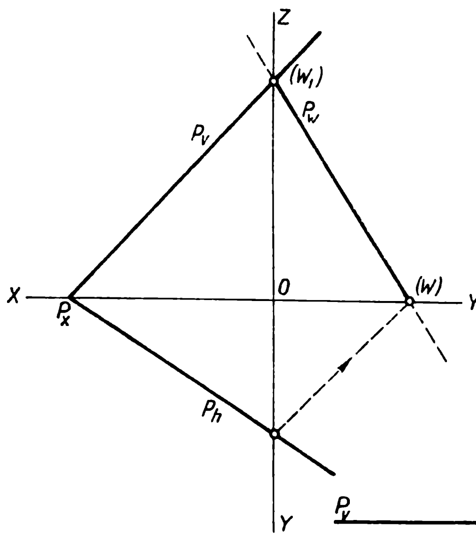



FIG. 260.

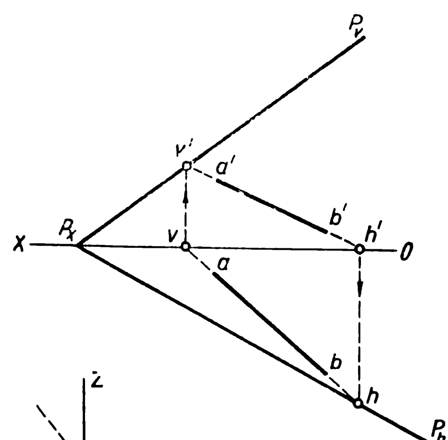



FIG. 261.

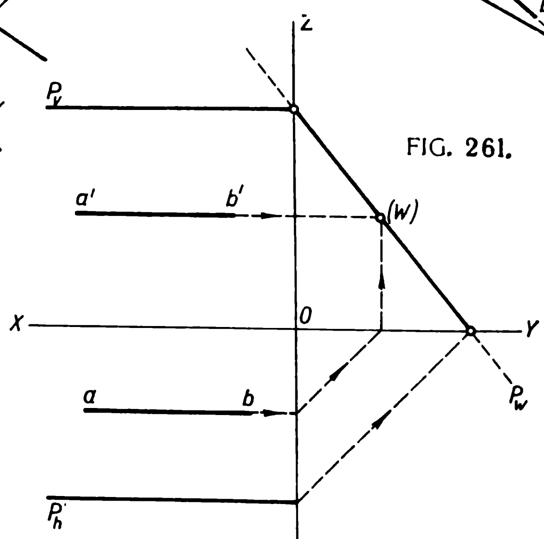



FIG. 262.

**Example 63**

Does the line $AB$ lie in the plane $P$? (Fig. 261.)

*Solution.* Construct the horizontal ($h$, $h'$) and vertical ($v$, $v'$) traces of the line $AB$. As is obvious from the drawing, they lie on the like traces of the plane; hence, the line $AB$ *lies* in the given plane (according to the appropriate theorem).

**Example 64**

Does the line $AB$ lie in the plane $P$? (Fig. 262.)

*Solution.* The given line ($ab$, $a'b'$) has only one (profile) trace. Construct the profile traces of the line and the plane. We see that the profile trace ($W$) of the line lies on the profile trace ($P_w$) of the plane. Hence, the line $AB$ *lies* in the plane $P$.

Below is an orthographic solution of the same problem without the use of the profile plane of projection.

**Example 65**

In the plane $P$ draw an arbitrary line passing through the second, first and fourth quadrants (Fig. 263).

*Solution.* As is known, the traces of the desired line must lie on like traces of the plane. Any line passing through the second, first and fourth quadrants has its

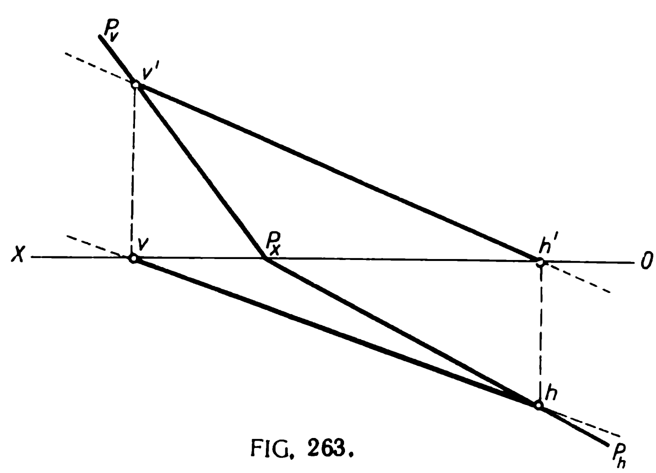



FIG. 263.

horizontal trace in the front half-plane of the horizontal plane of projection and the vertical trace in the upper half-plane of the vertical plane of projection; the traces are situated on the orthographic drawing in the following way: the vertical trace above the coordinate axis, and the horizontal one below it.

Assume an arbitrary point (trace) ($h$, $h'$) on the horizontal trace ($P_h$) of the plane and an arbitrary point (trace) ($v$, $v'$) on the vertical trace ($P_v$) of the plane. Then draw the projections of the desired line: the horizontal one through the points $h$ and $v$ and the vertical one through the points $h'$ and $v'$.

**Example 66**

In the plane $P$ draw an arbitrary line passing through the third and fourth quadrants (Fig. 264).

*Solution.* A line passing through the third and fourth quadrants has only one (vertical) trace found in the lower half-plane of the vertical plane of projection, which trace is orthographically located below the coordinate axis. Let us assume

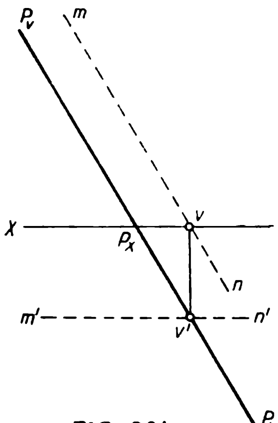



FIG. 264.

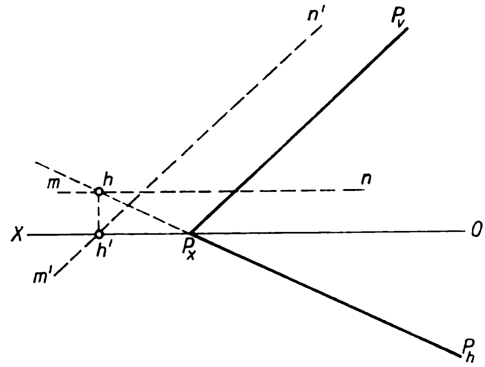



FIG. 265.

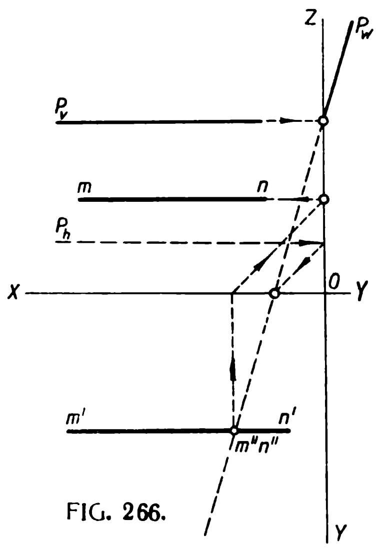



FIG. 266.

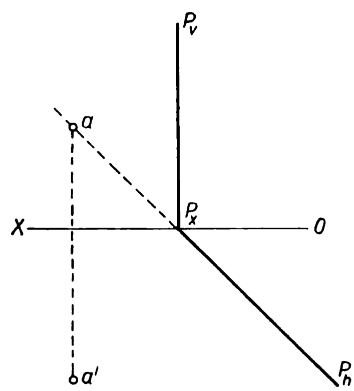



FIG. 267.

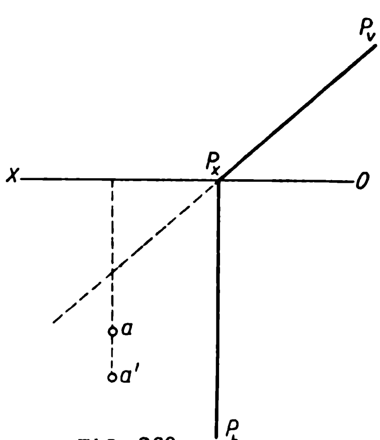



FIG. 268.

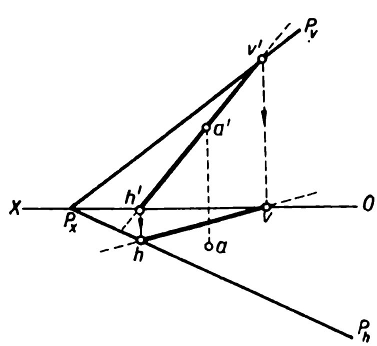



FIG. 269.

a point ($v$, $v'$) on the vertical trace ($P_v$) of the plane and draw through it the projections of the desired horizontal line: the vertical one through the point $v'$ and parallel to the coordinate axis, and the horizontal one through the point $v$ and parallel to the horizontal trace ($P_h$) of the plane.

**Example 67**

In the plane $P$ draw an arbitrary line passing through the second and third quadrants (Fig. 265).

*Solution.* A line passing through the second and third quadrants has only one (horizontal) trace found in the rear half-plane of the horizontal plane of projection, which trace is orthographically located above the coordinate axis. Let us assume a point ($h$, $h'$) on the extension of the horizontal trace ($P_h$) of the plane and draw through it the projections of the desired frontal line: the horizontal one through the point $h$ and parallel to the coordinate axis, and the vertical one through the point $h'$ and parallel to the vertical trace ($P_v$) of the given plane.

**Example 68**

In the plane $P$ draw a line-segment parallel to the coordinate axis and situated in the third quadrant (Fig. 266).

*Solution.* The profile projection of the required line represented orthographically as a point must lie on the profile trace of the given plane. Let us construct the profile trace ($P_w$) of the plane and take an arbitrary point, denoting it by $m''n''$. Having the profile projection ($m''n''$) of the desired line, construct its horizontal ($mn$) and vertical ($m'n'$) projections in accordance with the general rule.

**Example 69**

Given: a plane $P$ and a point $A$. Does the point $A$ lie in the plane $P$? (Fig. 267.)

*Solution.* As is known, the horizontal projection of any point contained in a horizontal projecting plane lies on the horizontal trace of the plane. In this particular case the horizontal projection ($a$) of the given point lies on the extension of the horizontal trace ($P_h$) of the plane. Hence, the given point ($a$, $a'$) lies in the plane $P$.

**Example 70**

Given: a point $A$ and a plane $P$. Does the point lie in the plane? (Fig. 268.)

*Solution.* As is known, the vertical projection of any point belonging to a vertical projecting plane lies on the vertical trace of the plane. In this particular case the vertical projection ($a'$) of the point does not lie on the vertical trace ($P_v$) of the given plane. Hence, the given point ($a$, $a'$) *does not lie* in the plane $P$.

**Example 71**

Given: a plane $P$ and a point $A$. Does the point lie in the plane? (Figs. 269 to 271.)

*Solution.* Since the given plane is oblique, we use an auxiliary line.

(1) Draw an arbitrary vertical projection ($h'v'$) of the auxiliary line through the vertical projection ($a'$) of the given point. From $h'v'$ find the horizontal projection ($hv$) of the auxiliary line. Since the horizontal projection ($a$) of the given point does not lie on the line $hv$, the given point ($a$, $a'$) *does not lie* in the plane $P$.

We may begin solving the problem by drawing the horizontal projection ($hv$) of the auxiliary line through the horizontal projection ($a$) of the point, and so on.

(2) Solving the problem with the aid of a horizontal line: draw the vertical projection of the horizontal line through the vertical projection ($a'$) of the given point and parallel to the coordinate axis to intersect the vertical trace ($P_v$) of the given plane at the point $v'$. From the latter point find point $v$ on the coordinate axis and, through it, draw the horizontal projection of the horizontal line parallel to the horizontal trace ($P_h$) of the plane. The point ($a$, $a'$) does not lie on the horizontal line of the plane; hence, the given point ($a$, $a'$) *is not contained* in the plane $P$.

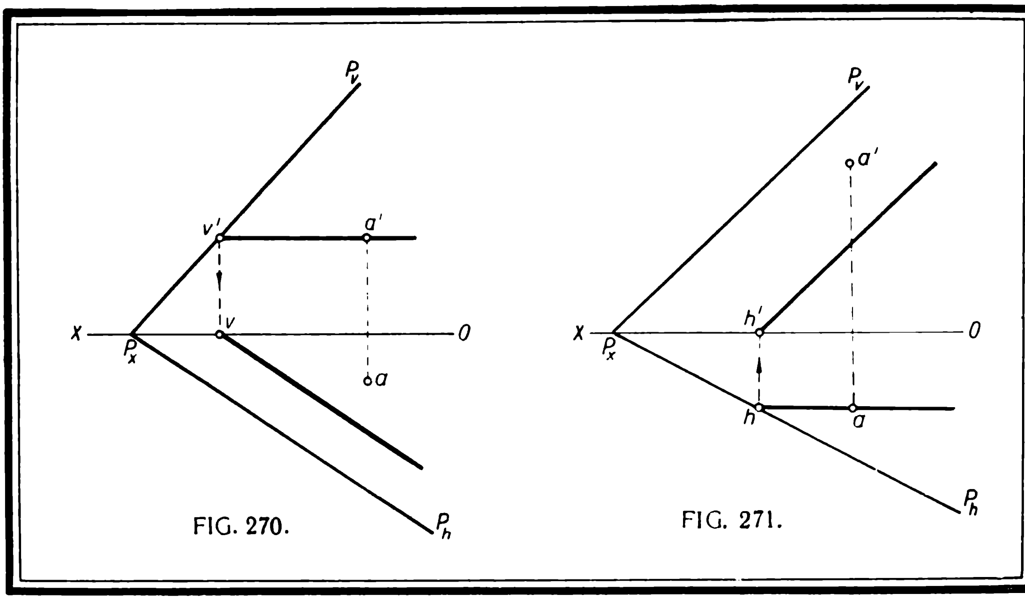



FIG. 270. FIG. 271.

We may begin solving the problem by drawing the horizontal projection of the horizontal line through the horizontal projection ($a$) of the given point and parallel to the horizontal trace of the plane, and so on.

(3) Solving the problem with the aid of a frontal line: draw the horizontal projection of the frontal line through the horizontal projection ($a$) of the given point and parallel to the coordinate axis to intersect the horizontal trace ($P_h$) of the plane at the point $h$. From the latter determine $h'$ on the coordinate axis and draw, through it, the vertical projection of the frontal line parallel to the vertical trace ($P_v$) of the given plane. As is obvious, the point ($a$, $a'$) does not lie on the frontal line; hence, the point ($a$, $a'$) ***does not lie*** in the plane $P$.

We may begin solving the problem by drawing the vertical projection of the frontal line through the vertical projection ($a'$) of the point and parallel to the vertical trace of the plane, and so on.

**Example 72**

Given: a plane $P$ and a point $A$. Does the point lie in the plane? (Figs. 272, 273.)

*Solution.* Let us use an auxiliary line ($mk$, $m'k'$) lying in the plane $P$. Draw, through the vertical projection ($a'$) of the point, the vertical projection of the auxiliary line passing through $k'$ to intersect the coordinate axis at the point ($m$, $m'$). Through $m$ and $k$ draw the horizontal projection of the auxiliary line. The point ($a$, $a'$) does not lie on the auxiliary line ($mk$, $m'k'$); hence, the given point ($a$, $a'$) *is not contained* in the plane $P$.

The problem may be solved in a different way. As is known, the profile projection of any point lying in the profile projecting plane is found on the profile trace of the plane. Find the profile trace ($P_w$) of the plane and the profile projection ($a''$) of the point. It is clear from the above that the given point ($a$, $a'$) ***does not lie*** in the plane $P$.

**Example 73**

Given: a plane $P$ and a line-segment $MN$. Does the segment belong to the plane? (Fig. 274.)

*Solution.* Solve this problem without using the profile plane of projection. Assume an arbitrary point $a'$ on the vertical projection ($m'n'$) of the given line-segment and draw through it the vertical projection ($h'v'$) of an auxiliary line contained in the plane $P$. From $h'v'$ find the horizontal projection ($hv$) of the auxiliary line, and from

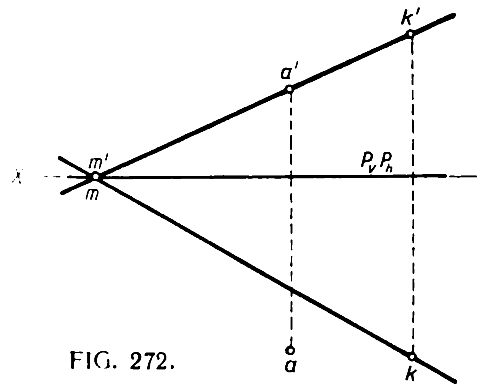



FIG. 272.

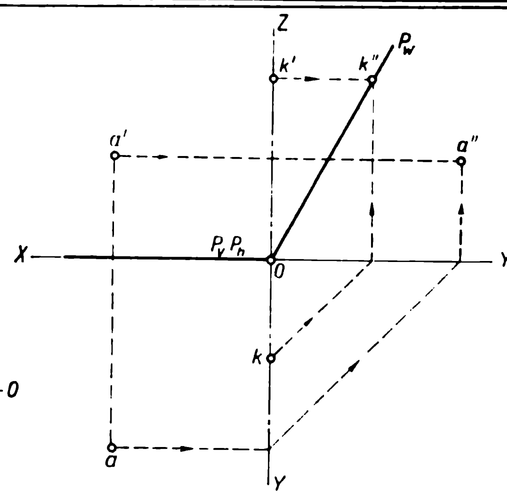



FIG. 273.

$a'$ the point $a$ on it. The point ($a$, $a'$) lies on the line ($hv$, $h'v'$). Hence, the line-segment ($mn$, $m'n'$) passes through the point ($a$, $a'$) lying in the plane $P$ and thus *lies* in this plane.

**Example 74**

Given: a plane $P$ and the horizontal projection ($a$) of a point $A$ lying in the plane. Find its vertical projection $a'$ (Fig. 275).

*Solution.* Since the given plane is oblique, we make use of an auxiliary line (say, a horizontal line). Draw the horizontal projection of the horizontal line through the point $a$ and parallel to the horizontal trace ($P_h$) to intersect the coordinate axis at point $v$. From the point $v$ find the point $v'$ on the vertical trace ($P_v$) of the plane and draw, through it, the vertical projection of the horizontal line parallel to the coordinate axis. Knowing $a$, find $a'$ on the vertical projection of the horizontal line.

If the vertical projection of the point is given, we begin solving the problem by drawing the vertical projection of the horizontal line through the vertical projection of the point and parallel to the coordinate axis, and so on.

**Example 75**

Given: a plane $P$ and the vertical projection ($a'$) of a point $A$ lying in this plane. Find its horizontal projection $a$ (Fig. 276).

*Solution.* Solve the problem with the aid of a frontal line. Draw the vertical projection of the frontal line through the point ($a'$) and parallel to the vertical trace ($P_v$) of the plane to intersect the coordinate axis at the point $h'$. The latter helps us to find the point $h$ on the horizontal trace ($P_h$) of the given plane, through which point we draw the horizontal projection of the frontal line parallel to the coordinate axis. From $a'$ we find the point $a$ on the horizontal projection of the frontal line.

If the horizontal projection of the point is given, we begin solving the problem by drawing the horizontal projection of the frontal line through the horizontal projection of the point and parallel to the coordinate axis, and so on.

**Example 76**

Given: a plane $P$ and the vertical projection ($a'$) of a point $A$ lying in this plane. Find its horizontal projection $a$ (Fig. 277).

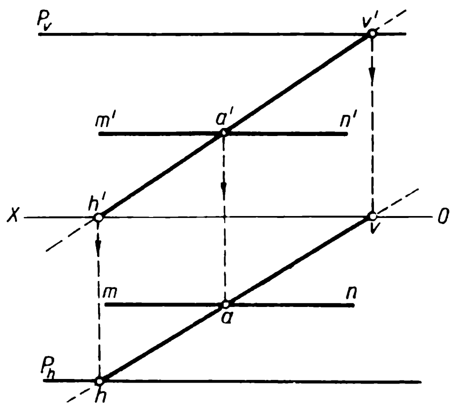



FIG. 274.

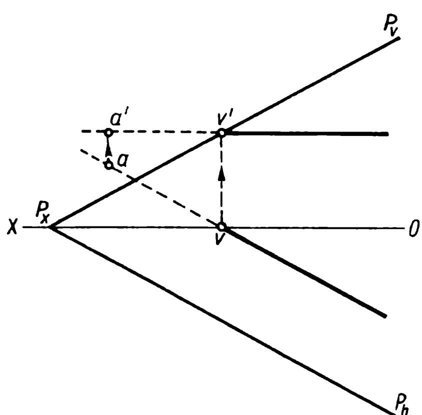



FIG. 275.

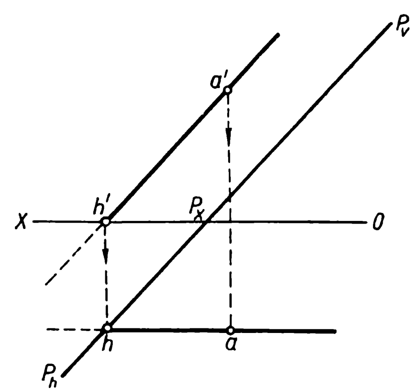



FIG. 276.

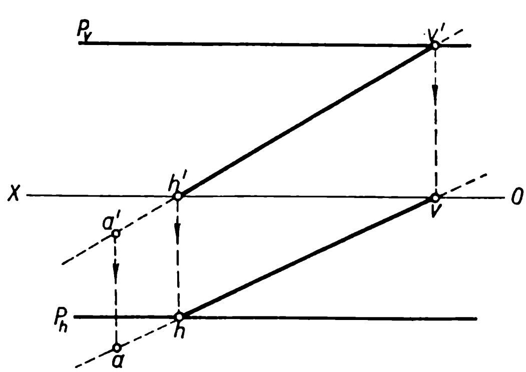



FIG. 277.

*Solution*. Draw the vertical projection ($h'v'$) of an auxiliary line contained in the plane $P$ through the vertical projection ($a'$) of the given point. From $h'v'$ find the horizontal projection ($hv$) of the line and, on it, the horizontal projection ($a$) of the point using its vertical projection ($a'$).

This problem may be solved in a different way. As is known, the profile projection ($a''$) of a point contained in a plane must lie on the profile trace ($P_w$) of the plane. Having the profile trace ($P_w$) of the plane and knowing the vertical projection ($a'$) of the point, find the profile projection ($a''$) of the point, and then, following the general rule, determine the horizontal projection ($a$) of the point from its projections $a'$ and $a''$ (make the drawing).

**Example 77**

Given: a plane $P$ and the horizontal projection ($a$) of a point $A$ lying in this plane. Find the vertical projection $a'$ (Figs. 278, 279).

*Solution*. Through the horizontal projection ($a$) of the point draw the horizontal projection ($ak$) of an auxiliary line intersecting the coordinate axis at point ($m$, $m'$). With the aid of the horizontal projection ($km$) of the line-segment find its vertical projection ($k'm'$) and, on it, the required vertical projection ($a'$) of the given point.

This problem may be solved in a different way. Knowing that the profile projection ($a''$) of the point must lie on the profile trace ($P_w$) of the plane, find this trace, and, on it, the profile projection ($a''$) of the point, and finally, the desired vertical projection ($a'$) of the point.

**Example 78**

Given: the vertical trace ($P_v$) of a plane $P$ and a point $A$ lying in this plane. Find the horizontal trace ($P_h$) of the plane (Fig. 280).

*Solution*. The vertical trace ($P_v$) of the given plane is perpendicular to the coordinate axis, and, hence, the given plane is a horizontal projecting plane. Since the given point ($a$, $a'$) lies in the plane $P$, draw the desired horizontal trace ($P_h$) of the plane through the horizontal projection ($a$) of the point and the point $P_x$.

**Example 79**

Given: the horizontal trace ($P_h$) of a plane $P$ and a point $A$ lying in this plane. Find the vertical trace ($P_v$) of the plane (Fig. 281).

*Solution*. The horizontal trace ($P_h$) of the plane is perpendicular to the coordinate axis, and, hence, the given plane is a vertical projecting plane. Since the given point ($a$, $a'$) lies in the plane $P$, draw the desired vertical trace ($P_h$) of the plane through the vertical projection ($a'$) of the given point and $P_x$.

**Example 80**

Given: the vertical trace ($P_v$) of a plane $P$ and a point $A$ lying in this plane. Find the horizontal trace ($P_h$) of the plane (Figs. 282, 283).

*Solution*. The vertical trace ($P_v$) of the plane is parallel to the coordinate axis, hence the given plane is a profile projecting plane. Since the given point ($a$, $a'$) lies in the plane $P$, its profile projection ($a''$) must lie on the profile trace ($P_w$) of the given plane.

Draw the profile trace ($P_w$) of the plane through the profile trace of the line $P_v$ and the profile projection $a''$ of the given point. Knowing the trace $P_w$ of the plane, draw the desired horizontal trace ($P_h$) of the plane through the horizontal projection of its profile trace and parallel to the coordinate axis.

The same problem may be solved without using the profile plane of projection, but in this case use is made of an auxiliary line. In the plane $P$ draw the vertical projection ($h'v'$) of the auxiliary line through the vertical projection ($a'$) of the point. Then find its horizontal projection ($hv$) from the vertical projection ($h'v'$) and draw the required horizontal trace ($P_h$) of the plane through the horizontal trace of the line and parallel to the coordinate axis.

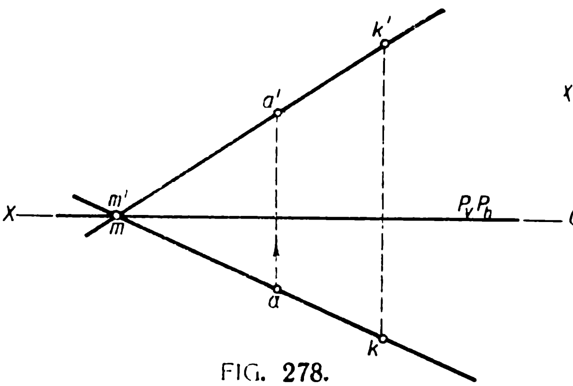



FIG. 278.

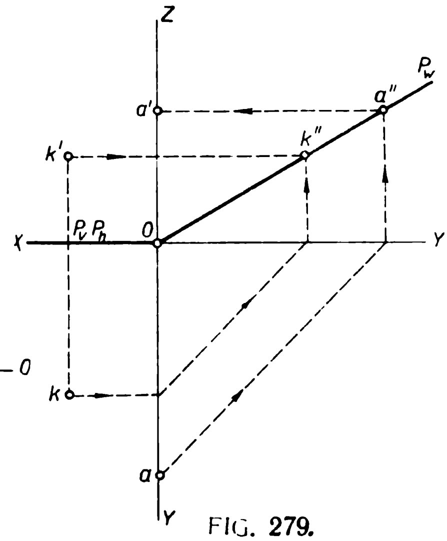



FIG. 279.

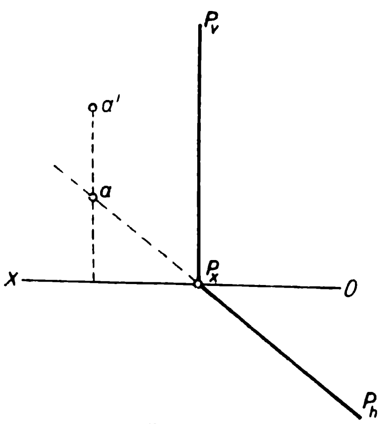



FIG. 280.

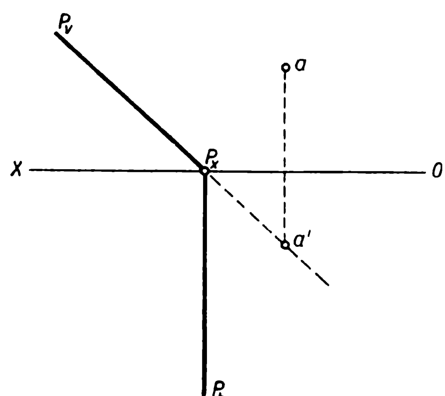



FIG. 281.

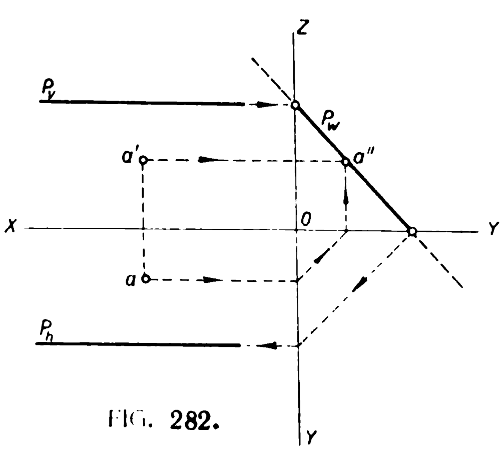



FIG. 282.

**Example 81**

Find the vertical trace ($P_h$) of a plane $P$, given its horizontal trace ($P_v$) and a point $A$ lying in the plane (Figs. 284 to 286).

*Solution.* Since the given plane is oblique, make use of an auxiliary line

(1) In the plane $P$ draw the horizontal projection ($hv$) of the auxiliary line through the horizontal projection ($a$) of the point and from it define the vertical projection ($h'v'$) of the line. Then draw the desired vertical trace ($P_v$) of the plane through the points $v$ and $P_x$.

The simplest way of solving this problem is by making use of principal lines of a plane: a horizontal or a frontal line.

(2) Draw the projections of a horizontal line of the plane through the projections of the point ($a$, $a'$): the horizontal one through the point $a$ and parallel to the horizontal trace ($P_h$) of the plane, and the vertical one through the point $a'$ and parallel to the coordinate axis. Find the trace ($v$, $v'$) of the horizontal line and draw the desired vertical trace ($P_v$) of the plane through the points $v'$ and $P_x$.

(3) Through the horizontal projection ($a$) of the given point draw the horizontal projection of a frontal line (how?) to intersect the horizontal trace ($P_h$) of the given plane at point $h$. Find $h'$ on the coordinate axis and draw, through the points $h'$ and $a'$, the vertical projection of the frontal line and, parallel to it, the required vertical trace ($P_v$) of the plane through the point $P_x$.

**Example 82**

Find the horizontal trace ($P_h$) of a plane, given its vertical trace ($P_v$) and a point $A$ lying in the plane (Figs. 287, 288).

*Solution.* 1. Draw the vertical projection of a horizontal line in the plane $P$ through the vertical projection ($a'$) of the given point and parallel to the coordinate axis to intersect the vertical trace ($P_v$) of the plane at the point $v'$. Find the point $v$ on the coordinate axis and draw, through the points $a$ and $v$, the horizontal projection of the horizontal line and, parallel to it and through the point $P_x$, the desired horizontal trace ($P_h$) of the plane.

2. Through the projections of the point ($a$, $a'$) draw the projections of a frontal line in the plane: the vertical one through the point $a'$ and parallel to the vertical trace ($P_v$) of the plane, and the horizontal one through the point $a$ and parallel to the coordinate axis. Find the horizontal trace ($h$, $h'$) of the frontal line and draw the desired horizontal trace ($P_h$) of the plane through the points $h$ and $P_x$.

**Example 83**

Construct the traces of the horizontal projecting plane, given a point $A$ lying in it and the vanishing point $P_x$ (Fig. 289).

*Solution.* Since the desired plane is a horizontal projecting plane, the horizontal projection ($a$) of the point $A$ must lie on the horizontal trace ($P_h$) of the plane, and the vertical trace ($P_v$) must be perpendicular to the coordinate axis. Draw the traces of the plane: the horizontal one ($P_h$) through the points $P_h$ and $a$, the vertical one ($P_v$) through the point $P_x$ and perpendicular to the coordinate axis.

*Conclusion*: If a horizontal projecting plane passes through a certain point, then the horizontal trace of the plane passes through the horizontal projection of the point.

**Example 84**

Given: a point $A$. Pass through it a plane $P$ perpendicular to the horizontal plane of projection and inclined at an angle of 30° to the vertical plane of projection (Fig. 290).

*Solution.* The angle formed by a horizontal projecting plane and the vertical plane of projection is determined by the angle at which the horizontal trace of the plane is inclined to the coordinate axis. Thus, through the horizontal projection ($a$) of the point, we draw the horizontal trace ($P_h$) of the plane at the given angle (30°)

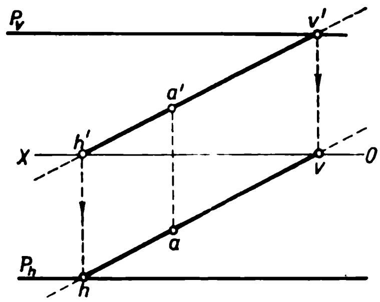



FIG. 283.

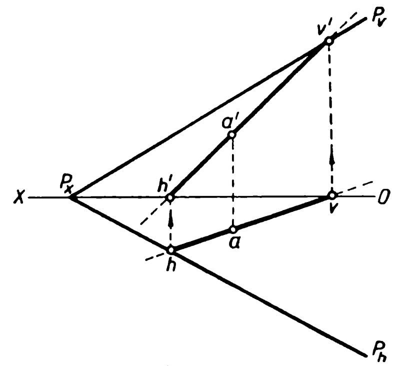



FIG. 284.

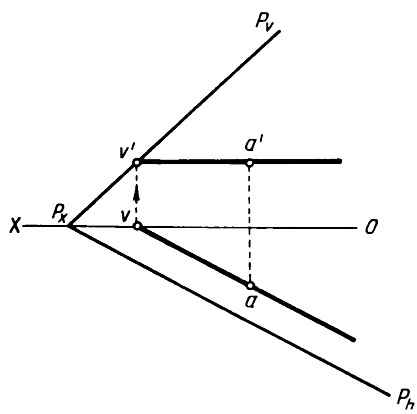



FIG. 285.

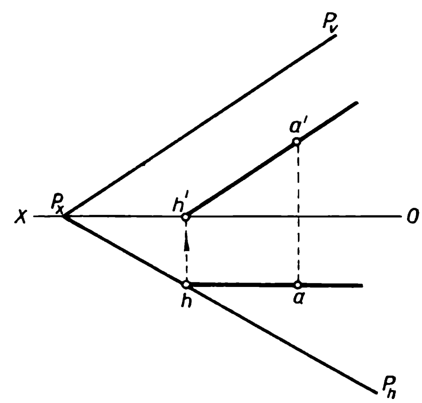



FIG. 286.

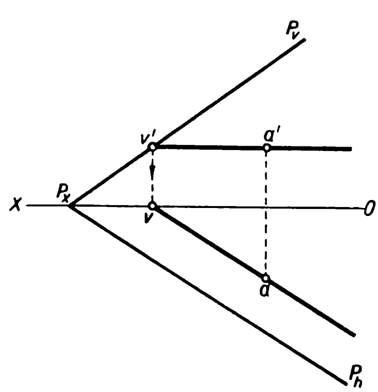



FIG. 287.

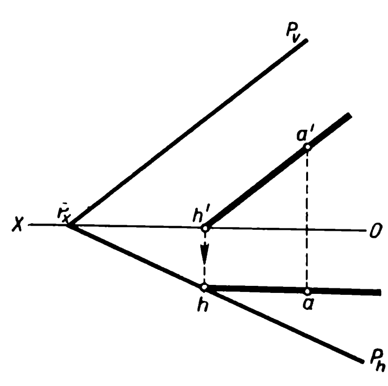



FIG. 288.

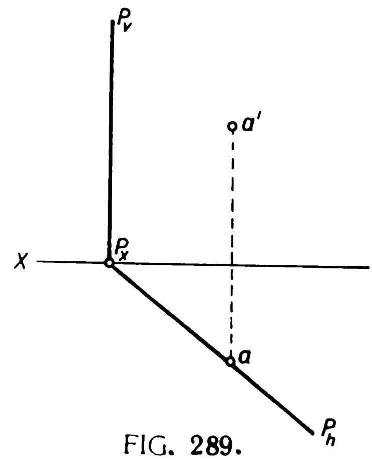



FIG. 289.

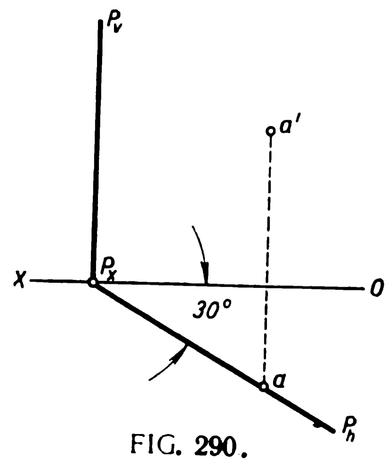



FIG. 290.

to the coordinate axis to intersect it at point $P_x$. Then, through $P_x$, draw the vertical trace ($P_v$) of the plane perpendicular to the coordinate axis. (We give one alternative solution.)

**Example 85**

Through a point $A$ pass a plane $P$ perpendicular to the vertical plane of projection (Fig. 291).

*Solution.* Since the point ($a$, $a'$) must lie in a vertical projecting plane, its vertical projection must lie on the vertical trace ($P_v$) of the plane; furthermore, the horizontal trace ($P_h$) must be perpendicular to the coordinate axis. As is known, an indefinite number of planes can be passed through a given point; that is why we assume an arbitrary point $P_x$ on the coordinate axis and draw the traces of the plane: the vertical one ($P_v$) through the points $a'$ and $P_x$, and the horizontal one ($P_h$) through the point $P_x$ and perpendicular to the coordinate axis.

*Conclusion*: If a vertical projecting plane passes through a point, then the vertical trace of the plane passes through the vertical projection of the point.

**Example 86**

Given: a point $A$. Pass through it a plane $P$ perpendicular to the vertical plane of projection and inclined at an angle of 60° to the horizontal plane (Fig. 292).

*Solution.* The angle formed by a vertical projecting plane and the horizontal plane of projection is determined by the angle at which the vertical trace of the plane is inclined to the coordinate axis. Thus, through the vertical projection ($a'$) of the point, we draw the vertical trace ($P_v$) of the plane at an angle of 60° to the coordinate axis to intersect it in the point $P_x$. Then, through $P_x$, draw the horizontal trace ($P_h$) of the plane perpendicular to the coordinate axis. (We give one alternative solution.)

**Example 87**

Through a point $A$ pass a plane $P$ parallel to the coordinate axis and passing through the first, second, and third quadrants (Fig. 293).

*Solution.* Since the plane $P$ is a profile projecting plane, its profile trace ($P_w$) must pass through the profile projection ($a''$) of the given point. The traces ($P_h$ and $P_v$) of the plane must be parallel to the coordinate axis. Any plane passing through the first, second and third quadrants has the horizontal trace ($P_h$) lying in the rear

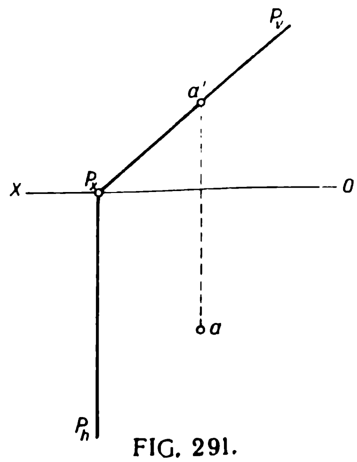



FIG. 291.

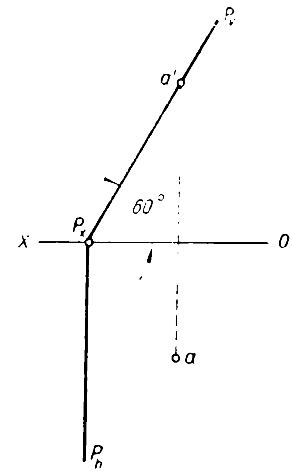



FIG. 292.

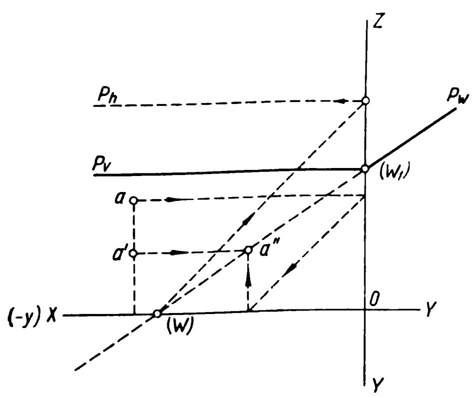



FIG. 293.

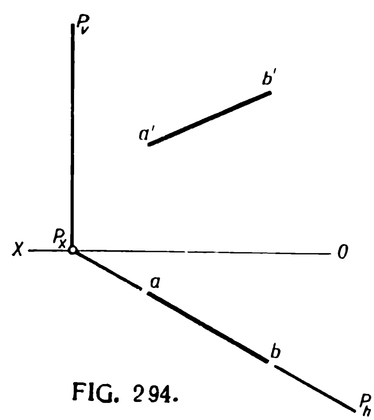



FIG. 294.

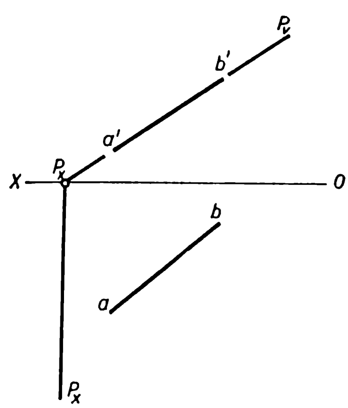



FIG. 295.

half-plane of the horizontal plane of projection and the vertical trace ($P_v$) in the upper half-plane of the vertical plane of projection. Both traces are orthographically represented above the coordinate axis. Find the profile projection ($a''$) of the given point and draw, through it, arbitrarily the profile trace ($P_w$) of the plane to intersect the $z$- and $y$-axis at the points $W$ (the profile trace of the line $P_h$) and $W_1$ (the profile trace of the line $P_v$); then determine $P_h$ and $P_v$ (see the accompanying drawing).

*Note.* On p. 102 we explain how an oblique plane is passed through a given point.

**Example 88**

Put the line-segment $AB$ into the horizontal projecting plane $P$ (Fig. 294).

*Solution.* As is known, the horizontal projection of any line contained in a horizontal projecting plane coincides with the horizontal trace of the plane. The converse is also true: the horizontal trace ($P_h$) of a plane containing an arbitrary line coincides with the horizontal projection ($ab$) of the given line. Therefore we extend the horizontal projection ($ab$) of the line-segment to intersect the coordinate axis at point $P_x$. Erect a perpendicular at this point; it will be the vertical trace ($P_v$) of the plane.

**Example 89**

Put the line-segment $AB$ into the vertical projecting plane $P$ (Fig. 295).

*Solution.* As is known, the vertical projection of an arbitrary line lying in a vertical projecting plane coincides with the vertical trace of the plane. The converse is also true: the vertical trace ($P_v$) of a plane containing an arbitrary line will coincide with the vertical projection ($a'b'$) of the given line. Hence, extend the vertical projection ($a'b'$) of the line-segment to intersect the coordinate axis at the point $P_x$. Through this point draw the horizontal trace ($P_h$) of the plane perpendicular to the coordinate axis.

**Example 90**

Put the line-segment $AB$ into the plane $P$, which is parallel to the coordinate axis (Fig. 296).

*Solution.* As is known, the profile projection of an arbitrary line contained in a profile projecting plane coincides with the profile trace of the plane. The converse

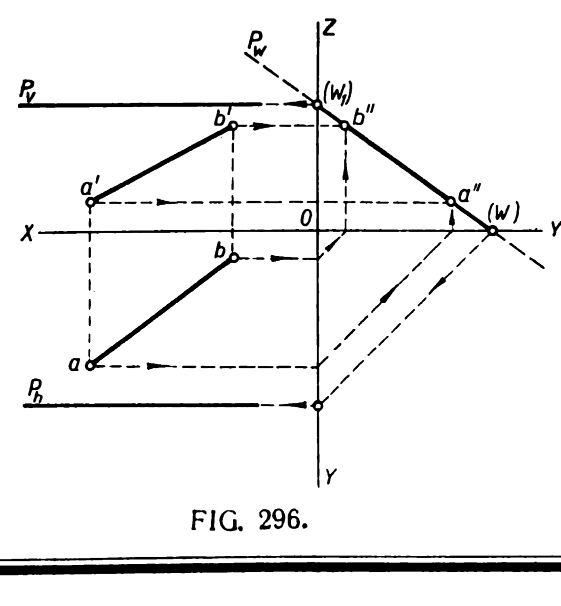



FIG. 296.

is also true: the profile trace ($P_w$) of a plane containing an arbitrary line coincides with the profile projection ($a''b''$) of the given line-segment. Hence, find the profile projection ($a''b''$) of the line-segment and extend it to intersect the $z$- and $y$-axes at points $W$ and $W_1$. Then draw the horizontal ($P_h$) and vertical ($P_v$) traces of the plane parallel to the coordinate axis (see the drawing).

*Note*. An alternative solution to this problem is possible without the use of the profile plane of projection.

*Conclusion*. It is possible to draw through an arbitrary line: one horizontal projecting plane, one vertical projecting plane, and one profile projecting plane (why?).

**Example 91**

Put the line-segment $AB$ into the oblique plane $P$, given the vanishing point ($P_x$) of its traces (Fig. 297).

*Solution*. As is known, the traces of a line lying in a plane are found on the like traces of the plane, and vice versa: the traces of a plane containing a line must pass through the like traces of the line.

Find the trace ($v$, $v'$) of the line and draw the vertical trace ($P_v$) of the plane through the points $P_x$ and $v'$, and its horizontal trace ($P_h$) through the point $P_x$ and parallel to the horizontal projection ($ab$) of the given line-segment.

**Example 92**

Put the line-segment $AB$ into the oblique plane $P$ (Fig. 298).

*Solution*. Find the trace ($h, h'$) of the line and, assuming an arbitrary (why?) point $P_x$ on the coordinate axis, draw the traces of the plane: the horizontal one ($P_h$) through the points $P_x$ and $h$, and the vertical one ($P_v$) through the point $P_x$ and parallel to the vertical projection ($a'b'$) of the line-segment (why?).

**Example 93**

Construct the traces of the oblique plane $P$ passing through the point $A$, given the vanishing point $P_x$ (Figs. 299, 300).

*Solution*: 1. Let us use a horizontal line in the plane. Draw the projections of an arbitrary horizontal line through the like projections of the point ($a$, $a'$) and find the trace ($v$, $v'$) of the line. Draw the vertical trace ($P_v$) of the plane through the points $P_x$ and $v'$, and the horizontal trace ($P_h$) through the point $P_x$ and parallel to the horizontal projection of the horizontal line (why?).

2. Use a frontal line in the plane. Draw the projections of an arbitrary frontal line through the like projections of the given point ($a$, $a'$) and find its trace ($h$, $h'$). Draw the horizontal trace ($P_h$) of the plane through the points $P_x$ and $h$, and the vertical trace ($P_v$) through the point $P_x$ and parallel to the vertical projection of the frontal line (why?).

*Note*. Any line can be drawn through the point $A$.

**Example 94**

Construct the traces of the oblique plane $P$ passing through the point $A$ (Figs. 301 to 303).

*Solution*: 1. Draw the projections of an arbitrary line through the like projections of the point ($a$, $a'$) and find the traces ($h$, $h'$) and ($v$, $v'$) of the line. Since an indefinite number of planes can be passed through a given point, assume an arbitrary point $P_x$ on the coordinate axis and draw the traces of the plane: the horizontal one ($P_h$) through the points $P_x$ and $h$, and the vertical one ($P_v$) through the points $P_x$ and $v'$.

2. Use a horizontal line in the plane. Draw the projections of an arbitrary horizontal line through the like projections of the point ($a$, $a'$) and find its trace ($v$, $v'$). Assume an arbitrary point $P_x$ on the coordinate axis and draw the traces of the plane: the vertical one ($P_v$) through the points $P_x$ and $v'$, and the horizontal one ($P_h$) through the point $P_x$ and parallel to the horizontal projection of the horizontal line (why?).

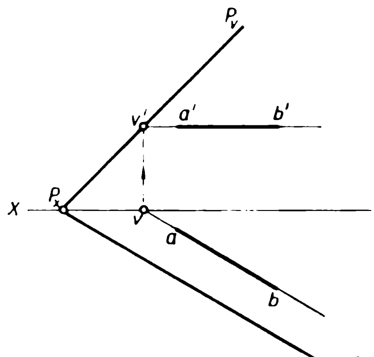



FIG. 297.

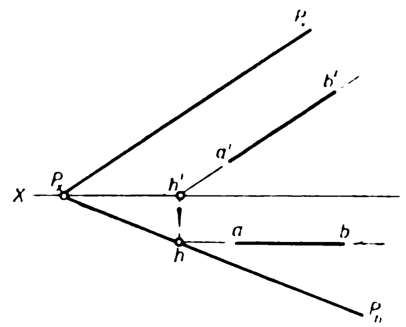



FIG. 298.

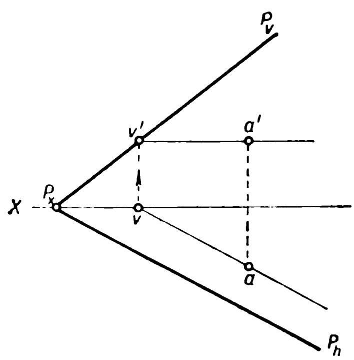



FIG. 299.

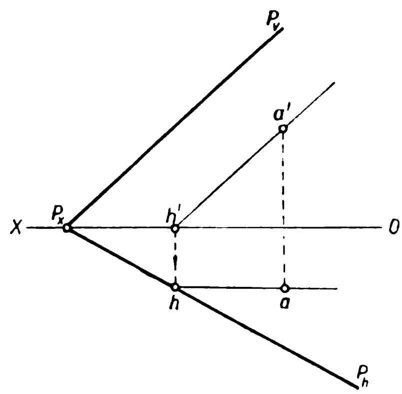



FIG. 300.

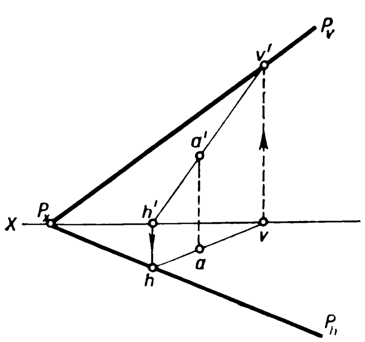



FIG. 301.

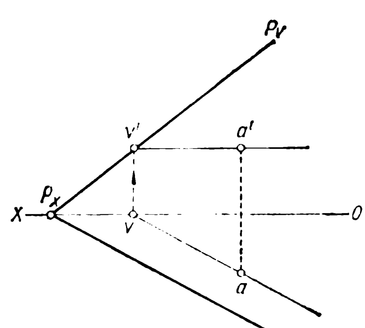



FIG. 302.

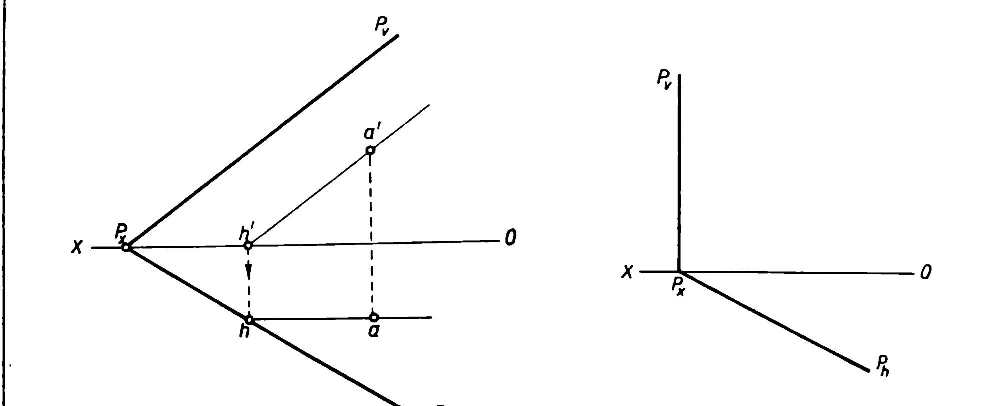



FIG. 303. FIG. 304.

3. Use a frontal line in the plane. Draw the projections of an arbitrary frontal line through the like projections of the point ($a$, $a'$) and find the trace ($h$, $h'$) of the line. Assume an arbitrary point $P_x$ on the coordinate axis and draw the traces of the plane: the horizontal one ($P_h$) through the points $P_x$ and $h$ and the vertical one ($P_v$) through the point $P_x$ and parallel to the vertical projection of the frontal line (why?).

## PROBLEMS

199. Enumerate the lines which can be drawn in the plane $P$ (Figs. 304 to 311).
200. Find the traces of a plane specified by: two intersecting lines, two parallel lines, a line and a point (Figs. 312 to 317).
201. Find the profile trace of the plane $P$ (Figs. 318 to 327).
202. Draw in the plane $P$ an arbitrary line passing through the given quadrants (Figs. 328 to 332).
203. Construct in the plane $P$ the locus of points 15 mm distant from the horizontal plane of projection (Figs. 332, 333).
204. Construct in the plane $P$ the locus of points 15 mm distant from the vertical plane of projection (Figs. 332, 333).
205. Construct in the plane $P$ a point $A$, given its coordinates (Figs. 332, 333):

| | | | | | | |
|---|---|---|---|---|---|---|
| $y$ | 15 | −15 | −20 | −15 | 25 | 20 |
| $z$ | 25 | 25 | 20 | −25 | −15 | −20 |

206. Determine the lacking projections of the points lying in the plane $P$ (Figs. 334 to 341).
207. Given: the horizontal or vertical projection of the triangle $ABC$ contained in the plane $P$. Determine the vertical or horizontal projection of the triangle without using the principal lines of the plane; using horizontal and frontal lines of the plane (Figs. 342, 343).
208. Given: one of the projections of the triangle $ABC$ lying in a profile projecting plane. Determine the other projection without using the profile plane of projection; using the profile plane of projection (Figs. 344, 345).

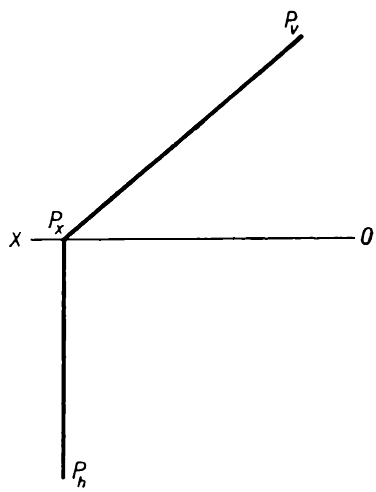



FIG. 305.

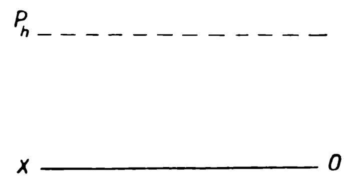



FIG. 306.

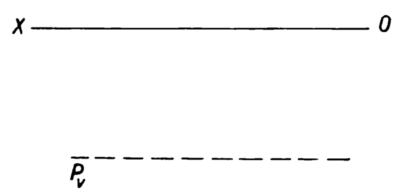



FIG. 307.

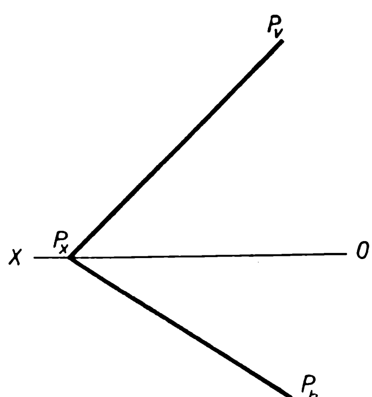



FIG. 308.

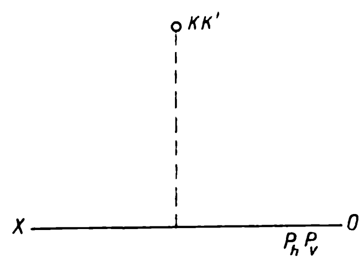



FIG. 309

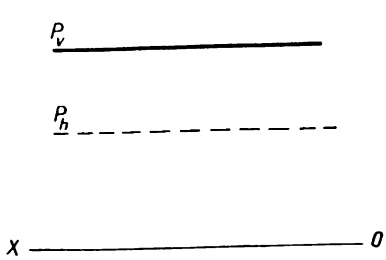



FIG. 310.

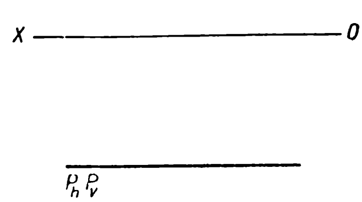



FIG. 311.

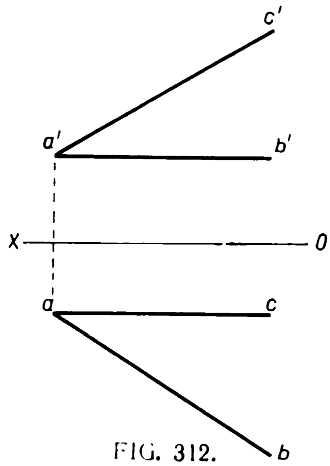



FIG. 312.

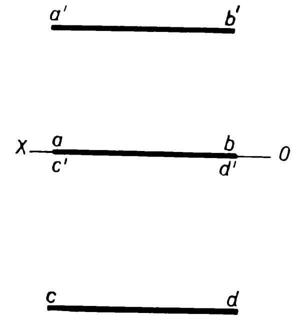



FIG. 313.

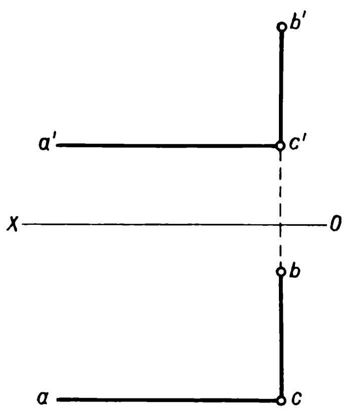



FIG. 314.

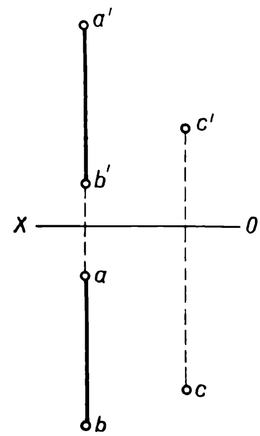



FIG. 315.

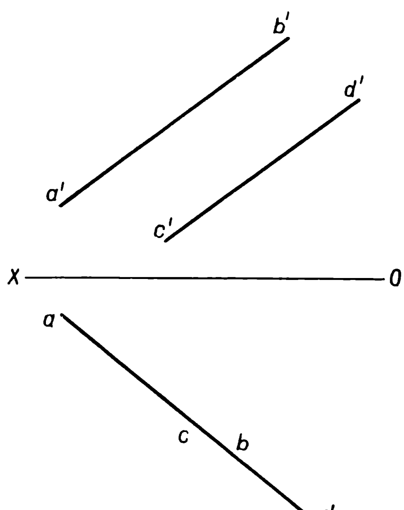



FIG. 316.

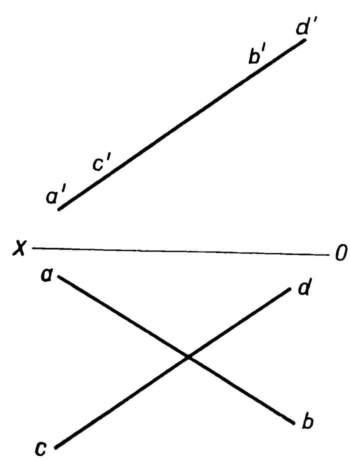



FIG. 317.

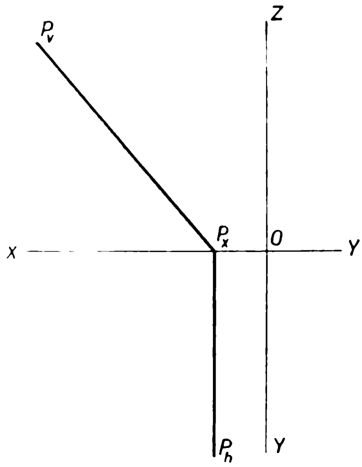


FIG. 318.

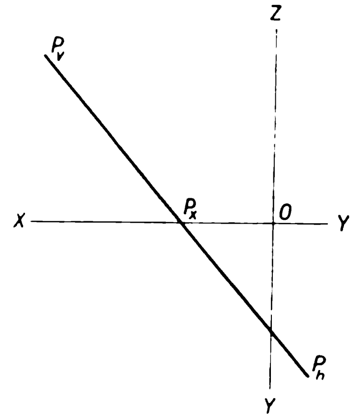


FIG. 319.

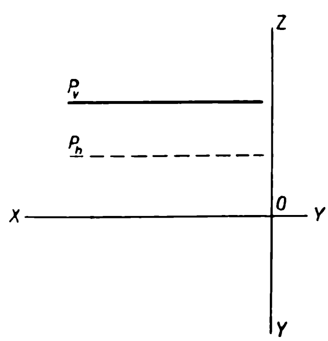


FIG. 320.

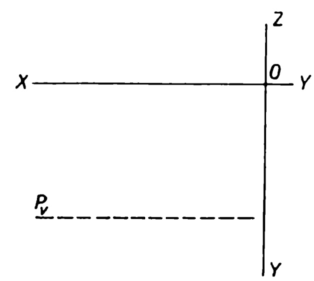


FIG. 321.

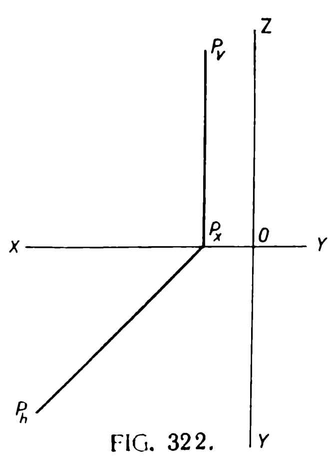


FIG. 322.

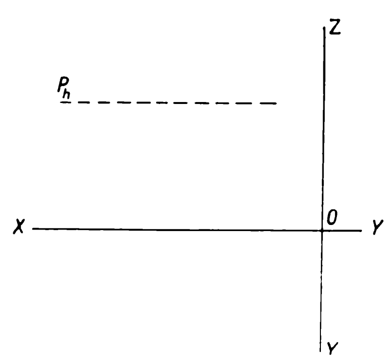


FIG. 323.

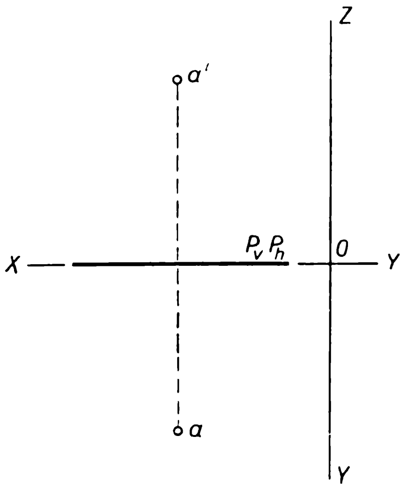



FIG. 324.

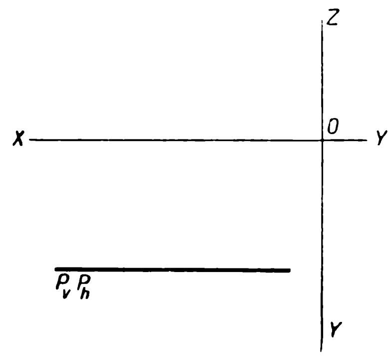



FIG. 325.

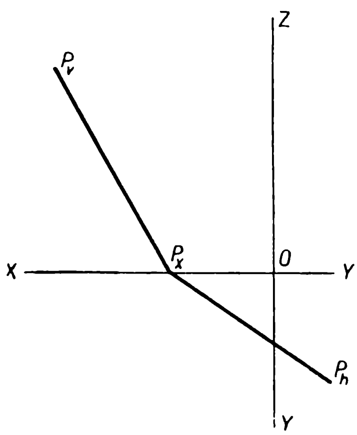



FIG. 326.

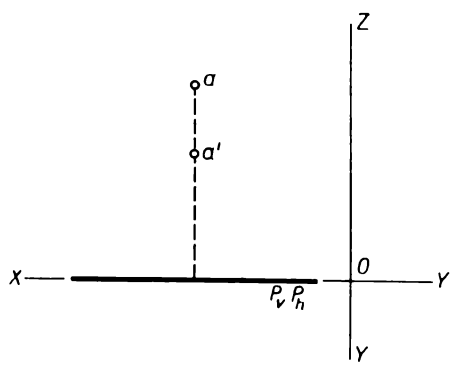



FIG. 327.

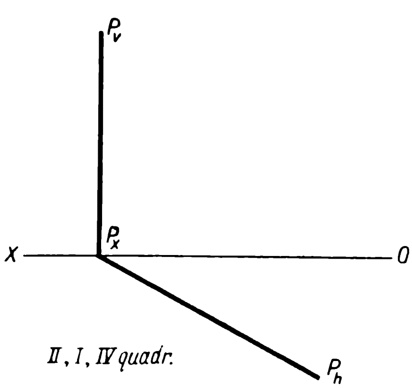



FIG. 328.

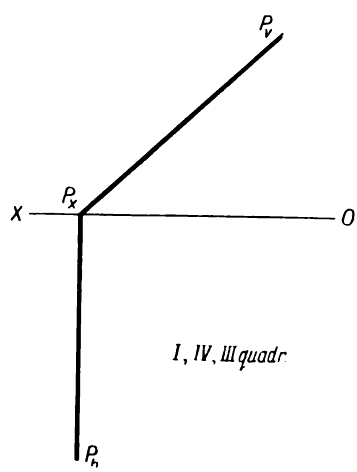



FIG. 329.

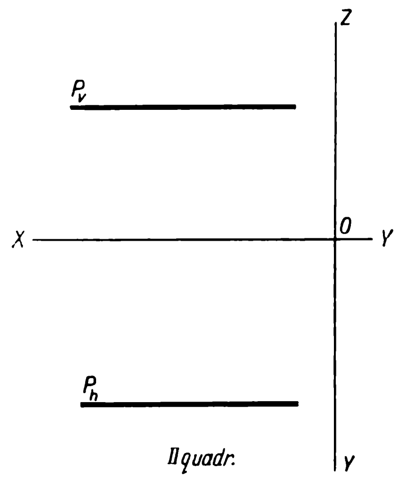



FIG. 330.

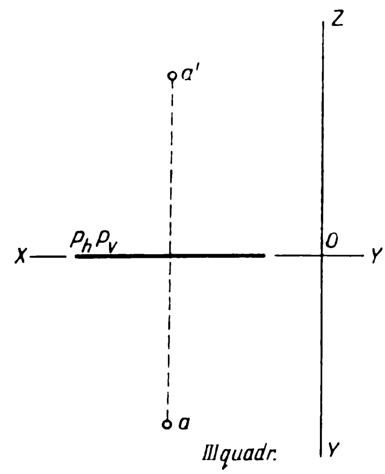



FIG. 331.

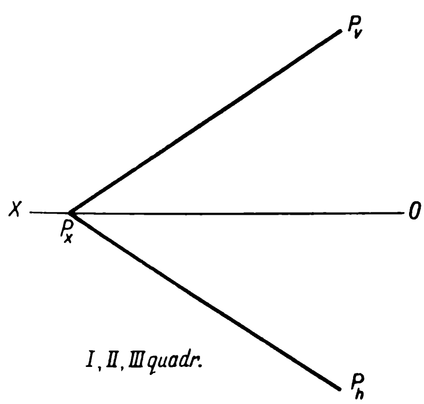



FIG. 332.

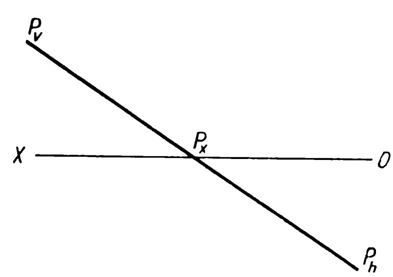



FIG. 333.

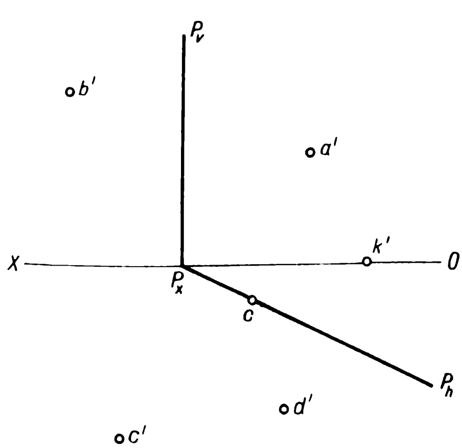



FIG. 334.

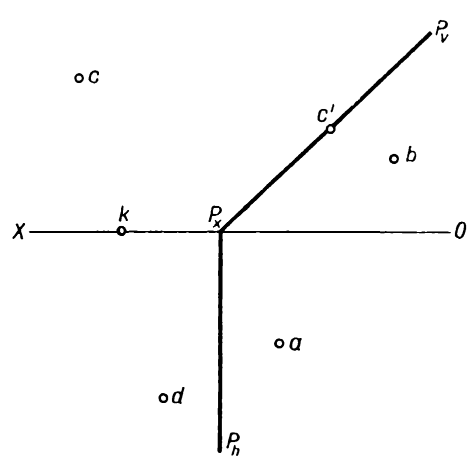



FIG. 335.

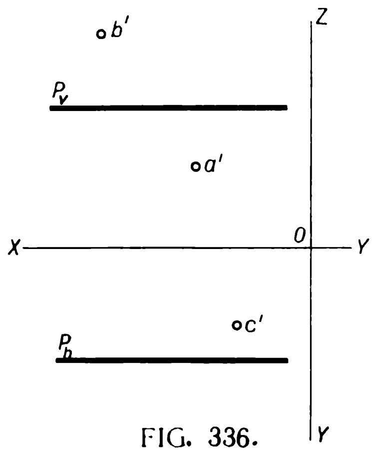



FIG. 336.

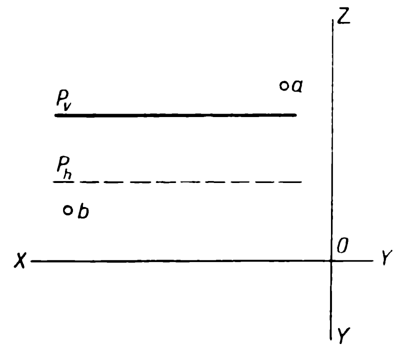



FIG. 337.

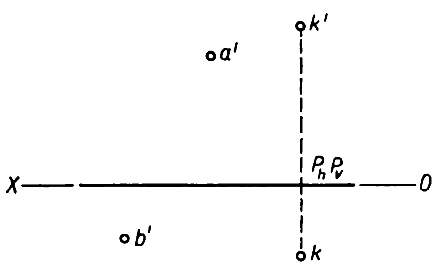



FIG. 338.

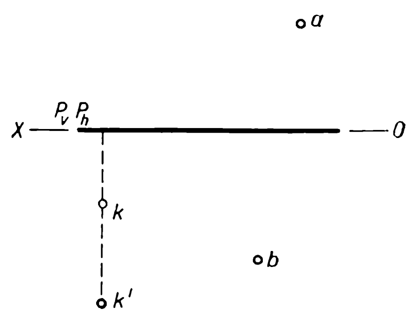



FIG. 339.

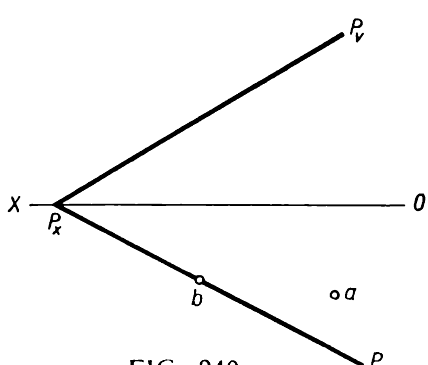



FIG. 340.

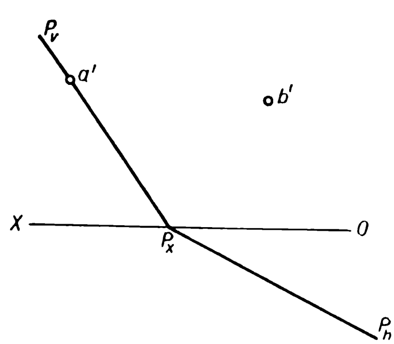



FIG. 341.

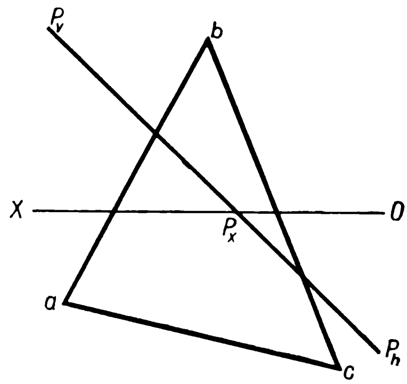



FIG. 342.

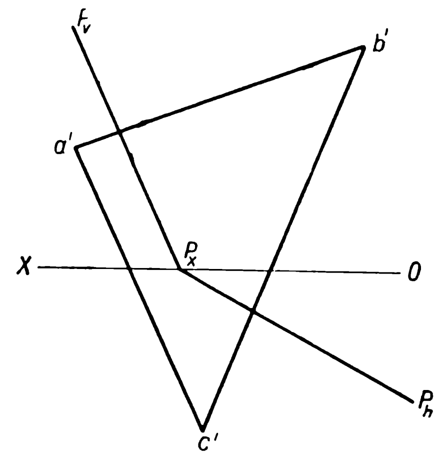



FIG. 343.

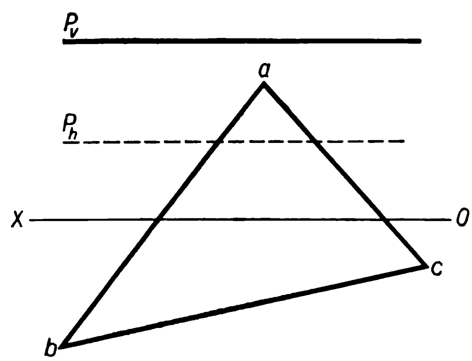



FIG. 344.

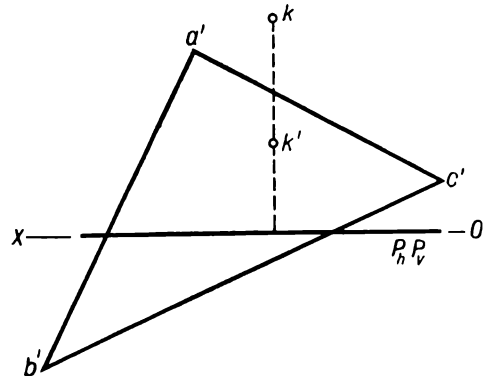



FIG. 345.

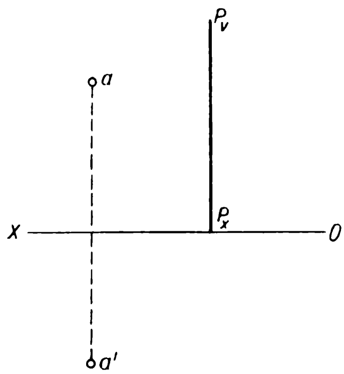



FIG. 346.

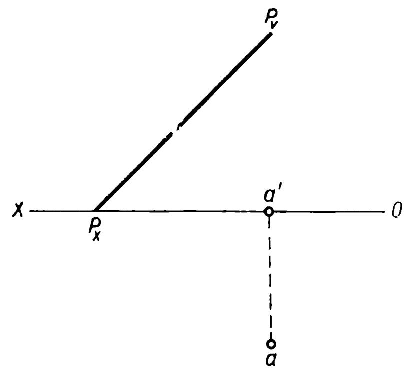



FIG. 347.

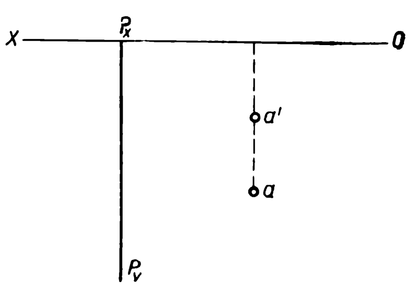



FIG. 348.

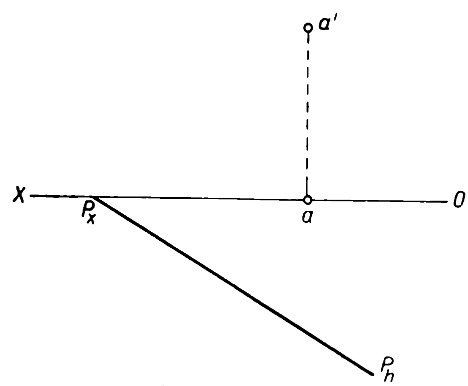



FIG. 349.

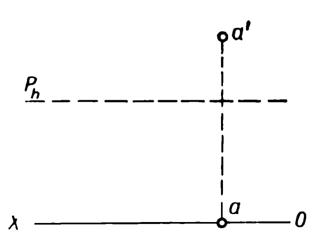



FIG. 350.

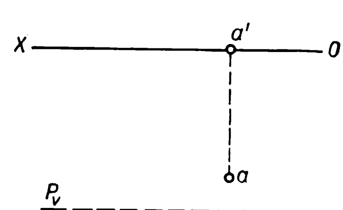



FIG. 351.

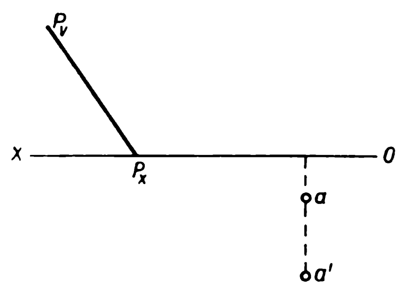



FIG. 352.

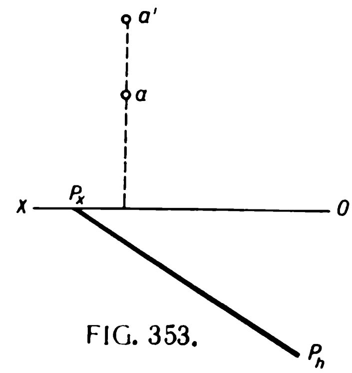



FIG. 353.

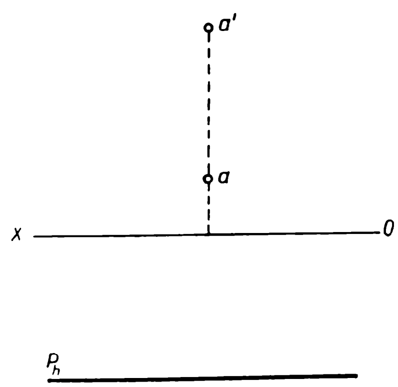



FIG. 354.

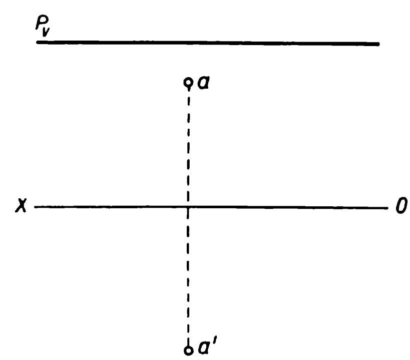



FIG. 355.

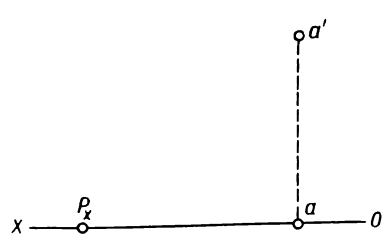



FIG. 356.

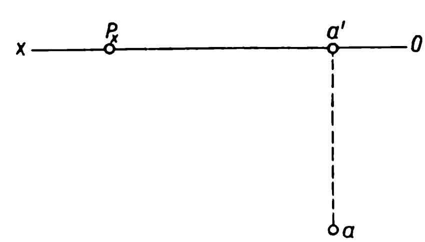



FIG. 357.

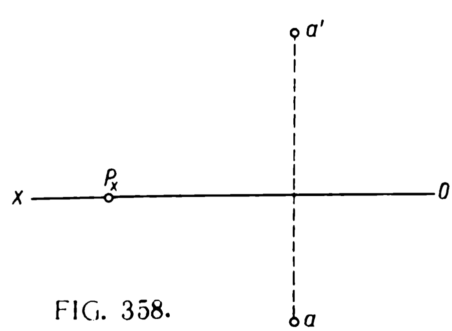



FIG. 358.

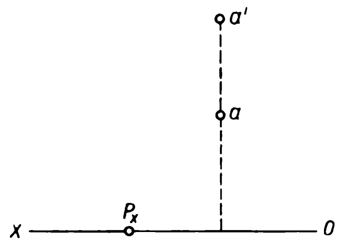



FIG. 359.

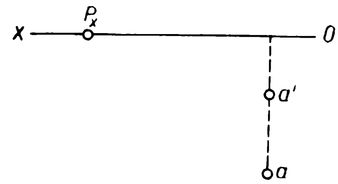



FIG. 360.

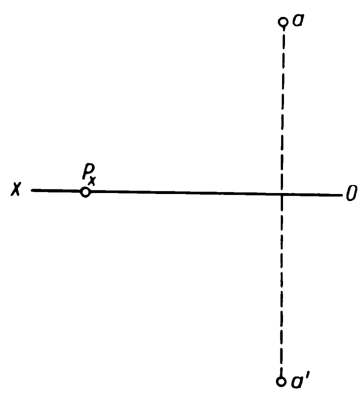



FIG. 361.

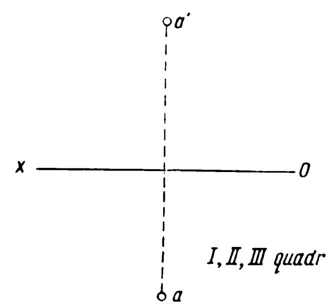



FIG. 362.

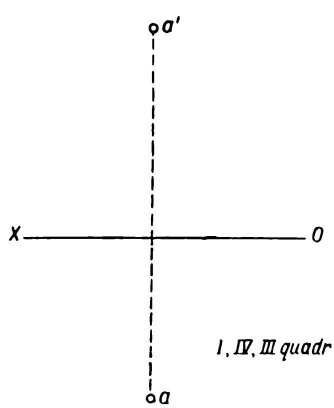



FIG. 363.

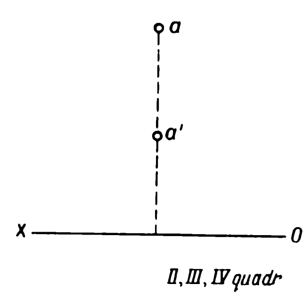



FIG. 364.

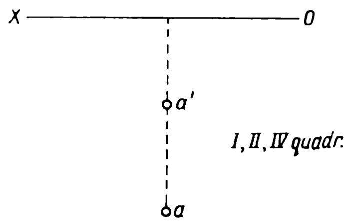



FIG. 365.

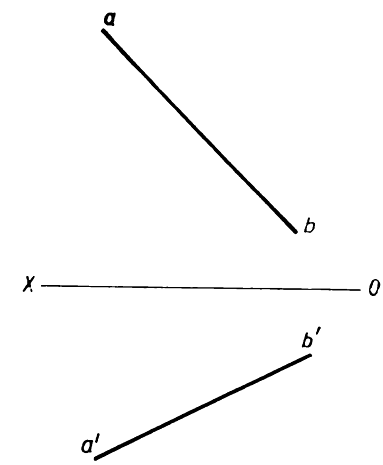



FIG. 366.

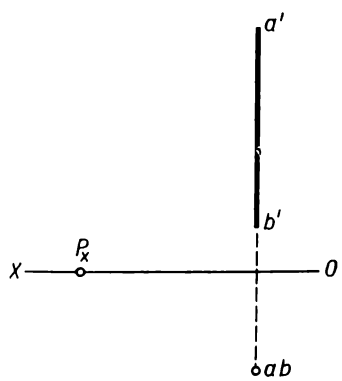



FIG. 367.

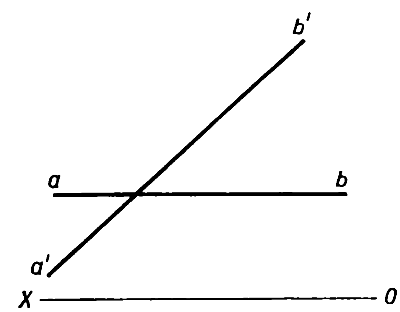



FIG. 368.

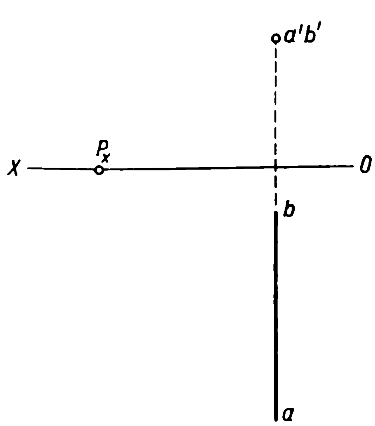



FIG. 369.

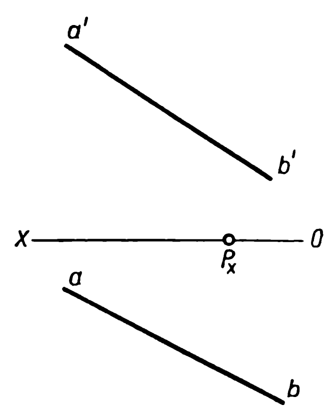



FIG. 370.

209. Determine the lacking trace of the plane $P$ specified by one of its traces and a point lying in this plane (Figs. 346 to 355).

210. Construct the traces of the plane $P$ passing through the point $A$, given the vanishing point and knowing that the plane is:

(1) oblique (Figs. 356 to 359);
(2) horizontal projecting (Fig. 360);
(3) vertical projecting (Fig. 361).

211. Construct the traces of the profile projecting plane $P$ passing through the point $A$ and through the indicated quadrants (Figs. 362 to 365). The problem is indeterminate.

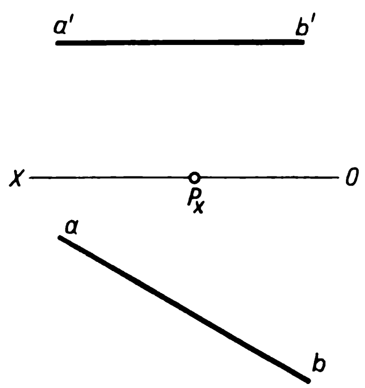



FIG. 371.

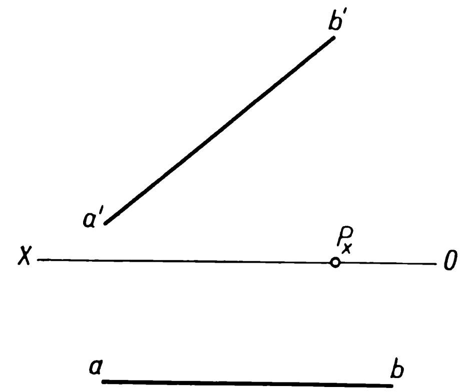



FIG. 372.

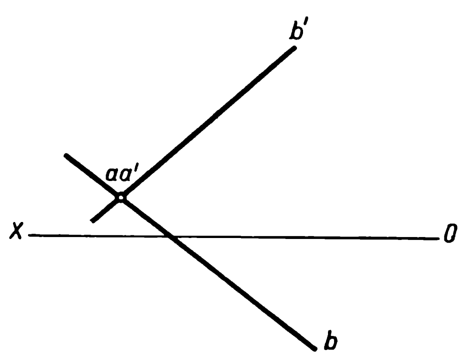



FIG. 373.

212. Through a given point $A$ (—30, —20) pass a horizontal projecting plane $P$ inclined at an angle of 45° to the vertical plane of projection.

213. Through a given point $A$ (30, —20) pass a vertical projecting plane $P$ inclined at an angle of 30° to the horizontal plane of projection.

214. Through a given point $A$ (—20, 30) pass a profile projecting plane inclined: at an angle of 30° to the horizontal plane of projection; at an angle of 45° to the vertical plane of projection.

215. Construct the traces of the plane $P$ passing through the line $AB$, given that the plane is:

(1) horizontal projecting (Figs. 366, 367);
(2) vertical projecting (Figs. 368, 369);
(3) oblique (Figs. 370 to 372).

216. Construct the traces of a profile projecting plane passing through the line $AB$: using the profile plane of projection; without using the profile plane of projection (Fig. 373).

CHAPTER XIII

# INTERSECTION OF PLANES SPECIFIED BY TRACES

The line of intersection of two planes is a straight line for the construction of which it is sufficient to determine two points that are common to the two planes, or one point and the direction of the line of intersection of the planes.

In a particular case a line may be specified by its traces.

The traces of the line of intersection of two planes are found at the intersection of the like traces of these planes, namely:

the horizontal trace of the line of intersection is found at the intersection of the horizontal traces of the planes;

the vertical trace of the line of intersection is found at the intersection of the vertical traces of the planes;

the profile trace of the line of intersection is found at the intersection of the profile traces of the planes.

Depending on the relative positions of intersecting planes in space, the line of their intersection, represented orthographically in a two-plane system ($H$ and $V$), may have the following traces: horizontal and vertical; only vertical; neither horizontal nor vertical.

If the line of intersection has two traces (horizontal and vertical), then it is an oblique line or a profile line, or a line intersecting the coordinate axis. If the line of intersection has only a horizontal trace, then it is parallel to the vertical plane of projection (a frontal principal line); in a particular case it may be perpendicular to the horizontal plane of projection.

If the line of intersection has only a vertical trace, then it is parallel to the horizontal plane of projection (a horizontal principal line); in a particular case it may be perpendicular to the vertical plane of projection.

If the line of intersection has only a profile trace, then it is parallel to the coordinate axis.

If the line of intersection is a projecting one, i.e. perpendicular to a projecting plane, *it should be borne in mind* that one of the projections of the line coincides with its trace, namely: if the line of intersection is perpendicular to the plane $H$, then its horizontal projection coincides with the horizontal trace of the line; if it is perpendicular to the plane $V$, then its vertical projection coincides with the vertical trace of the line.

If the line of intersection is perpendicular to the plane $W$ (i. e. parallel to the coordinate axis), then its profile projection coincides with the profile trace of the line.

If the like traces (horizontal or vertical, or both) of the planes fail to intersect within the limits of the drawing, it is necessary to find one or two *arbitrary* points belonging to the line of intersection of the given planes.

An arbitrary point belonging to the line of intersection of a pair of planes is determined by introducing an *auxiliary* plane (see examples given below).

EXAMPLES

**Example 95**

Find the line of intersection of the planes $P$ and $Q$ (Fig. 374).

*Solution.* The given planes intersect along an oblique line passing through the traces (points $h$, $h'$ and $v$, $v'$) found at the intersection of the horizontal and vertical traces of the planes. Draw the projections of the desired line: the horizontal one

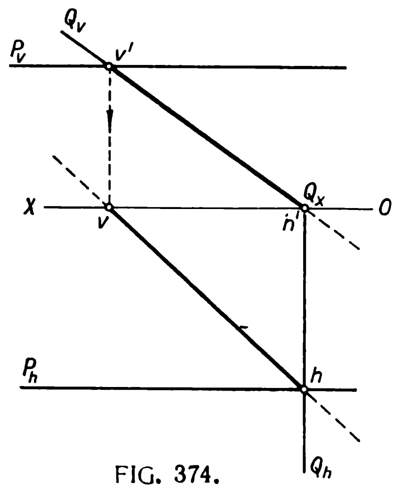



FIG. 374.

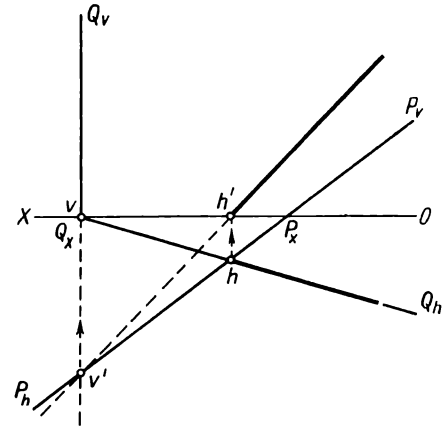



FIG. 375.

through the points $h$ and $v$ and the vertical one through the points $h'$ and $v'$; the latter coincides with the vertical trace ($Q_v$) of the plane $Q$ (why?). The line passes through the second, first, and fourth quadrants (why?).

**Example 96**

Find the line of inter: ection of the planes $P$ and $Q$ (Fig. 375).

*Solution.* The given planes intersect along an oblique line passing through the traces (points $h$, $h'$ and $v$, $v'$) found at the intersection of the horizontal and vertical traces of the planes. Draw the projections of the desired line: the horizontal one through the points $h$ and $v$ (it coincides with the horizontal trace $Q_h$ of the plane $Q$) (why?), and the vertical one through the points $h'$ and $v'$. The line passes through the first, fourth, and third quadrants (why?).

**Example 97**

Find the line of intersection of the planes $P$ and $Q$ (Fig. 376).

*Solution.* The given planes intersect along a profile principal line passing through the traces (points $h$, $h'$ and $v$, $v'$) found at the intersection of the horizontal and vertical traces of the planes and situated on a common perpendicular to the coordinate axis. Draw the projections of the desired line: the horizontal one through the points $h$ and $v$, and the vertical one through the points $h'$ and $v'$. Construct the profile projection ($h''v''$) as well.

What quadrants does the line pass through and what does the *coincidence* of its orthographic traces imply?

**Example 98**

Find the line of intersection of the planes $P$ and $Q$ (Fig. 377).

*Solution.* The given planes intersect along a horizontal principal line passing through the trace (point $v$, $v'$) found at the intersection of the vertical traces of the planes. Draw the projections of the desired line: the vertical one through the point $v'$ and parallel to the coordinate axis, which coincides with the vertical trace ($Q_v$) of the plane $Q$ (why?), and the horizontal one through the point $v$ and parallel to the horizontal trace ($P_h$) of the plane $P$. The line passes through the first and second quadrants (why?).

*Conclusion.* An oblique plane intersects a plane parallel to the plane $H$ *along a horizontal line.*

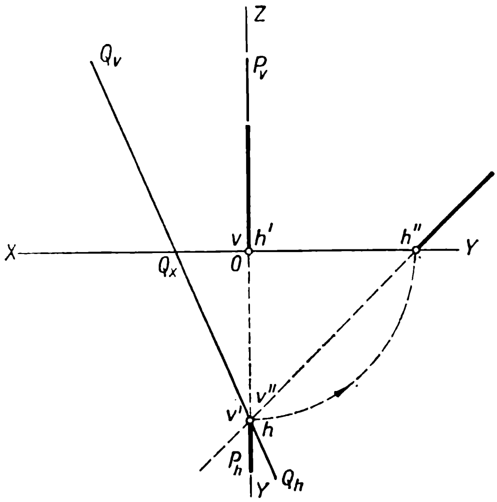


FIG. 376.

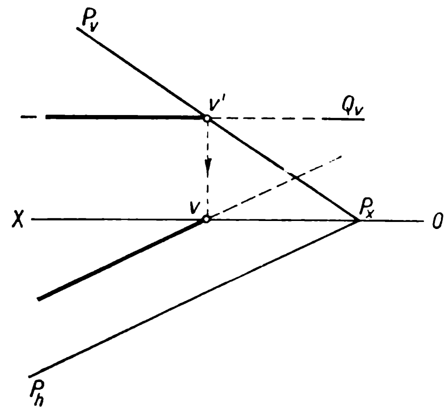


FIG. 377.

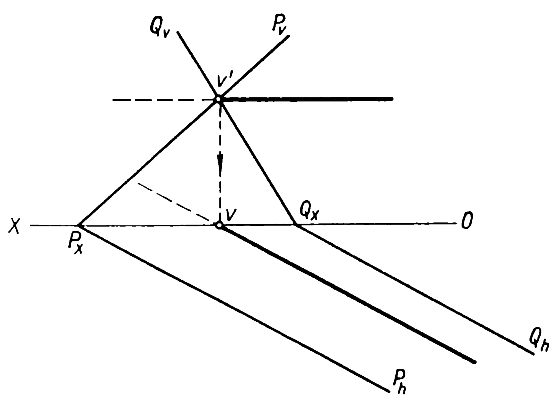


FIG. 378.

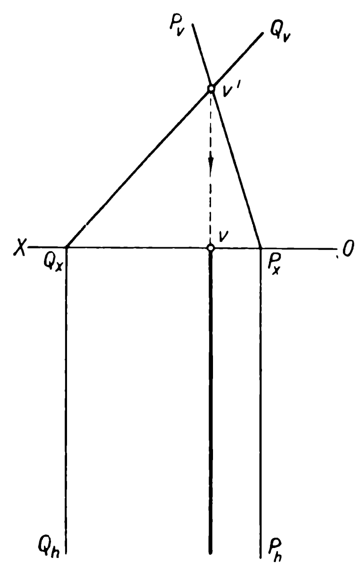


FIG. 379.

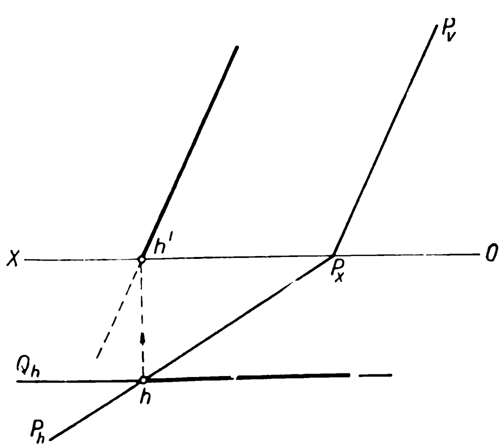


FIG. 380.

**Example 99**

Find the line of intersection of the planes $P$ and $Q$ (Fig. 378).

*Solution.* See the preceding example.

*Conclusion.* Two oblique lines, whose vertical traces intersect and whose horizontal traces are parallel to each other, intersect *along a horizontal principal line.*

**Example 100**

Find the line of intersection of the planes $P$ and $Q$ (Fig. 379).

*Solution.* The given planes intersect along a horizontal principal line, which is perpendicular to the vertical plane of projection (why?).

The vertical projection of the desired line coincides with the point $v'$ (why?), the horizontal projection passing through the point $v$ perpendicular to the coordinate axis. The line passes through the first and second quadrants.

*Conclusion.* Two vertical projecting planes intersect along a line perpendicular to the vertical plane of projection.

**Example 101**

Find the line of intersection of the planes $P$ and $Q$ (Fig. 380).

*Solution.* The given planes intersect along a frontal principal line passing through the trace (point $h$, $h'$) found at the intersection of the horizontal traces of the planes. Draw the projections of the desired line: the horizontal one through the point $h$ and parallel to the coordinate axis (it coincides with the horizontal trace $Q_h$ of the plane $Q$) (why?), and the vertical one through the point $h'$ and parallel to the vertical trace ($P_v$) of the plane $P$. The line passes through the first and fourth quadrants (why?).

*Conclusion.* An oblique plane intersects a plane parallel to the plane $V$ *along a vertical principal line.*

**Example 102**

Find the line of intersection of the planes $P$ and $Q$ (Fig. 381).

*Solution.* See the preceding example.

*Conclusion.* Two oblique planes, whose horizontal traces intersect and vertical traces are parallel to each other, intersect along a frontal principal line.

**Example 103**

Find the line of intersection of the planes $P$ and $Q$ (Fig. 382).

*Solution.* The horizontal projecting planes $P$ and $Q$ intersect along a frontal principal line which is perpendicular to the plane $H$. The horizontal projection of the desired line coincides with the point $h$ (why?), the vertical projection passing through the point $h'$ perpendicular to the coordinate axis. The line passes through the first and fourth quadrants (why?).

*Conclusion.* Two horizontal projecting planes intersect along a line perpendicular to the plane $H$.

**Example 104**

Find the line of intersection of the planes $P$ and $Q$ (Figs. 383, 384).

*Solution.* The given planes intersect along a line $MN$ which is parallel to the coordinate axis (why?).

*First method.* The profile projection ($m''n''$) of the desired line coincides with the profile trace of the line, which, as is known, is situated at the intersection of the profile traces ($P_w$ and $Q_w$) of the planes. Find the profile traces of the given planes. They intersect along the profile projection ($m''n''$) of the desired line. Knowing the profile projection of the line, find its horizontal ($mn$) and vertical ($m'n'$) projections, which must be parallel to the coordinate axis (why?).

*Second method.* Since the desired line $MN$ lies in the plane $Q$, its vertical projection ($m'n'$) coincides with the vertical trace ($Q_v$) of the plane (why?). Having the vertical projection ($m'n'$) of the line, we can find its horizontal projection ($mn$) without using the profile plane of projection. For this purpose assume an arbitrary

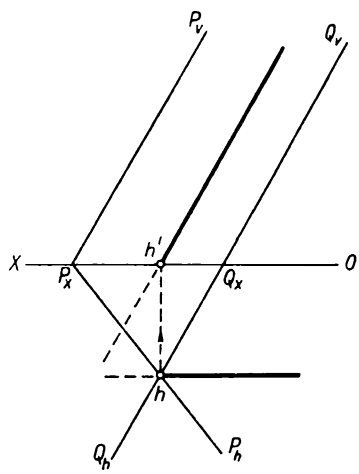


FIG. 381.

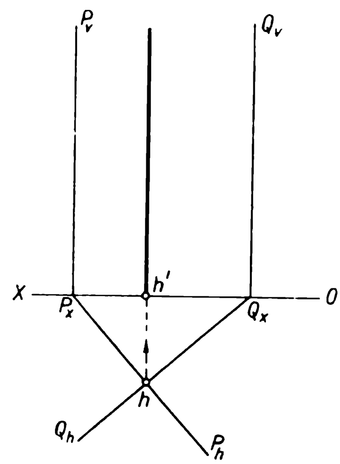


FIG. 382.

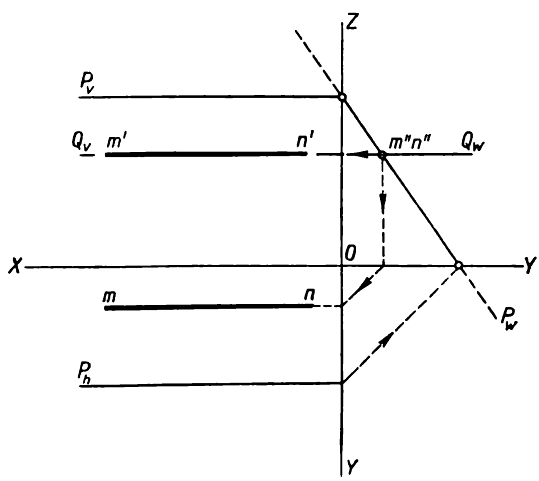


FIG. 383.

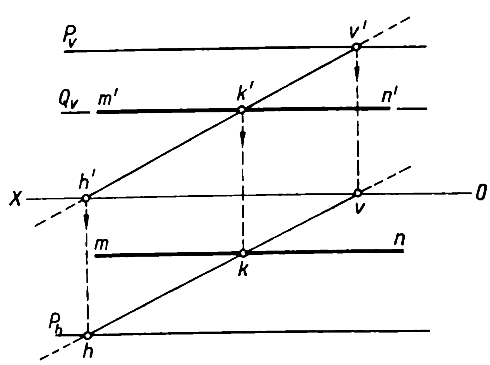


FIG. 384.

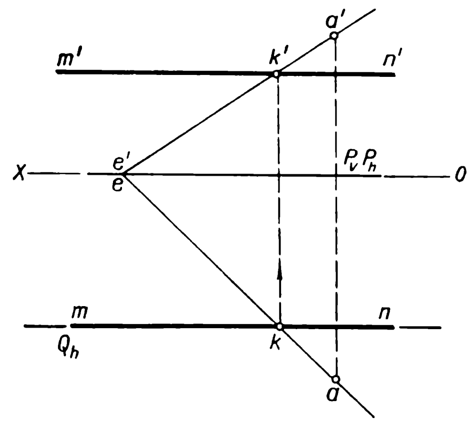



FIG. 385.

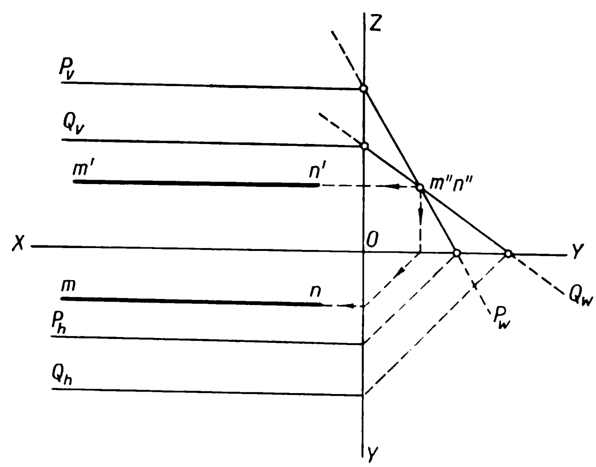



FIG. 386.

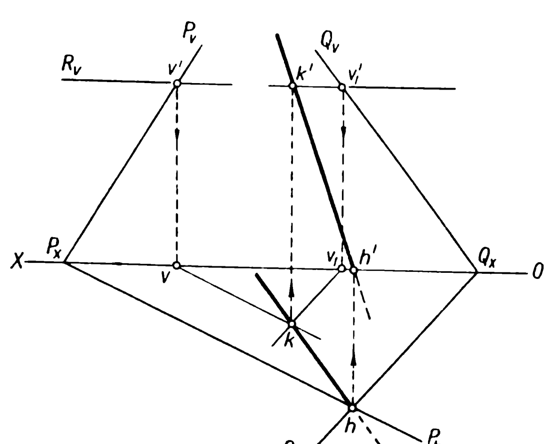



FIG. 387.

point $k'$ on $m'n'$ and find the horizontal projection ($k$) of the point, knowing that the point ($k$, $k'$) lies in the plane $P$ as well. Draw the horizontal projection ($mn$) of the desired line through the point $k$ and parallel to the coordinate axis.

**Example 105**

Find the line of intersection of the planes $P$ and $Q$ (Fig. 385).

*Solution. First method*—see the preceding example.

*Second method.* Since the desired line lies in the plane $Q$, the horizontal projection ($mn$) of the line coincides with the horizontal trace ($Q_h$) of the plane (why?). Having the horizontal projection ($mn$) of the line, we can find its vertical projection ($m'n'$) without using the profile plane of projection. For this purpose assume a point $k$ on $mn$ and find the vertical projection ($k'$) of the point, knowing that the point ($k$, $k'$) lies in the plane $P$ as well. Draw the vertical projection ($m'n'$) of the desired line through the point $k'$ and parallel to the coordinate axis.

**Example 106**

Find the line of intersection of the planes $P$ and $Q$ (Fig. 386).

*Solution.* The given planes intersect along the line $MN$ which is parallel to the coordinate axis (why?). Find the profile projection ($m''n''$) of the line at the intersection of the profile traces of the planes, and then, from the profile projection of the line, determine its horizontal ($mn$) and vertical ($m'n'$) projections, which are parallel to the coordinate axis.

**Example 107**

Find the line of intersection of the planes $P$ and $Q$ (Fig. 387).

*Solution.* The given planes intersect along an oblique line passing through the point (trace $h$, $h'$) of intersection of the horizontal traces of the planes. The point (trace $v$, $v'$) of intersection of the vertical traces of the planes is *out of reach*, as these traces of the planes do not intersect within the limits of the drawing.

Instead of the point ($v$, $v'$), it is necessary to find another, arbitrary point belonging to the line of intersection and thus common to both given planes. To this end introduce an auxiliary plane $R$, say, parallel to the plane $H$, which, as is known, intersects each of the given planes along a horizontal principal line. Their intersection yields an auxiliary point ($k$, $k'$), which is common to the given planes. Now draw the projections of the desired line: the horizontal one through the points $h$ and $k$, and the vertical one through the points $h'$ and $k'$.

Through what quadrants does the line of intersection pass?

*Note.* If necessary, we can, using the above method, find *two* random points belonging to the line of intersection by *consecutively* introducing two planes. The *simplest* way is to pass them parallel to the plane $H$ or $V$.

**Example 108**

Find the line of intersection of the planes $P$ and $Q$ (Fig. 388).

*Solution.* The given planes intersect along a line intersecting the coordinate axis at the point ($m$, $m'$), which is found at the intersection of the horizontal and vertical traces of the planes. It is well known that the points (traces) ($h$, $h'$) and ($v$, $v'$), as coincident ones, do not specify a straight line and it is necessary to find one more point lying on this line and common to the given planes. The point ($a$, $a'$), by hypothesis, lies in the plane $P$. Let us first check whether it lies in the plane $Q$ too. This can be done, say, with the aid of a horizontal principal line. The point ($a$, $a'$), indeed, lies in the plane $Q$ as well, i.e. it belongs to both given planes. Draw the projections of the desired line: the vertical one through the points $m'$ and $a'$, and the horizontal one through $m$ and $a$. The line passes through the first and third quadrants (why?).

**Example 109**

Find the line of intersection of the planes $P$ and $Q$ (Fig. 389).

*Solution.* The given planes intersect along a line passing through the point ($l$, $l'$), which coincides with the points ($h$, $h'$) and ($v$, $v'$) of intersection of the like

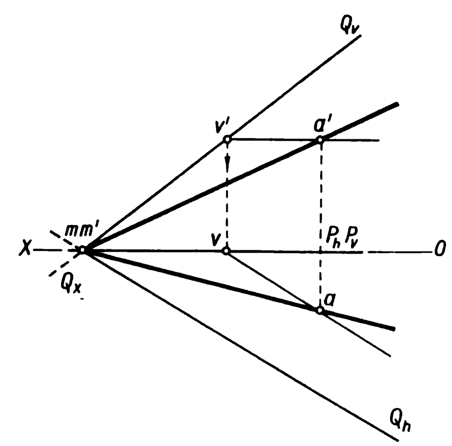



FIG. 388.

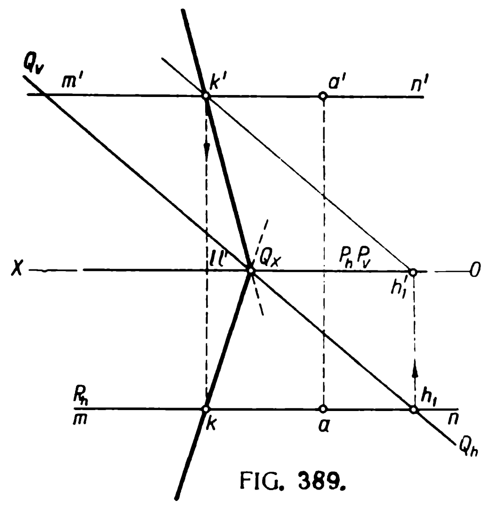



FIG. 389.

traces of the planes. By hypothesis, the point $(a, a')$ lies in the plane $P$; it *does not lie* in the plane $Q$, which fact may be verified by drawing a frontal principal line. Hence, to determine the straight line it is necessary to find one more point belonging to it. Let us introduce an auxiliary plane $R$ parallel to the vertical plane of projection and passing through the point $(a, a')$. The horizontal trace $(R_h)$ passes through the point $a$ and parallel to the coordinate axis. The plane $R$ intersects the plane $P$ along a straight line $(mn, m'n')$ parallel to the coordinate axis and passing through the point $(a, a')$, and the plane $Q$ along a frontal principal line passing through the point $(h_1, h_1')$. The intersection of $MN$ and the frontal line yields the point $(k, k')$ through which the line of intersection must also pass. Draw the projections of the desired line: the horizontal one through the points $l$ and $k$, and the vertical one through the points $l'$ and $k'$.

**Example 110**

Find the line of intersection of the planes $P$ and $Q$ without using the profile plane of projection (Fig. 390).

*Solution.* The given planes intersect along a line $MN$, which is parallel to the coordinate axis (why?).

Knowing the direction of the desired line, it is necessary to find one more point belonging to it, for which purpose we introduce an auxiliary vertical projecting plane $R$ passing through the point $(a, a')$. The auxiliary plane $R$ intersects the plane $P$ along the line $(al, a'l')$, and the plane $Q$ along the vertical projecting line passing through the point $(v, v')$. Their intersection yields the point $(k, k')$. Then draw the projections $(mn, m'n')$ of the desired line through the like projections of the point $(k, k')$ and parallel to the coordinate axis. We see that the vertical projection $(m'n')$ of the desired line coincides with the vertical trace $(Q_v)$ of the plane $Q$ (why?).

**Example 111**

Find the line of intersection of the planes $P$ and $Q$ without using the profile plane of projection (Fig. 391).

*Solution.* The given planes intersect along the line $MN$, which is parallel to the coordinate axis (why?).

Knowing the direction of the desired line, it is necessary to find one more point belonging to it, for which purpose we introduce an arbitrary horizontal projecting plane $R$. The latter intersects the plane $Q$ along the line $(hv, h'v')$, and the plane $P$

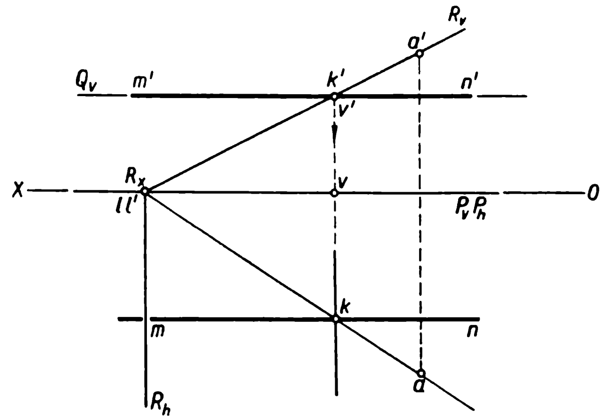



FIG. 390.

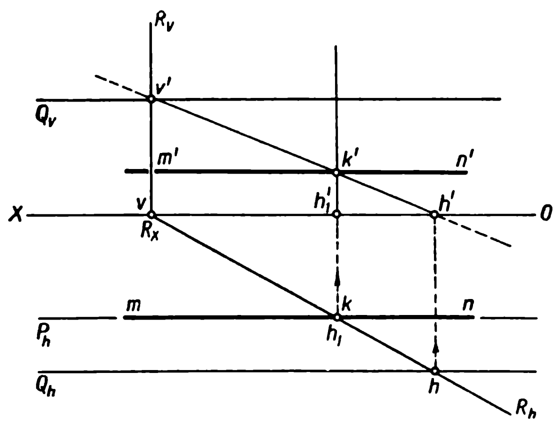



FIG. 391.

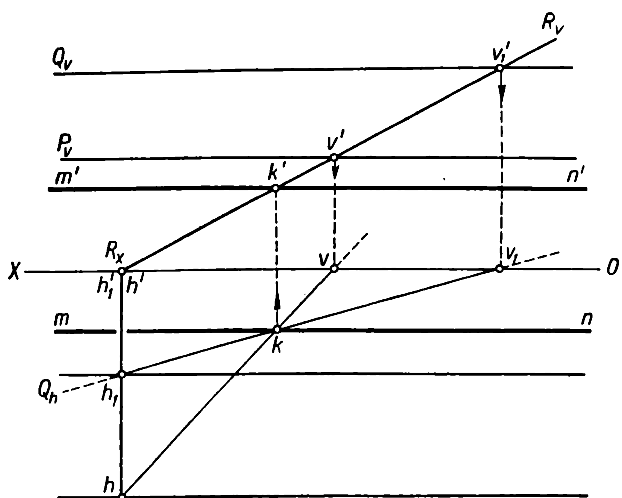



FIG. 392.

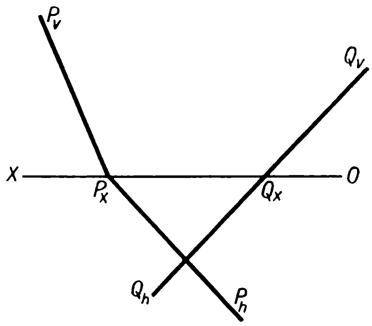



FIG. 393.

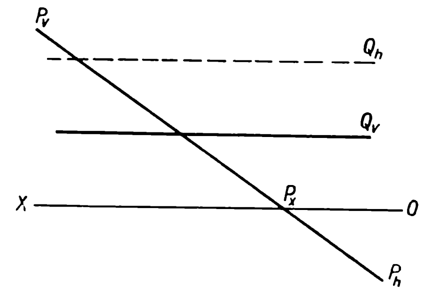



FIG. 394.

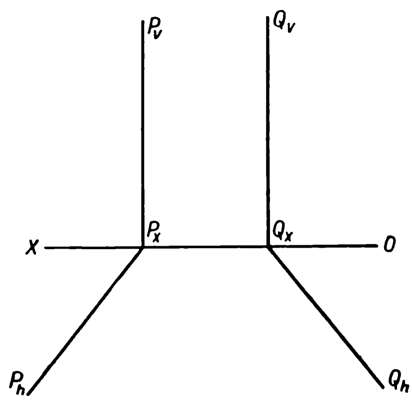



FIG. 395.

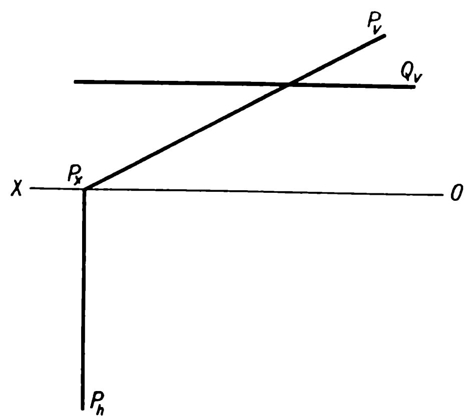



FIG. 396.

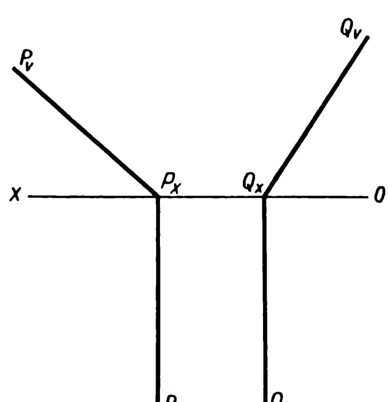



FIG. 397.

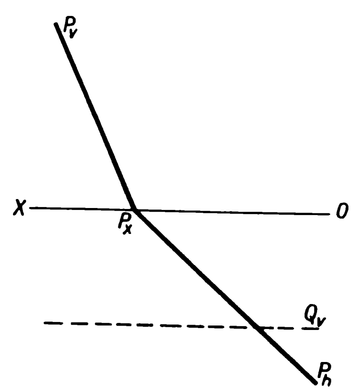



FIG. 398.

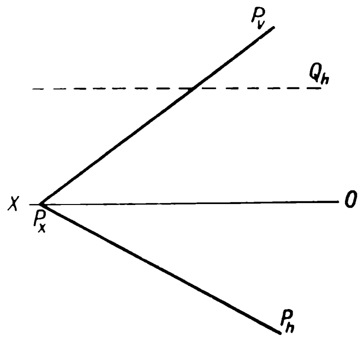


FIG. 399.

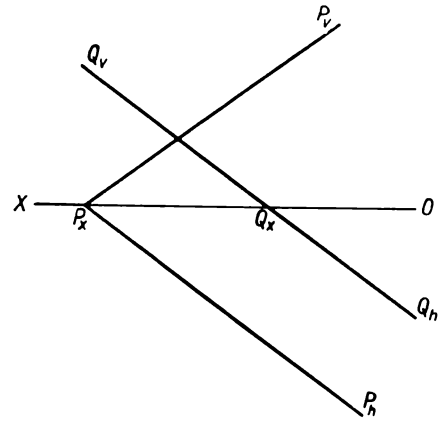


FIG. 400.

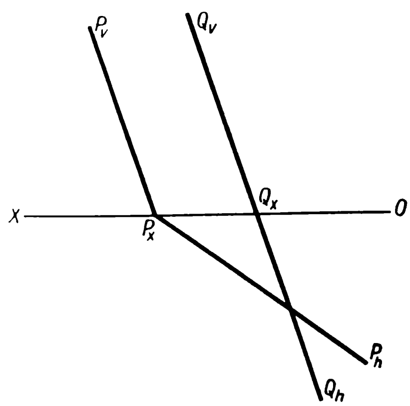


FIG. 401.

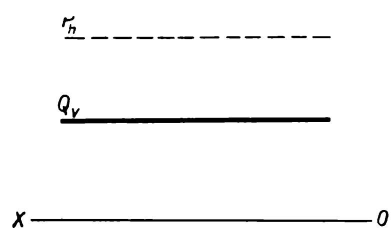


FIG. 402.

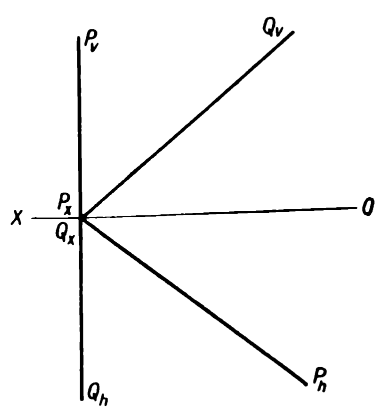


FIG. 403.

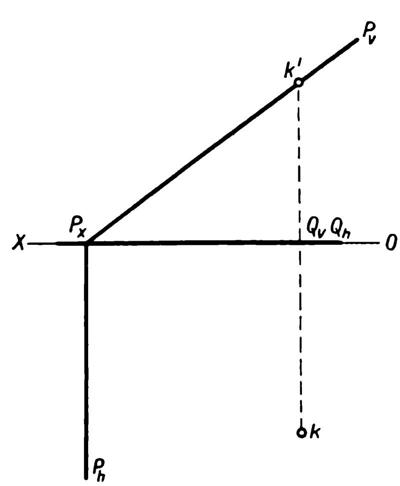


FIG. 404.

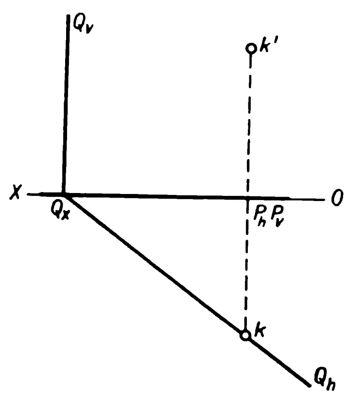



FIG. 405.

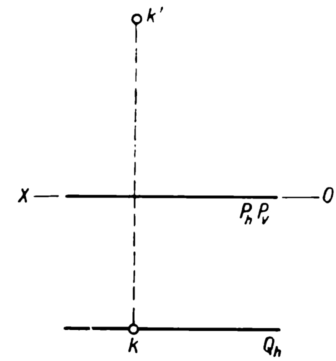



FIG. 406.

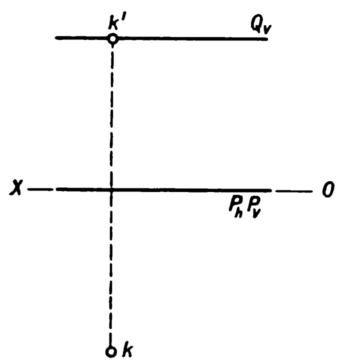



FIG. 407.

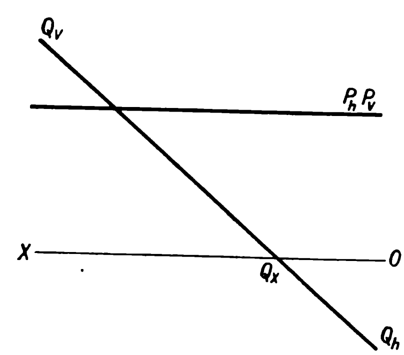



FIG. 408.

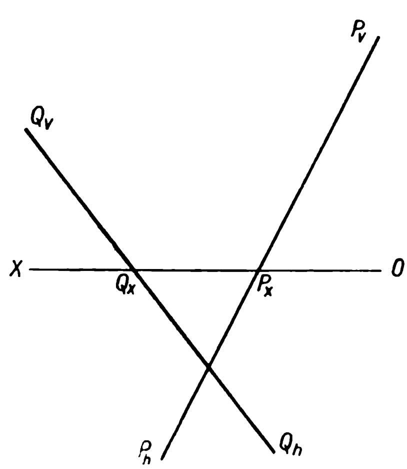



FIG. 409.

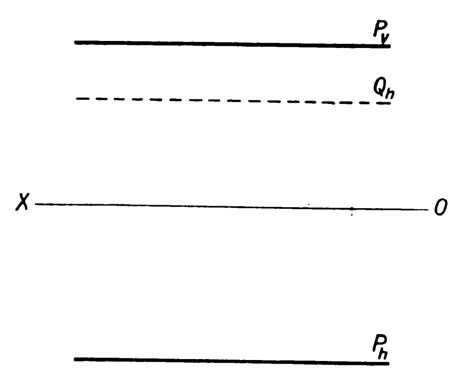



FIG. 410.

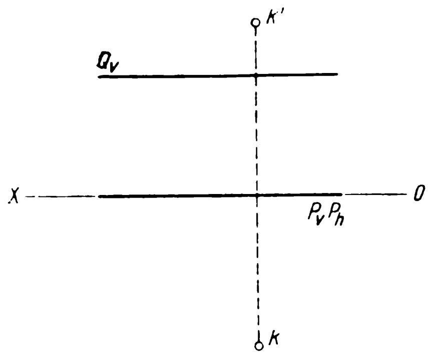



FIG. 411.

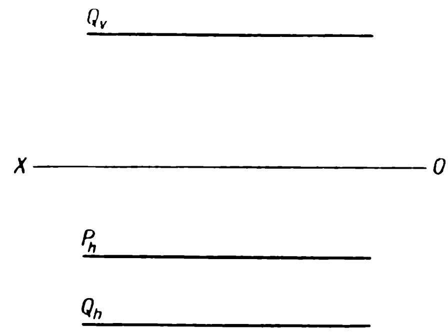



FIG. 412.

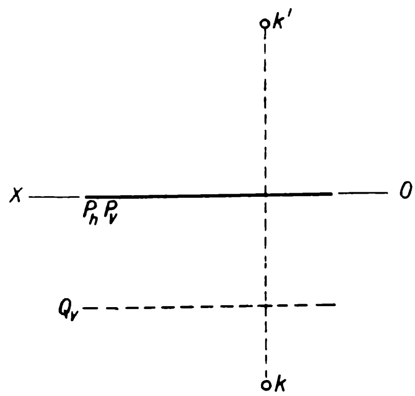



FIG. 413.

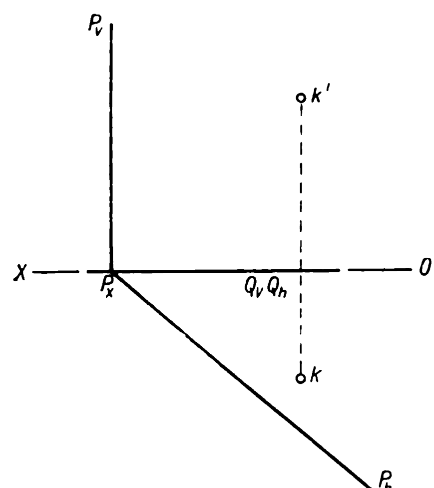



FIG. 414.

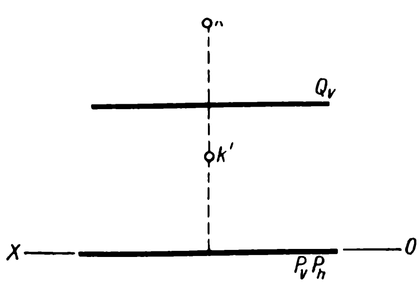



FIG. 415.

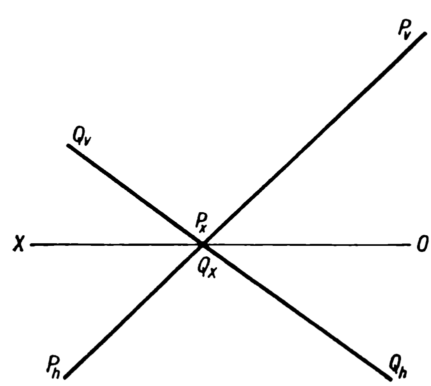



FIG. 416.

along a horizontal projecting line passing through the point ($h_1$, $h'_1$). The point of their intersection will be denoted as ($k$, $k'$). Draw the projections ($mn$, $m'n'$) of the desired line through the like projections of the point ($k$, $k'$) and parallel to the coordinate axis.

**Example 112**

Find the line of intersection of the planes $P$ and $Q$ without using the profile plane of projection (Fig. 392).

*Solution.* The given planes intersect along the line $MN$, which is parallel to the coordinate axis (why?). Knowing the direction of the desired line, it is necessary to find one more point belonging to it, for which purpose we introduce an auxiliary vertical projecting plane $R$, which intersects the plane $P$ along the line ($hv$, $h'v'$), and the plane $Q$ along the line ($h_1v_1$, $h'_1v'_1$). The point of their intersection will be denoted as ($k$, $k'$). Draw the projections ($mn$, $m'n'$) of the desired line through the like projections of the point ($k$, $k'$) and parallel to the coordinate axis.

## PROBLEMS

217. Find the line of intersection of the planes $P$ and $Q$:

(1) without introducing an auxiliary plane (Figs. 393 to 413);

(2) without introducing an auxiliary plane or using the profile plane of projection (Figs. 410 to 413);

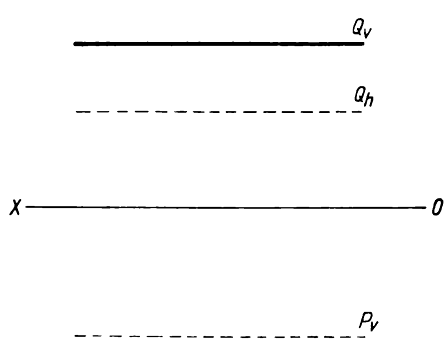



FIG. 417.

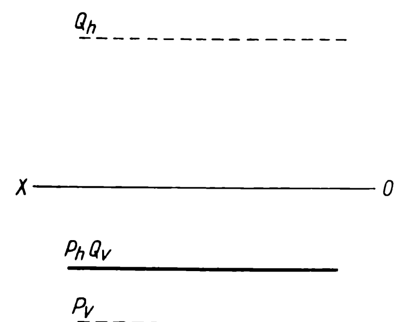



FIG. 418.

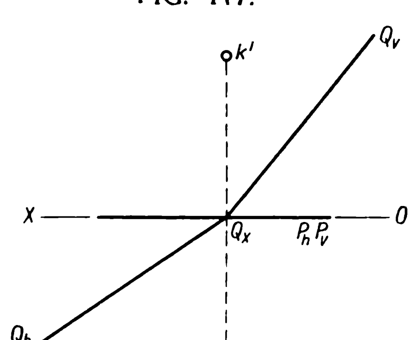



FIG. 419.

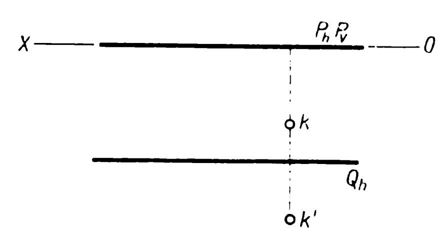



FIG. 420.

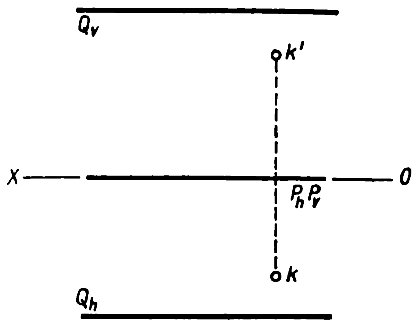



FIG. 421.

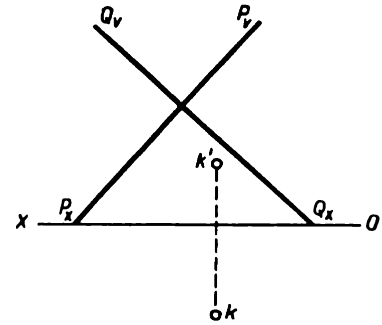



FIG. 422.

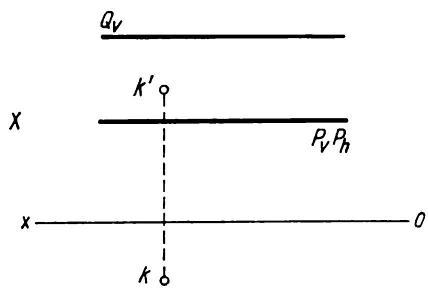



FIG. 423.

(3) by introducing an auxiliary plane (Figs. 414 to 421).

218. Find the horizontal traces of the intersecting planes $P$ and $Q$, given their vertical traces and a point $K$, which belongs to the line of their intersection (Figs. 422, 423).

CHAPTER XIV

# A LINE CUTTING A PLANE

To determine the point of intersection of a straight line and a plane proceed as follows:

(1) pass an arbitrary auxiliary plane through the given line (the *simplest* way is to pass a projecting plane);

(2) find the line of intersection of the given and auxiliary planes;

(3) the intersection of the two lines—the given and obtained ones—yields the desired point.

*Note*: If one of the intersecting elements—a plane or a line—is a projecting one, the above rule should not be followed, since in most cases the point of intersection can then be found in a simpler way (see examples).

EXAMPLES

**Example 113**

Find the point of intersection of the line $AB$ and the plane $P$ (Fig. 424).

*Solution.* We designate the desired point as $M$ ($m$, $m'$). Since the point $M$ lies in a horizontal projecting plane, its horizontal projection ($m$) must lie somewhere on the horizontal trace ($P_h$) of the plane. On the other hand, since the point $M$ lies also on the line-segment $AB$, the horizontal projection ($m$) must lie also on the

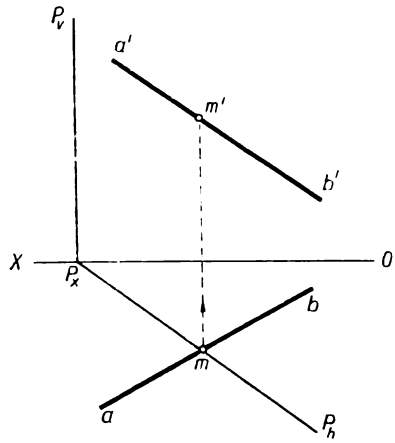



FIG. 424.

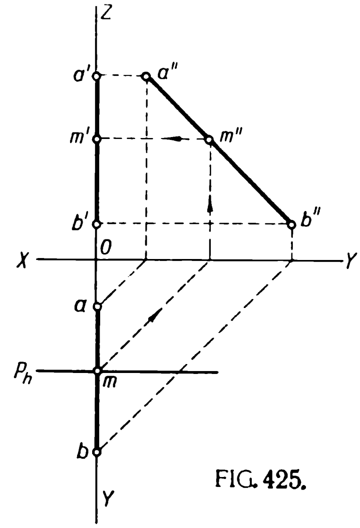



FIG. 425.

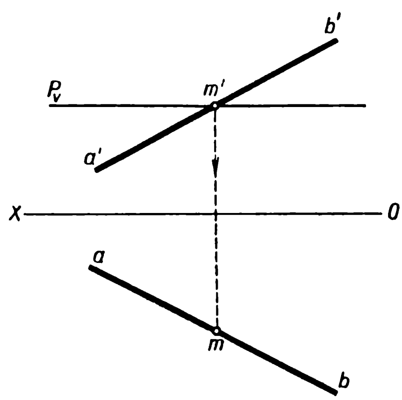



FIG. 426.

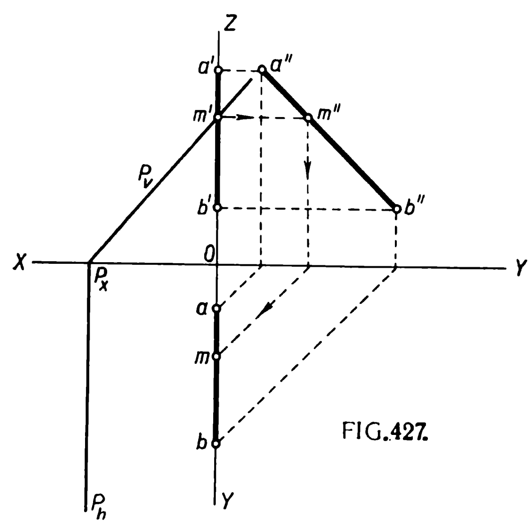



FIG. 427.

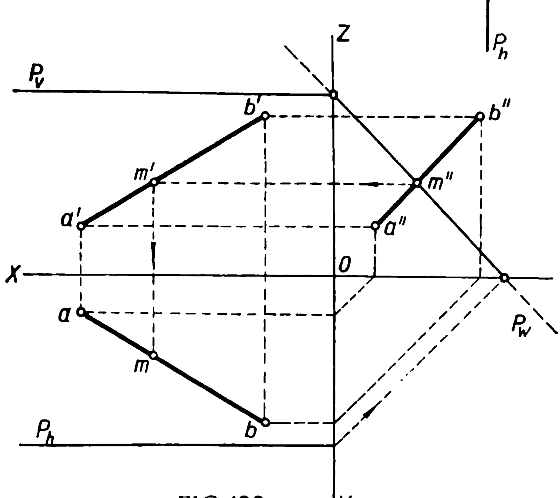



FIG. 428.

horizontal projection ($ab$) of the line. Hence, the horizontal projection ($m$) of the desired point must lie on the horizontal trace ($P_h$) of the plane and, at the same time, on the horizontal projection ($ab$) of the given line, that is at their intersection. Having the horizontal projection ($m$) of the required point, find its vertical projection ($m'$) on the vertical projection ($a'b'$) of the given line.

*Conclusion.* The horizontal projection of the point of intersection of *any* straight line and a horizontal projecting plane is found at the intersection of the horizontal trace of the plane with the horizontal projection of the line. (Conventional designation: $m \rightarrow P_h \times ab$.)

**Example 114**

Find the point of intersection of the line $AB$ and the plane $P$ (Fig. 425).

*Solution.* We designate the desired point as $M$ ($m$, $m'$). The given plane, which is parallel to the vertical plane of projection, is a horizontal projecting one. Hence, the horizontal projection ($m$) of the desired point is found at the intersection of $P_h$ and $ab$. Since the given line ($ab$, $a'b'$) is a profile one, use the profile plane of projection to find the point $m''$ on the line-segment $a''b''$, given the point $m$, and then the point $m'$ on the line-segment $a'b'$ from the point $m''$.

**Example 115**

Find the point of intersection of the line $AB$ and the plane $P$ (Fig. 426).

*Solution.* Designate the required point as $M$ ($m$, $m'$). Since the point $M$ lies in a vertical projecting plane, its vertical projection ($m'$) must lie *somewhere* on the vertical trace ($P_v$) of the given plane. On the other hand, since the point $M$ lies also on the line-segment $AB$, its vertical projection ($m'$) must lie *somewhere* also on the vertical projection ($a'b'$) of the given line. Hence, the vertical projection ($m'$) of the desired point must lie on the vertical trace ($P_v$) of the plane and, at the same time, on the vertical projection ($a'b'$) of the given line, that is at their intersection. Having the vertical projection ($m'$) of the required point, find its horizontal projection ($m$) on the horizontal projection ($ab$) of the given line.

*Conclusion.* The vertical projection of the point of intersection of any straight line and a vertical projecting plane is found at the intersection of the vertical trace of the plane with the vertical projection of the plane. (Conventional designation: $m' \rightarrow P_v \times a'b'$.)

**Example 116**

Find the point of intersection of the line $AB$ and the plane $P$ (Fig. 427).

*Solution.* Designate the desired point as $M$ ($m$, $m'$). The given plane $P$ is a vertical projecting one. Hence, the vertical projection ($m'$) of the desired point is found at the intersection of $P_v$ and $a'b'$. Since the given line is a profile one, make use of the profile plane of projection to find the point $m''$ on the line-segment $a''b''$, given the point $m'$, and then the point $m$ on the line-segment $ab$ from the point $m''$.

**Example 117**

Find the point of intersection of the line $AB$ and the plane $P$ (Fig. 428).

*Solution.* Designate the desired point as $M$ ($m$, $m'$). Since the point $M$ lies on a profile projecting plane its profile projection ($m''$) must lie *somewhere* on the profile trace ($P_w$) of the given plane. On the other hand, since the point $M$ lies also on the line-segment $AB$, its profile projection ($m''$) must also lie *somewhere* on the profile projection ($a''b''$) of the given line. Hence, the profile projection ($m''$) of the desired point must lie both on the profile trace ($P_w$) of the plane and the profile projection ($a''b''$) of the line, that is at their intersection. Find the profile trace of the plane and the profile projection of the line to obtain the profile projection ($m''$) of the desired point at their intersection. Having the profile projection ($m''$) of the required point, find its other two projections on the like projections of the line.

*Conclusion.* The profile projection of the point of intersection of any straight line and a profile projecting plane is found at the intersection of the profile trace

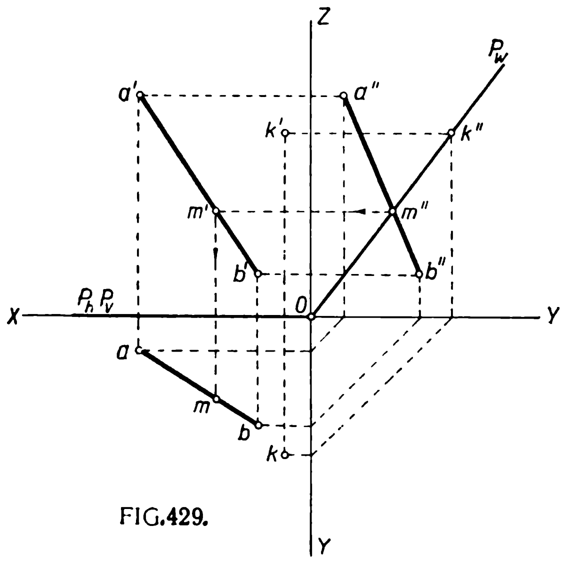



FIG. 429.

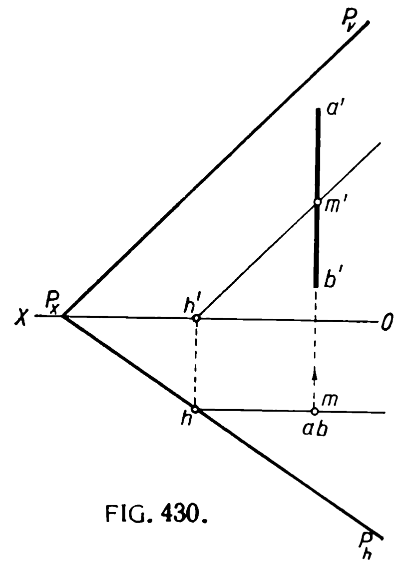



FIG. 430.

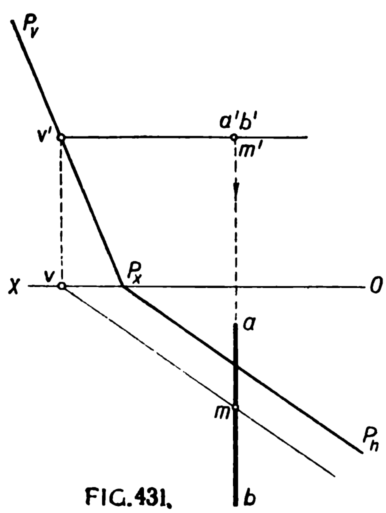



FIG. 431.

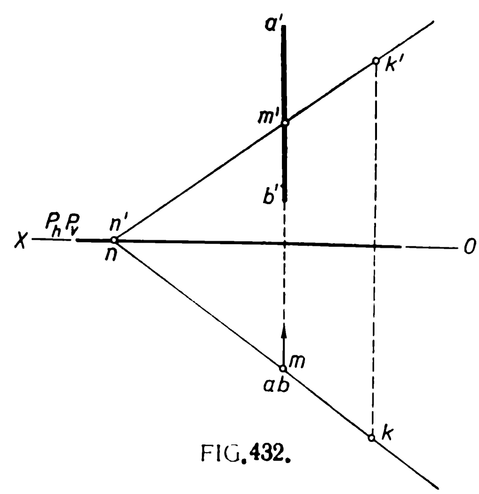



FIG. 432.

of the plane and the profile projection of the line. (Conventional designation: $m'' \rightarrow$ $\rightarrow P_w \times a''b''$.)

*Note.* This problem can be solved without using the profile plane of projection.

**Example 118**

Find the point of intersection of the line $AB$ and the plane $P$ (Fig. 429).

*Solution.* We designate the desired point as $M$ ($m$, $m'$). The given plane, which passes through the coordinate axis, is a profile projecting one. Hence, the profile projection ($m''$) of the required point is found at the intersection of the profile trace ($P_w$) of the given plane and the profile projection ($a''b''$) of the given line.

Knowing the profile projection ($m''$) of the desired point, find its other two projections on the like projections of the given line.

*Note.* This problem can be solved without using the profile plane of projection, but the above method is much simpler.

**Example 119**

Find the point of intersection of the line $AB$ and the plane $P$ (Fig. 430).

*Solution.* Designate the desired point as $M$ ($m$, $m'$). Since this point lies on a horizontal projecting line ($ab$, $a'b'$), its horizontal projection ($m$) must coincide with the horizontal projection ($ab$) of the line. Knowing the horizontal projection ($m$) of the point, find its vertical projection ($m'$), given that the point ($m$, $m'$) lies also in the given plane. To do so, use a frontal (or horizontal) principal line. Subsequent construction is obvious from the drawing.

**Example 120**

Find the point of intersection of the line $AB$ and the plane $P$ (Fig. 431).

*Solution.* Designate the desired point as $M$ ($m$, $m'$). Since this point lies on a vertical projecting line, its vertical projection ($m'$) coincides with the vertical projection ($a'b'$) of the given line. Knowing the vertical projection ($m'$) of the point, find its horizontal projection ($m$), given that the point ($m$, $m'$) lies in the given plane. To do so, use a horizontal frontal principal line. Subsequent construction is obvious from the drawing.

**Example 121**

Find the point of intersection of the line $AB$ and the plane $P$ (Fig. 432).

*Solution.* Designate the desired point as $M$ ($m$, $m'$). Since this point lies on a horizontal projecting line, its horizontal projection ($m$) coincides with the horizontal projection ($ab$) of the given line. Knowing the horizontal projection ($m$) of the point, find its vertical projection ($m'$), given that this point lies also in the given plane. To do so, use an auxiliary line ($nk$, $n'k'$). Subsequent construction is clear from the drawing.

**Example 122**

Find the point of intersection of the line $AB$ and the plane $P$ (Fig. 433).

*Solution.* Designate the desired point as $M$ ($m$, $m'$). Since this point lies on a vertical projecting line, its vertical projection ($m'$) coincides with the vertical projection ($a'b'$) of the given line. Knowing the vertical projection ($m'$) of the point, find its horizontal projection ($m$), given that the point lies also in the given plane. For this purpose use an auxiliary line ($hv$, $h'v'$). Though the plane $P$ is a profile projecting one, the problem is solved without using the profile plane of projection. Here we use not the property of points lying in a plane, but that of points of a projecting line.

**Example 123**

Find the point of intersection of the line $AB$ and the plane $P$ (Fig. 434).

*Solution.* Designate the desired point as $M$ ($m$, $m'$). Put the line $AB$ into a horizontal (or vertical) projecting plane $R$, which intersects the given plane along the line-segment ($hv$, $h'v'$). The intersection of the vertical projections ($h'v'$ and $a'b'$) of the lines will yield the vertical projection ($m'$) of the required point. Then find its horizontal projection ($m$) on the horizontal projection ($ab$) of the given line from the vertical projection ($m'$) of the point.

**Example 124**

Find the point of intersection of the line $AB$ and the plane $P$ (Fig. 435).

*Solution.* Designate the desired point as $M$ ($m$, $m'$). Put the given line $AB$ into a plane $R$, which is parallel to the plane $H$ and intersects the given plane $P$ along a horizontal principal line. The point of intersection of the horizontal projections of the horizontal line and the given line yields the horizontal projection ($m$) of the desired point. Now, from $m$ find $m'$ on $a'b'$.

The line-segment $AB$ can be put into a horizontal projecting plane, but it would complicate the solution.

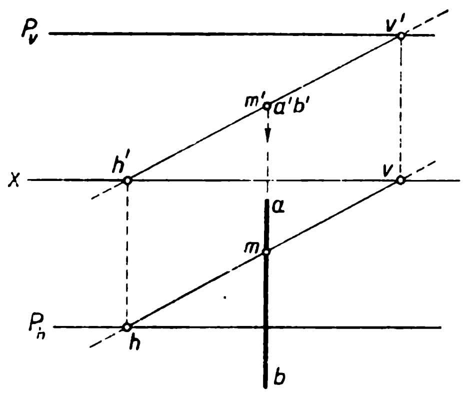



FIG. 433.

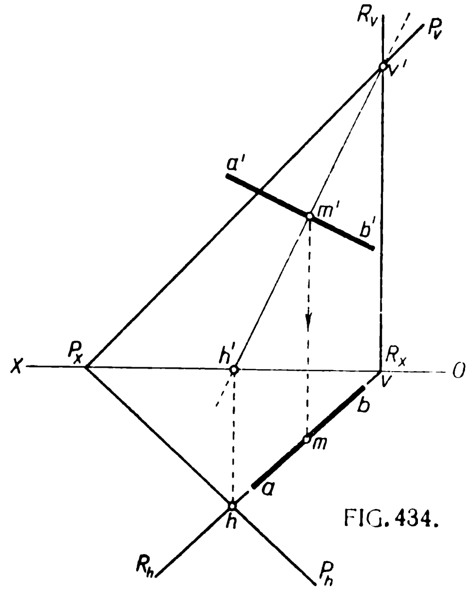



FIG. 434.

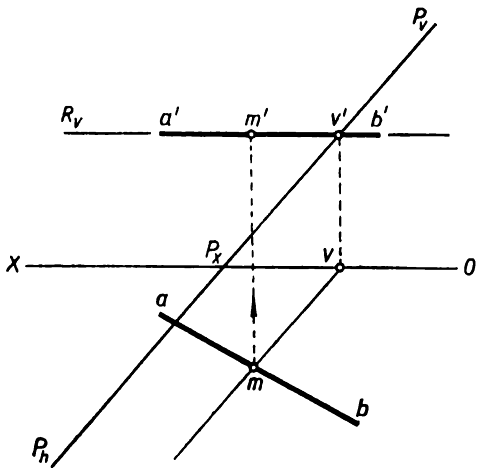



FIG. 435.

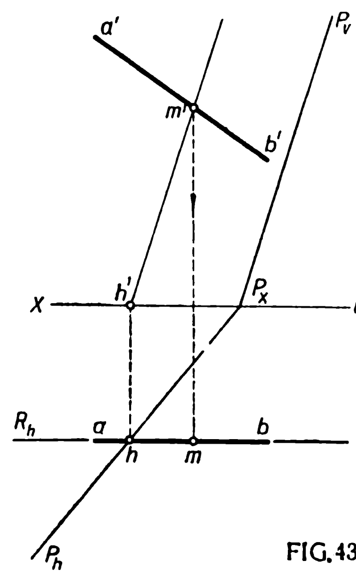



FIG. 436.

**Example 125**

Find the point of intersection of the line $AB$ and the plane $P$ (Fig. 436).

*Solution*. Designate the desired point as $M$ ($m$, $m'$). Put the given line $AB$ into a plane $R$, which is parallel to the plane $V$ and intersects the given plane $P$ along a frontal principal line. The point of intersection of the vertical projection of the vertical line and the given line yields the vertical projection ($m'$) of the required point. Now, from $m'$ find $m$ on $ab$.

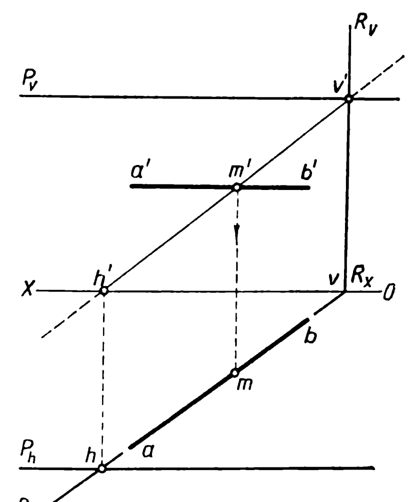



FIG. 437.

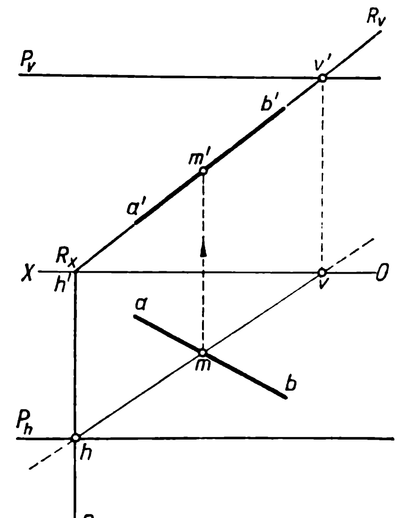



FIG. 438.

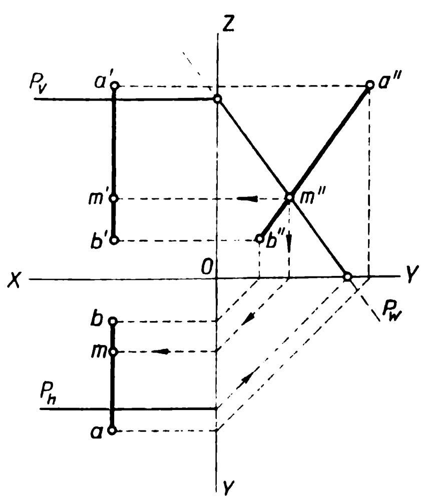



FIG. 439.

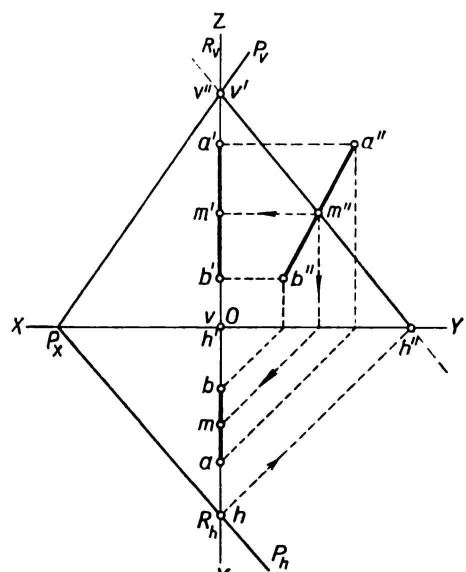



FIG. 440.

**Example 126**

Find the point of intersection of the line $AB$ and the plane $P$ (Fig. 437).

*Solution.* Designate the desired point as $M$ ($m$, $m'$). Put the given line $AB$ into a horizontal projecting plane $R$, which intersects the given plane $P$ along the line ($hv$, $h'v'$).

The point of intersection of the vertical projections of the given and auxiliary lines yields the vertical projection ($m'$) of the desired point. Then find its horizontal projection ($m$) on the horizontal projection ($ab$) of the given line.

**Example 127**

Find the point of intersection of the line $AB$ and the plane $P$ (Fig. 438).

*Solution.* Designate the desired point as $M$ ($m$, $m'$). Put the given line $AB$ into a vertical projecting plane $R$, which intersects the given plane along the line

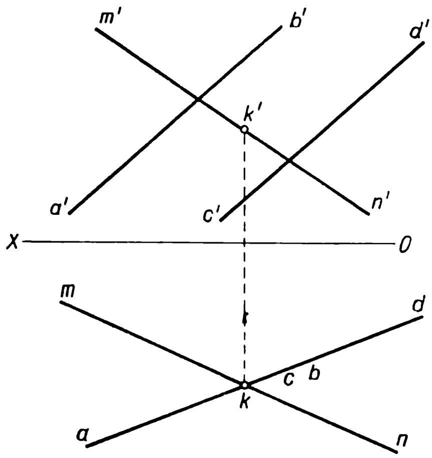



FIG. 441.

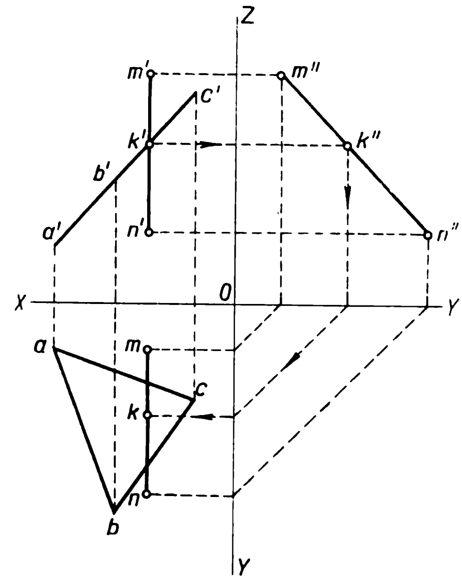



FIG. 442.

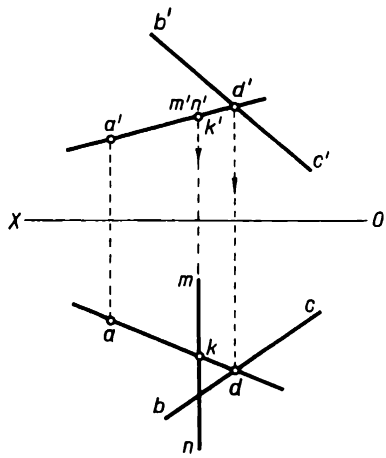



FIG. 443.

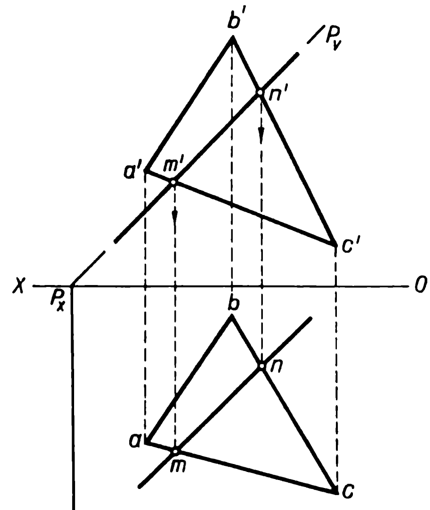



FIG. 444.

($hv$, $h'v'$). The point of intersection of the horizontal projections of the given and auxiliary lines yields the horizontal projection ($m$) of the required point. Then find its vertical projection ($m'$) on the vertical projection ($a'b'$) of the given line.

**Example 128**

Find the point of intersection of the line $AB$ and the plane $P$ (Fig. 439).

*Solution.* Designate the desired point as $M$ ($m$, $m'$). Since the given line is a profile one, the problem cannot be solved without making use of the profile plane of projection; hence, we proceed as in Example 117: construct the profile trace ($P_w$) of the given plane and the profile projection ($a''b''$) of the line. Their intersection yields the profile projection ($m''$) of the required point. Having $m''$, find the other two projections ($m$ and $m'$) on the like projections ($ab$, $a'b'$) of the given line.

**Example 129**

Find the point of intersection of the line $AB$ and the plane $P$ (Fig. 440).

*Solution.* Designate the desired point as $M$ ($m$, $m'$). Pass, through $AB$, a profile plane $R$, which will intersect the given plane $P$ along a profile line ($hv$, $h'v'$). Since both lines (the given and auxiliary ones) are profile lines, their intersection will yield the profile projection ($m''$) of the required point. Then find $m$ and $m'$ on the like projections ($ab$ and $a'b'$) of the given line.

**Example 130**

Find the point of intersection of the line $MN$ with the plane specified by the parallel lines $AB$ and $CD$ (Fig. 441).

*Solution.* Designate the desired point as $K$ ($k$, $k'$). Since the given plane is a horizontal projecting one (why?), find the horizontal projecton ($k$) of the point at the intersection of $mn$ and $ab$ (why?), or of $mn$ and $cd$, which is the same thing. Then find $k'$ on $m'n'$.

**Example 131**

Find the point of intersection of the line $MN$ and the plane of the triangle $ABC$ (Fig. 442).

*Solution.* Designate the desired point as $K$ ($k$, $k'$). Since the given plane is a vertical projecting one (why?), find the vertical projection ($k'$) of the point $K$ at the intersection of the line-segment $m'n'$ and the vertical projection ($a'b'c'$) of the given triangle (why?). Knowing $k'$, define the horizontal projection ($k$) on the horizontal projection ($mn$) of the line (how?).

**Example 132**

Find the point of intersection of the line $MN$ and the plane specified by the point $A$ and the line $BC$ (Fig. 443).

*Solution.* Designate the desired point as $K$ ($k$, $k'$). Since this point must lie on a vertical projecting line ($mn$, $m'n'$), its vertical projection ($k'$) must coincide with the point $m'n'$ (why?). Having the vertical projection of the point, determine its horizontal projection ($k$) on the line-segment $mn$, given that the point ($k$, $k'$) lies in the given plane. From there on, the solution is clear from the drawing.

**Example 133**

Find the line of intersection of the plane $P$ and the plane of the triangle $ABC$ (Fig. 444).

*Solution.*

*First method.* The line of intersection will be determined if we find two points belonging to the given planes. The intersections of the sides ($ac$, $a'c'$) and ($bc$, $b'c'$) of the triangle with the plane $P$ yield the required points ($m$, $m'$) and ($n$, $n'$). Join the latter to obtain the desired line ($mn$, $m'n'$).

*Second method.* Since $P$ is a vertical projecting plane, the vertical projection ($m'n'$) of the line of intersection coincides with the vertical trace ($P_v$) of the plane. Given that the desired line ($mn$, $m'n'$) belongs to the plane of the triangle $ABC$, determine the horizontal projection ($mn$) of the line from its vertical projection ($m'n'$).

**Example 134**

Find the line of intersection of a plane $P$ and the plane specified by a straight line $AB$ and a point $C$ (Fig. 445).

*Solution.*

*First method.* The line of intersection will be determined if we find two points belonging to the given planes. The point ($c$, $c'$) does not lie on the line of intersection (why?). To find such points it is advisable to re-specify the plane by two parallel

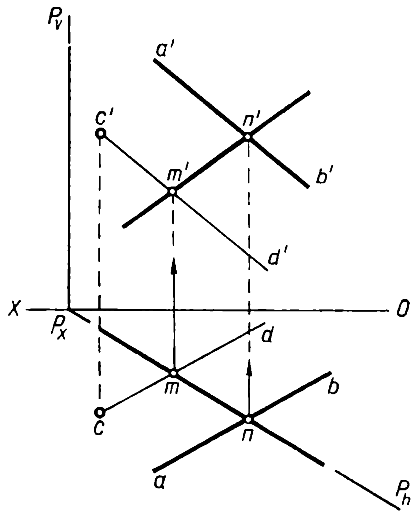



FIG.445.

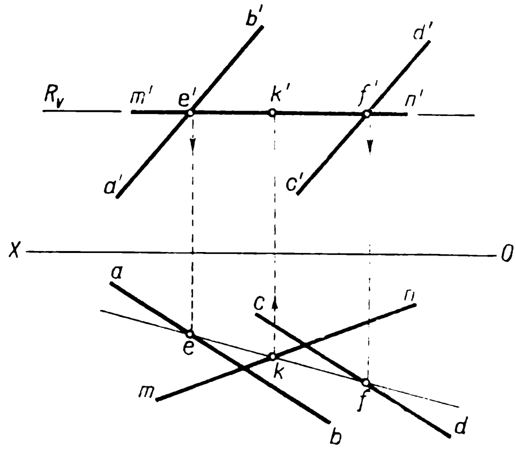



FIG.446.

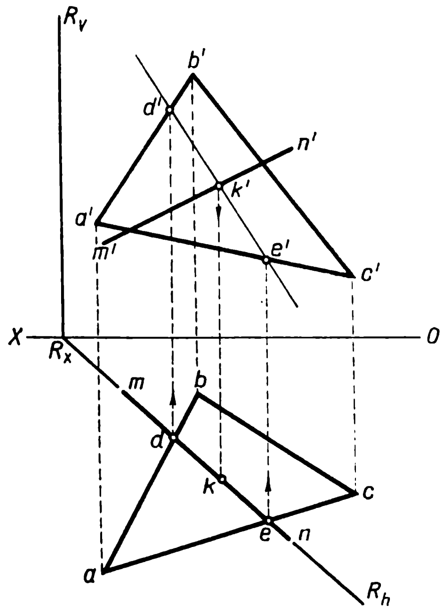



FIG.447.

lines $(ab, a'b')$ and $(cd, c'd')$, instead of by a line and a point, and then find the points $(m, m')$ and $(n, n')$ at which these lines cut the plane $P$. The line $(mn, m'n')$ passing through the points thus found is the desired line

*Second method.* Since $P$ is a horizontal projecting plane, the horizontal projection $(mn)$ of the line of intersection coincides with the horizontal trace $(P_h)$ of the plane. Now, given that the desired line $(mn, m'n')$ also belongs to the other plane, find the vertical projection $(m'n')$ of the line from its horizontal projection $(mn)$.

**Example 135**

Find the point of intersection of the line $MN$ and the plane specified by the two parallel lines $AB$ and $CD$ (Fig. 446).

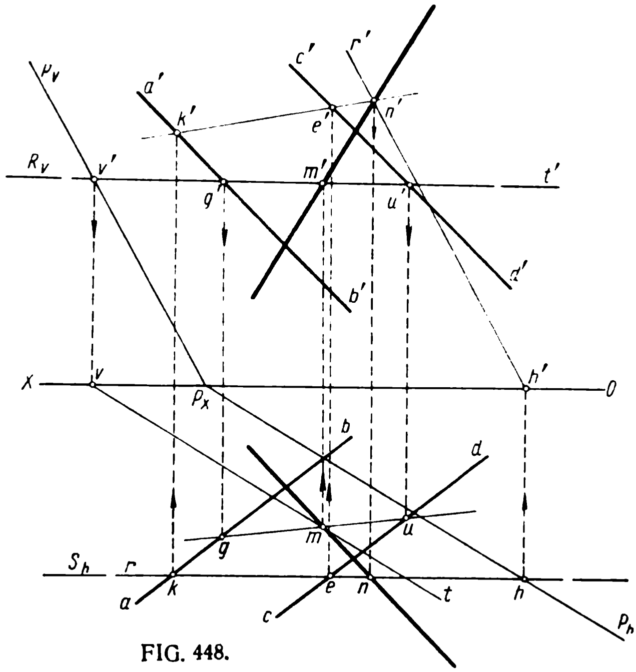



FIG. 448.

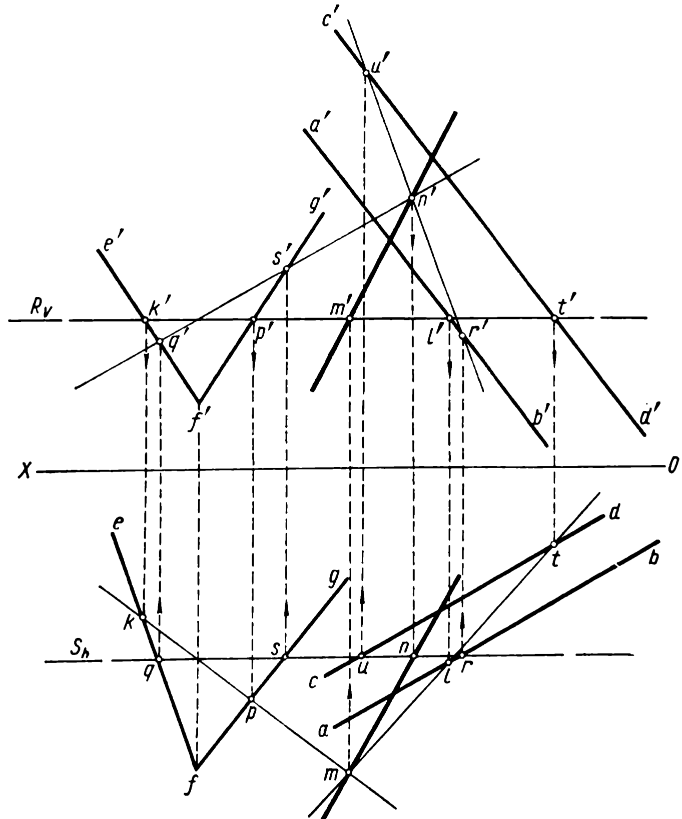



FIG. 449.

*Solution.* Since the given plane is an oblique one (why?), put the line-segment $MN$ into an auxiliary plane $R$, for instance, into one parallel to the horizontal plane of projection, and find the line $(ef, e'f')$ of intersection of the planes. The intersection of the lines $(mn, m'n')$ and $(ef, e'f')$ yields the desired point $(k, k')$. (See Example 133.)

**Example 136**

Find the point of intersection of the line $MN$ with the plane of the triangle $ABC$ (Fig. 447).

*Solution.* Pass an auxiliary horizontal projecting plane $R$ through the line-segment $MN$ and find the line $(de, d'e')$ of intersection of the planes. The intersection of the lines $(mn, m'n')$ and $(de, d'e')$ yields the desired point $(k, k')$.

**Example 137**

Find the line of intersection of the plane $P$ and the plane specified by the parallel lines $AB$ and $CD$ (Fig. 448).

*Solution.* This problem can be solved by three different methods.

*First method.* First re-specify the plane by its traces, and then proceed as in Examples 95 to 112.

*Second method.* Find the points at which the lines $AB$ and $CD$ cut the plane $P$. These points will determine the line of intersection.

*Third method.* Find the points which determine the line of intersection by consecutively introducing two auxiliary planes.

The third method is the simplest: introduce an auxiliary plane $R$, which is parallel to the horizontal plane of projection and intersects the plane $P$ along a horizontal principal line $(vt, v't')$, and a second plane along a horizontal line $(gu, g'u')$. Their intersection yields the point $(m, m')$. Then introduce another auxiliary plane $S$, which is, say, parallel to the vertical plane of projection and intersects the plane $P$ along a frontal line $(rh, r'h')$, and another plane along a frontal line $(ke, k'e')$. Their intersection yields the second point, i.e. $n$, $n'$. Join $m$ to $n$ and $m'$ to $n'$ to obtain the desired line $(mn, m'n')$.

**Example 138**

Find the line of intersection of the planes specified by the intersecting lines $FE$ and $FG$ and the parallel lines $AB$ and $CD$ (Fig. 449).

*Solution.* Find the points $(m, m')$ and $(n, n')$ common to the given planes, for which purpose introduce an auxiliary plane $R$ parallel to the horizontal plane of projection and intersecting the given planes along horizontal lines $(kp, k'p')$ and $(lt, l't')$, whose intersection yields the point $(m, m')$. Then introduce another auxiliary plane $S$, which is parallel to the vertical plane of projection and intersects the given planes along vertical lines $(qs, q's')$ and $(ru, r'u')$; their intersection yields the point $(n, n')$. Join $(m, m')$ to $(n, n')$ to obtain the required line.

PROBLEMS

219. Find the point of intersection of the line $AB$ and the plane $P$ (Figs. 450 to 467).

220. Find the point of intersection of the line $AB$ and a plane specified other than by its traces (Figs. 468 to 473).

221. Find the line of intersection of the plane of the triangle $ABC$ and the plane $P$ and determine the quadrants through which the desired line passes (Fig. 474).

222. Find the line of intersection of the plane specified by the two parallel lines $KL$ and $MN$ and the plane specified by the intersecting lines $AB$ and $AC$. Determine the quadrants through which the desired line passes (Fig. 475).

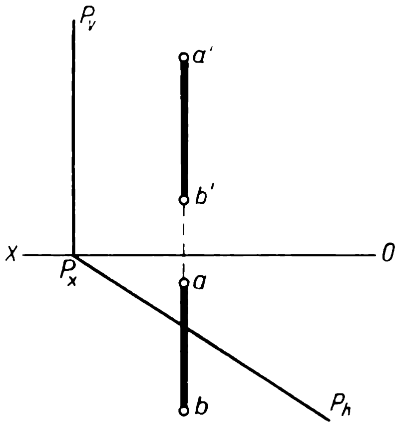



FIG. 450.

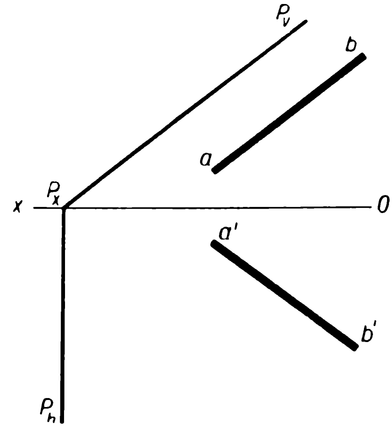



FIG. 451

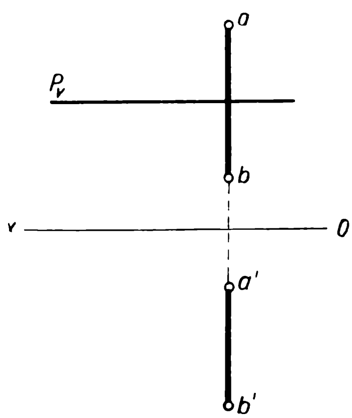



FIG. 452.

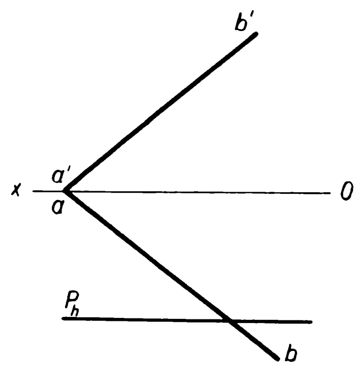



FIG. 453.

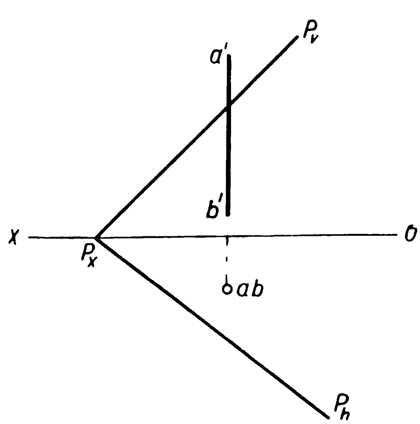



FIG. 454.

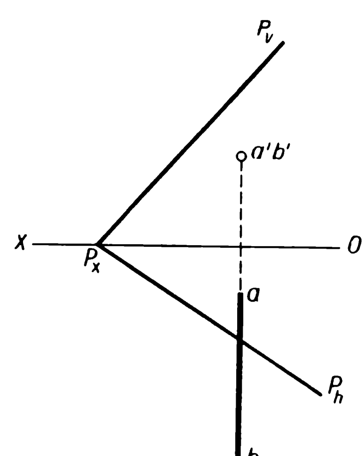



FIG. 455.

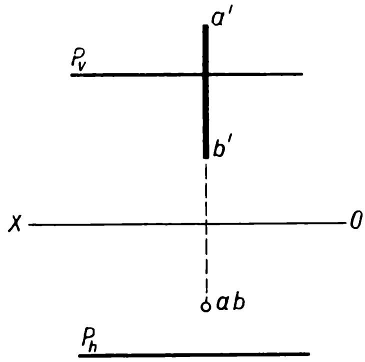



FIG. 456.

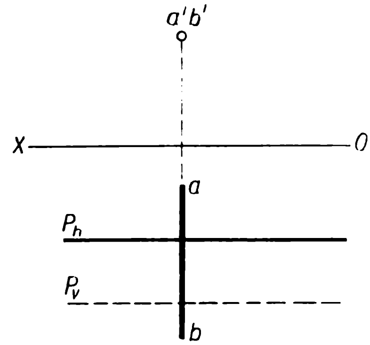



FIG. 457.

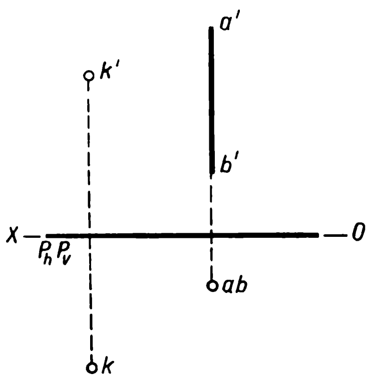



FIG. 458.

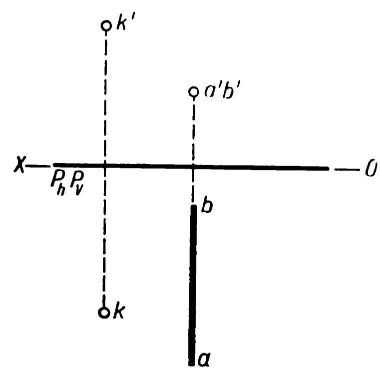



FIG. 459.

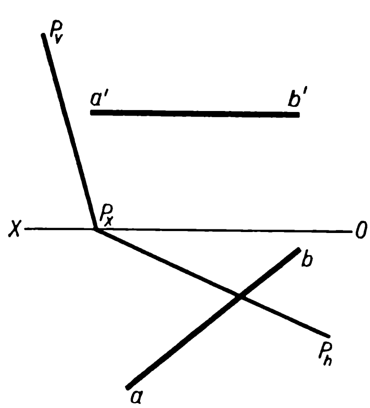



FIG. 460.

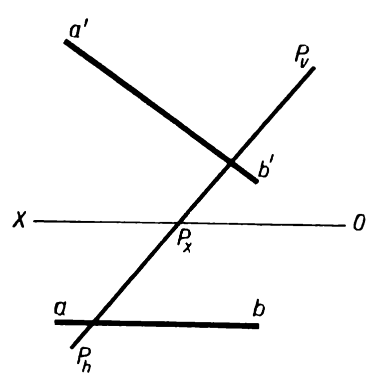



FIG. 461.

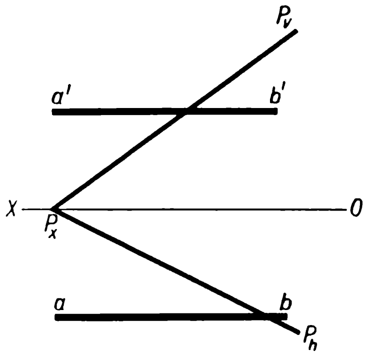


FIG. 462.

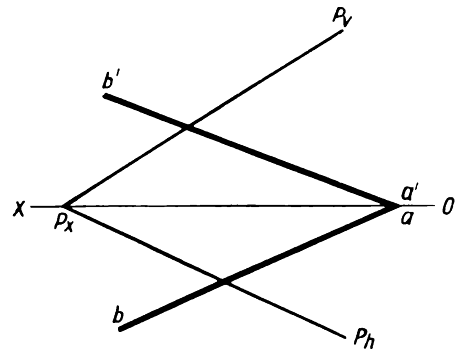


FIG. 463.

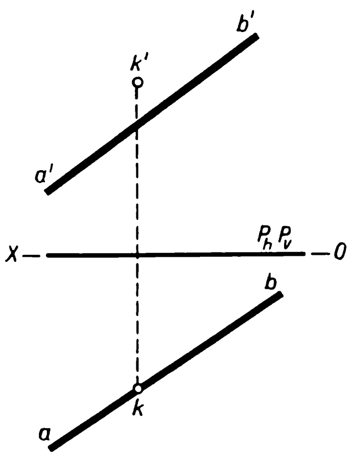


FIG. 464.

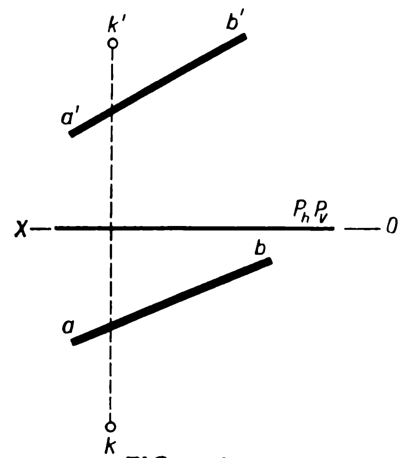


FIG. 465.

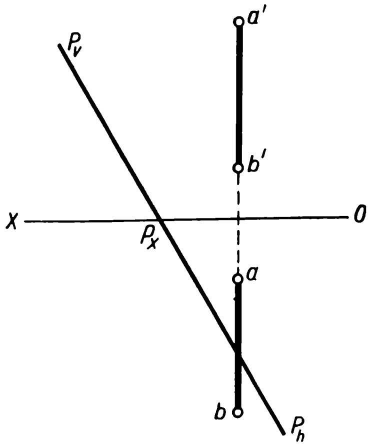


FIG. 466.

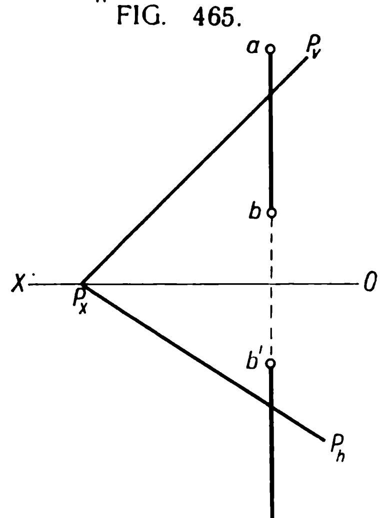


FIG. 467.

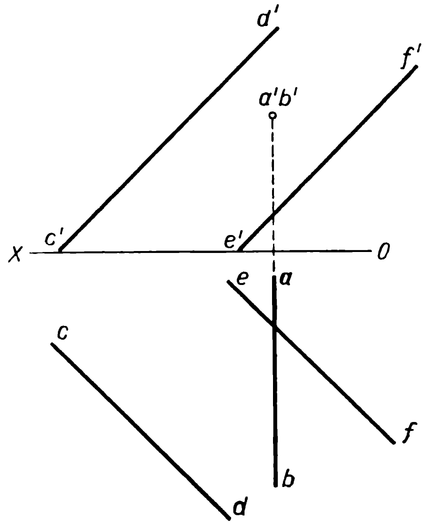



FIG. 468.

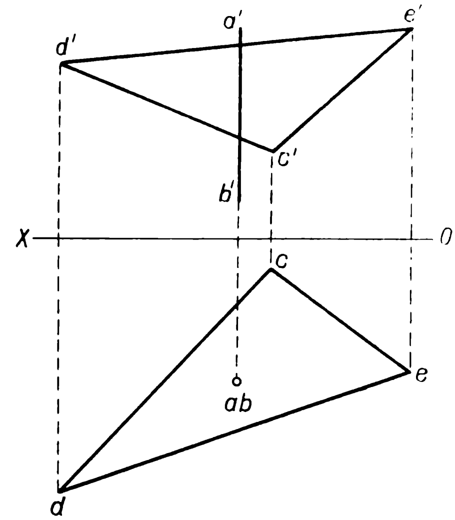



FIG. 469.

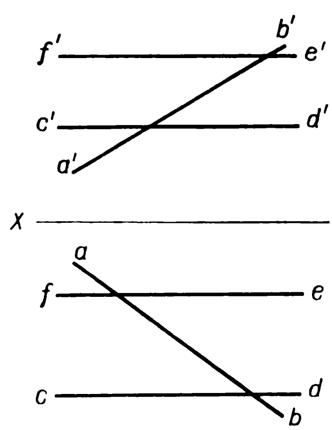



FIG. 470.

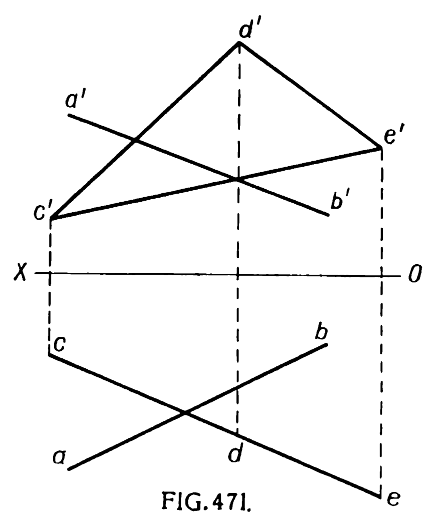



FIG. 471.

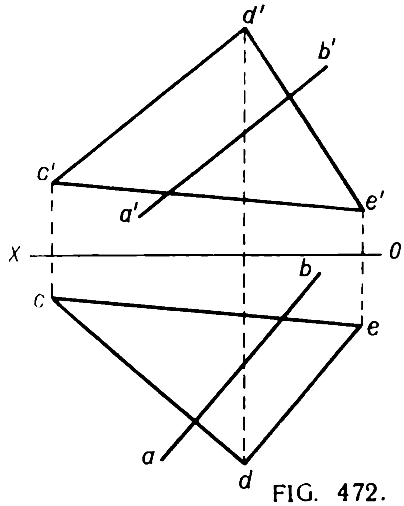



FIG. 472.

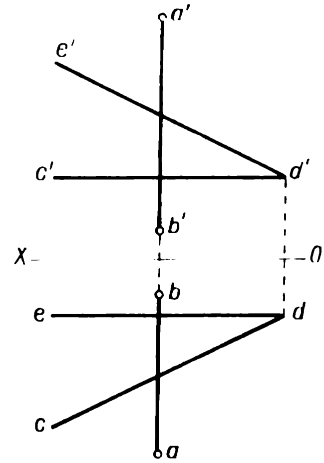



FIG. 473.

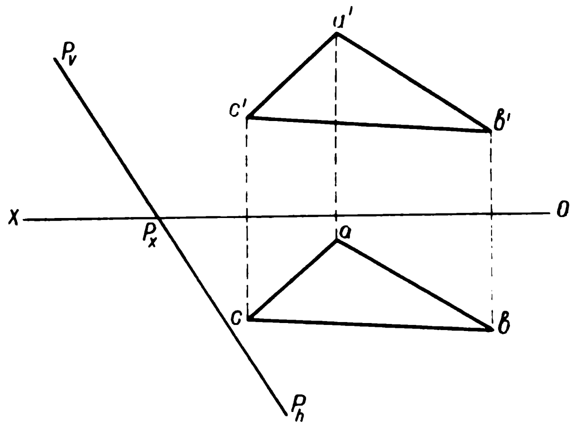



FIG. 474.

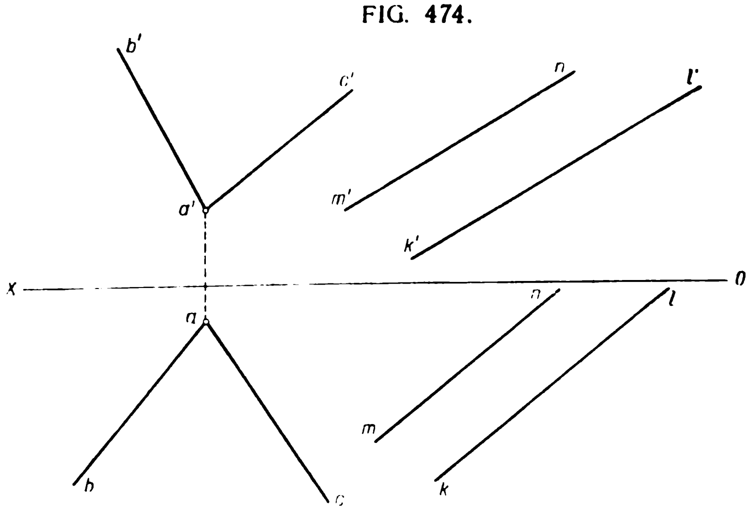



FIG. 475.

CHAPTER XV

# PARALLELISM OF A STRAIGHT LINE AND A PLANE. PARALLELISM OF PLANES

A straight line and a plane are parallel if one can draw in the plane a line parallel to the given line.

Two planes $P$ and $Q$ given by traces are parallel if their like traces are parallel.

The converse is not always true in a two-plane system ($H$ and $V$). For instance, two profile projecting planes are parallel only when their profile traces are parallel.

Principal (horizontal and vertical) lines of two parallel planes are also parallel.

This feature of the principal lines of parallel planes can be utilized conveniently for checking the parallelism of two planes when one of the planes or both are given other than by their traces (determination of the traces is not obligatory).

The parallelism of planes may also be checked by means of arbitrary lines.

EXAMPLES

**Example 139**

Given: a plane $P$ and a point $A$. Required: to draw through the point a line parallel to the plane (Fig. 476).

*Solution.* Assume any line (say, $hv$, $h'v'$) in the plane, and draw, through the point ($a$, $a'$), a line ($ab$, $a'b'$) parallel to it. Since the line ($hv$, $h'v'$) is assumed in the plane arbitrarily it is possible to draw through the given point an indefinite number of straight lines parallel to the plane $P$. But, as we know, through a point one can draw only one horizontal and only one vertical line parallel to a given plane (why?).

**Example 140**

Given: a point $A$ and a plane specified by the line $DE$ and the point $C$. Required: to draw through the point an arbitrary line parallel to the given plane (Fig. 477).

*Solution.* Assume any straight line ($ck$, $c'k'$) in the plane and, through the point ($a$, $a'$) draw a line ($ab$, $a'b'$) parallel to it. (The problem is indeterminate.)

**Example 141**

Given: a plane $P$ and a line $AB$. Check their parallelism (Fig. 478).

*Solution.* The line $AB$ is parallel to the plane $P$ if a line parallel to the given line $AB$ can be drawn in the plane. Draw the vertical projection ($h'v'$) of a line (lying in the plane) parallel to the line ($a'b'$) and find its horizontal projection ($hv$). If the latter is parallel to the line $ab$, then the line ($ab$, $a'b'$) is also parallel to the plane $P$, and vice versa. In this problem the line $AB$ and the plane $P$ are not parallel.

We may begin solving the problem by drawing the horizontal projection ($hv$) of the line, and so on.

**Example 142**

Given: a line $MN$ and a plane specified by the parallel lines $AB$ and $CD$. Are the line $MN$ and the plane parallel (Fig. 479)?

*Solution.* Draw arbitrarily the vertical projection ($e'k'$) of an auxiliary line (lying in the given plane and parallel to the line $m'n'$), and find its horizontal projection ($ek$). Since the lines ($ek$, $e'k'$) and ($mn$, $m'n'$) are not parallel (why?), the given line and plane are not parallel either.

We may also begin solving the problem by arbitrarily drawing the horizontal projection ($ek$) of the auxiliary line, and so on.

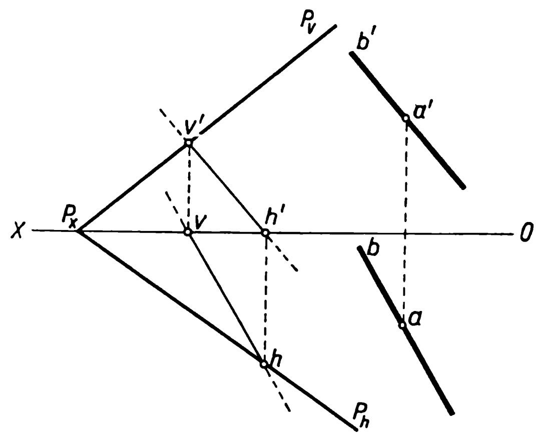



FIG. 476.

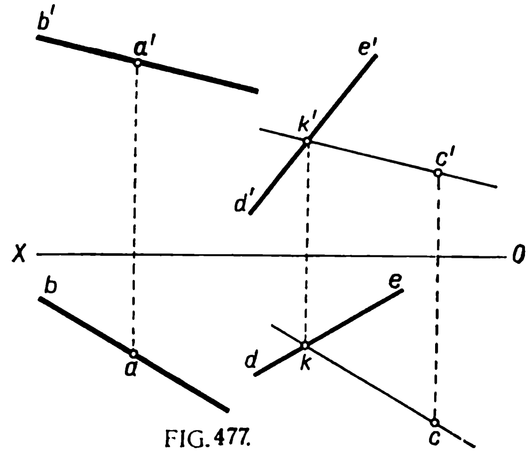



FIG. 477.

**Example 143**

Given: a line $AB$ and a point $C$. Draw through the point $C$ an oblique plane $P$ with coincident traces, which is parallel to the line $AB$ (Fig. 480).

*Solution.* For the plane $P$ to pass through the point $C$ parallel to the line $AB$, the plane should contain a line parallel to $AB$. Through the point $(c, c')$, draw a straight line $(mn, m'n')$ parallel to the line $(ab, a'b')$ and pass a plane through the latter. To do this, determine the traces $(h, h')$ and $(v, v')$ of the line $(mn, m'n')$ and draw, through the points $(h, h')$ and $(v, v')$, the traces $P_h$ and $P_v$ (coincident) of the desired plane.

**Example 144**

Given: straight lines $AB$ and $CD$. Through $AB$ pass a plane parallel to $CD$ (Fig. 481).

*Solution.* For a plane passing through the line $AB$ to be parallel to the line $CD$, the plane should contain a line parallel to the line $CD$. Draw a line $(mk, m'k')$ parallel to the line $(cd, c'd')$ through an arbitrary point $(k, k')$ lying on the line $(ab, a'b')$. The lines $(ab, a'b')$ and $(mk, m'k')$ determine the desired plane.

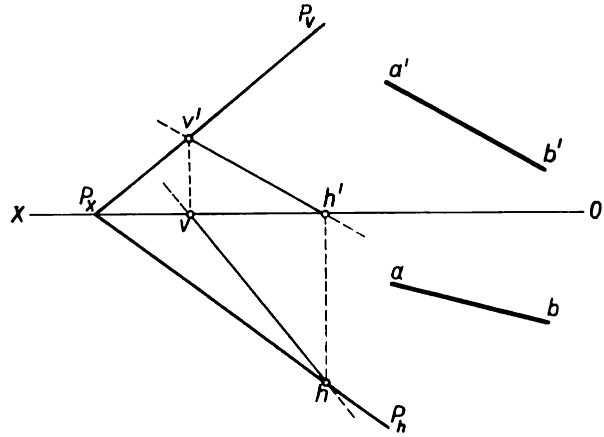



FIG. 478.

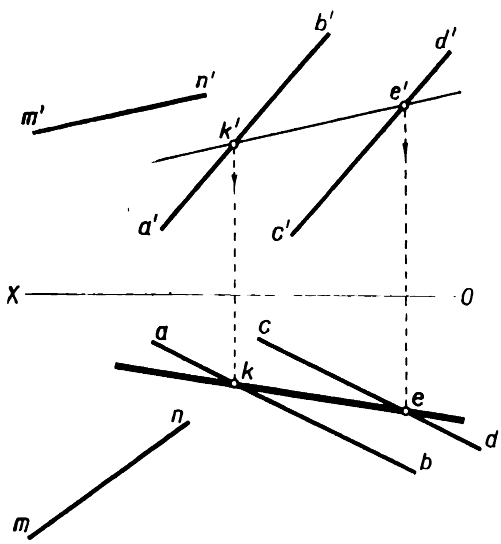



FIG. 479.

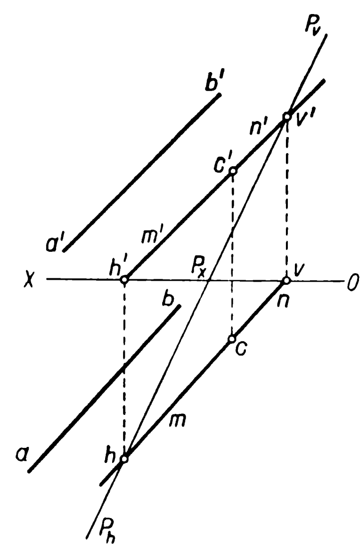



FIG. 480.

In the same manner we can pass through the line $CD$ a single plane parallel to the line $AB$.

These planes may be specified by traces, which are found by the above method. (How?)

*Conclusion.* Only one pair of parallel planes (*planes of parallelism*) can be passed through two skew lines.

**Example 145**

Given: a plane $P$ and the vanishing point of a plane $Q$, which is parallel to the plane $P$. Construct the traces of the plane $Q$ (Figs. 482, 483).

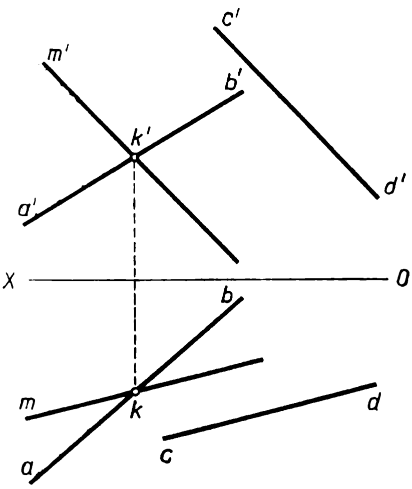



FIG. 481.

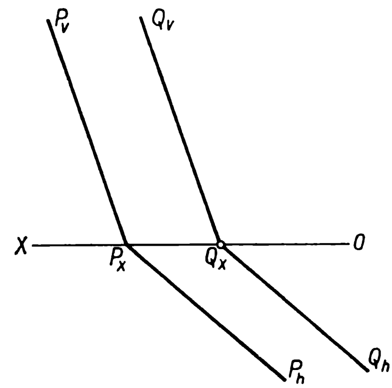



FIG. 482.

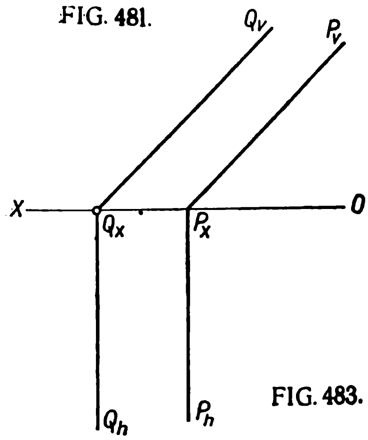



FIG. 483.

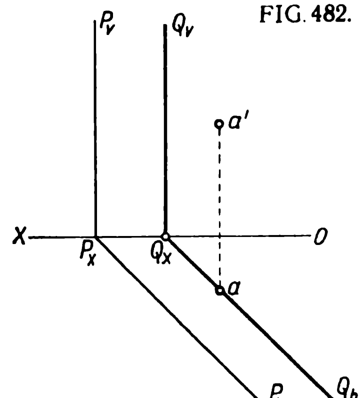



FIG. 484.

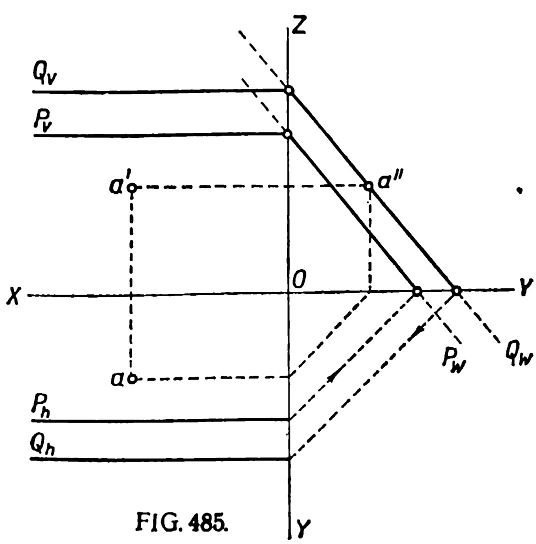



FIG. 485.

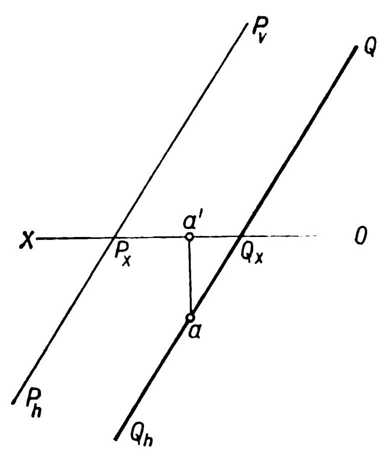



FIG. 486.

*Solution*. The required traces must be parallel to the like traces of the plane $P$. Through the point $Q_x$ draw the following traces: $Q_h$ parallel to $P_h$, and $Q_v$ parallel to $P_v$.

**Example 146**

Given: a plane $P$ and a point $A$. Required: pass through $A$ a plane $Q$ parallel to the plane $P$ (Fig. 484).

*Solution*. The desired plane ($Q$) is a horizontal projecting one. Since this plane must pass through the point $(a, a')$, first draw its horizontal trace ($Q_h$) through the point $a$ and parallel to the trace $P_h$ to intersect the coordinate axis at point $Q_x$; then, through the obtained point, draw the vertical trace ($Q_v$) parallel to $P_v$.

*Note*. If the point $Q_x$ is outside the limits of the drawing, then it is not obligatory to draw the vertical trace.

**Example 147**

Given: a plane $P$ and a point $A$. Required: pass, through point $A$, a plane $Q$ parallel to the plane $P$ (Fig. 485).

*Solution*. The desired plane $Q$ is a profile projecting one. As is known, the necessary condition for the parallelism of two profile projecting planes is the parallelism of their profile traces. Find the profile trace ($P_w$) of the plane $P$ and the profile projection ($a''$) of the point $A$. Since the plane $Q$ must pass through the point $(a, a')$, draw the profile trace ($Q_w$) of the plane $Q$ through the profile projection ($a''$) of the point and parallel to the trace $P_w$. Then, having the trace $Q_w$, find the traces $Q_h$ and $Q_v$, which are parallel to the coordinate axis.

*Note*. This problem may be solved without using the profile plane of projection (see Example 149, first method).

**Example 148**

Given: a plane $P$ and a point $A$. Required: pass through the point $A$ a plane $Q$ parallel to the plane $P$ (Fig. 486).

*Solution*. The desired plane $Q$ is an oblique one with coincident traces. Since the point $(a, a')$ through which the plane $Q$ passes lies in the horizontal plane of projection, it must lie also on the horizontal trace ($Q_h$) of the plane. Hence, through the point $a$ draw the horizontal trace ($Q_h$) of the plane to intersect the coordinate axis at point $Q_x$; the extension of $Q_h$ will be the vertical trace ($Q_v$) of the plane.

**Example 149**

Given: a plane $P$ and a point $A$. Required: pass through the point $A$ a plane $Q$ parallel to the plane $P$ (Figs. 487 to 489).

*Solution*. The general method is as follows: through the point $A$ an auxiliary line is drawn parallel to the given plane $P$ (see Example 139). Then this line is put into a plane parallel to $P$.

*First method*. Assume an arbitrary line $(hv, h'v')$ in the given plane $P$ and, through the point $(a, a')$, draw a line parallel to it. Find the traces $(h_1, h_1')$ and $(v_1, v_1')$ of this line and draw through them the traces of the required plane $Q$: the horizontal one ($Q_h$) through the point $h_1$ and parallel to the trace $P_h$, and the vertical one ($Q_v$) through the point $v_1'$ and parallel to the trace $P_v$. The traces $Q_h$ and $Q_v$ of the desired plane must intersect on the coordinate axis at the point $Q_x$. It is easier to solve the problem with the aid of the principal lines (horizontal or frontal) of the plane.

*Second method*. Draw a horizontal line of the required plane $Q$ through the point $(a, a')$ and parallel to an arbitrary horizontal line of the plane $P$. Its horizontal projection must pass through the point $a$ and parallel to the trace $P_h$, and its vertical projection through the point $a'$ and parallel to the coordinate axis. Find the trace $(v, v')$ of this horizontal line and draw the traces of the desired plane: first the vertical one ($Q_v$) through the point $v'$ and parallel to the trace $P_v$ to intersect the coordinate axis at the point $Q_x$, and then the horizontal trace ($Q_h$) through the latter point and parallel to trace the $P_h$.

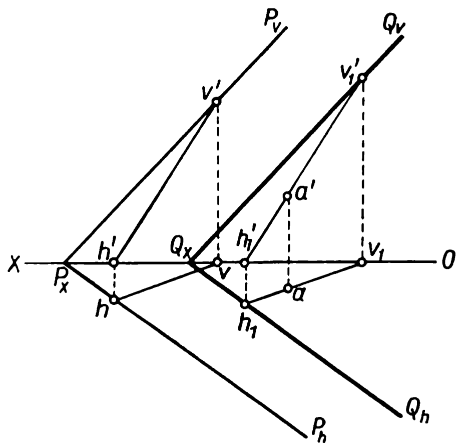



FIG.487.

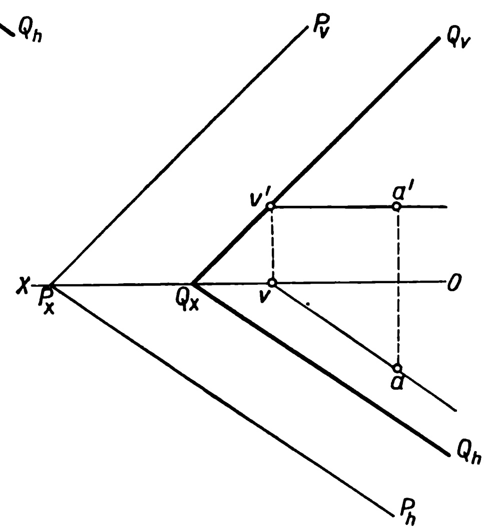



FIG.488.

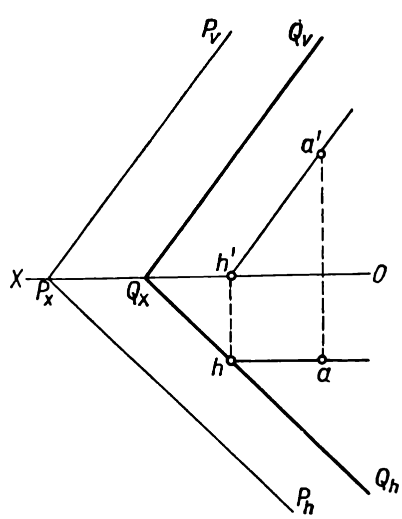



FIG.489.

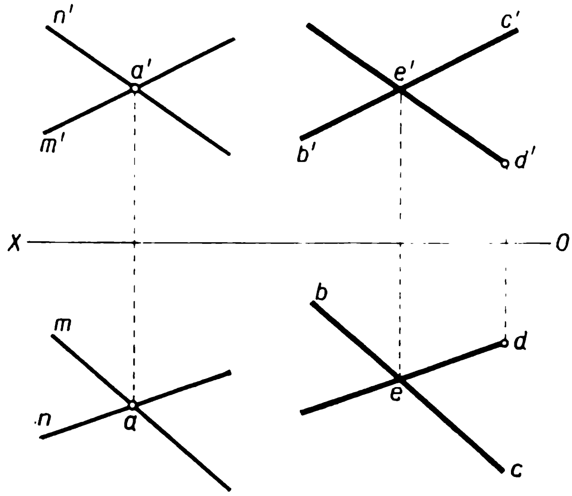



FIG. 490.

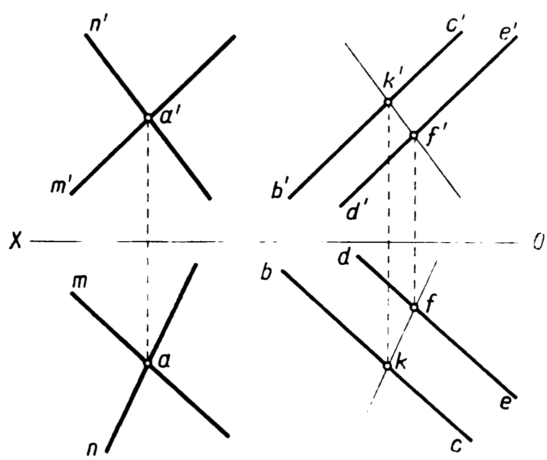



FIG. 491.

*Third method.* Draw a frontal line of the desired plane $Q$ through the point $(a, a')$ and parallel to an arbitrary frontal line of the plane $P$. The horizontal projection of the first line must pass through the point $a$ and parallel to the coordinate axis, and the vertical one, through the point $a'$ and parallel to the trace $P_v$. Find the trace $(h, h')$ of this frontal line and draw the traces of the required plane: first the horizontal one $(Q_h)$ through the point $h$ and parallel to the trace $P_h$ to intersect the coordinate axis at the point $Q_x$, and then the vertical trace $(Q_v)$ through the latter point and parallel to the trace $P_v$.

*Notes*: 1. Through the given point it is possible to draw horizontal and frontal lines of the desired plane without drawing these lines on the given plane (why?).

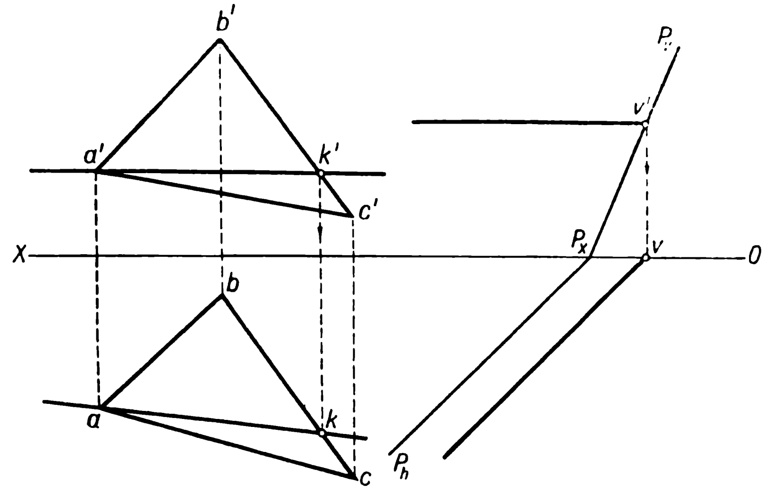



FIG. 492.

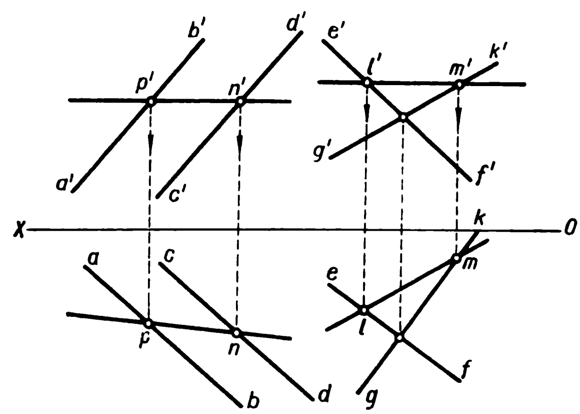



FIG. 493.

2. Sometimes, when a horizontal or frontal line is used, the point $Q_x$ is found outside the limits of the drawing. In such cases the traces $Q_h$ and $Q_v$ of the required plane should be drawn independently of the point $Q_x$, for which purpose both frontal and horizontal lines are used.

**Example 150**

Given: a point $A$ and a plane specified by the line $BC$ and the point $D$. Required: pass through $A$ a plane parallel to the given plane (Fig. 490).

*Solution.* First re-specify the plane by two intersecting lines $BC$ and $DE$. Then, through the point ($a$, $a'$), draw the lines ($am$, $a'm'$) and ($an$, $a'n'$) parallel to the lines ($bc$, $b'c'$) and ($de$, $d'e'$), respectively. The desired plane is specified by the intersecting lines $AM$ and $AN$. It may alternatively be specified by traces which are constructed according to the general rule (how?).

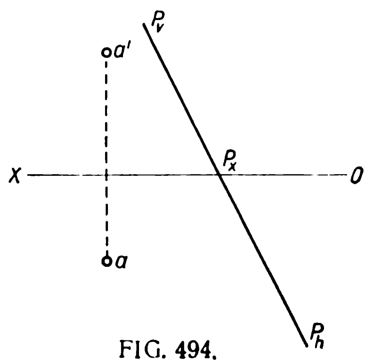


FIG. 494.

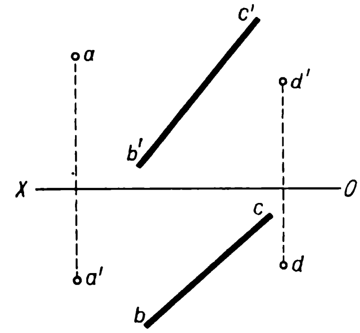


FIG. 495.

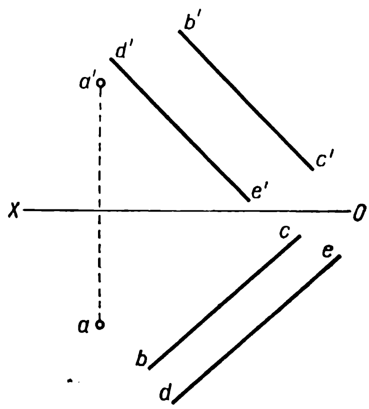


FIG. 496.

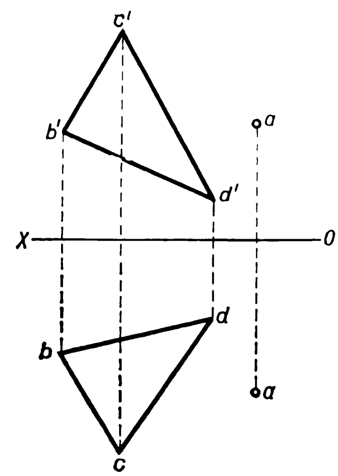


FIG. 497.

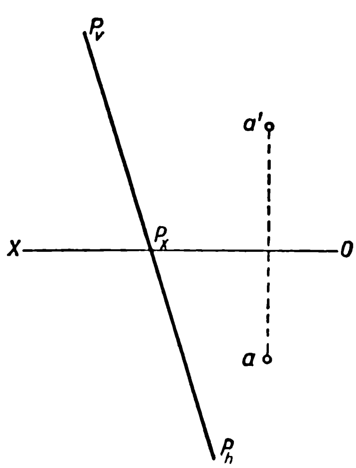


FIG. 498.

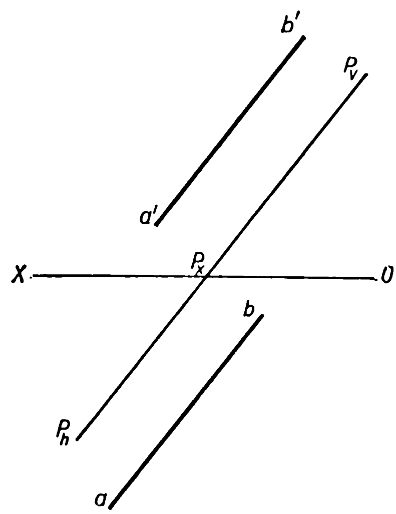


FIG. 499.

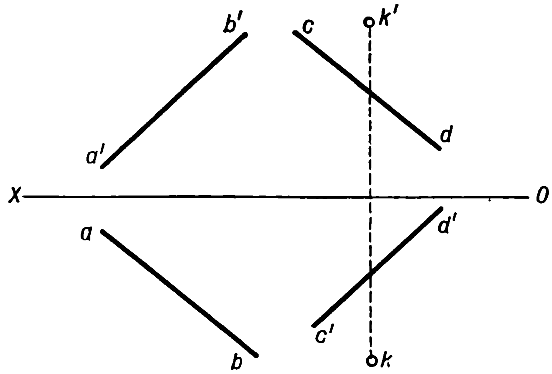



FIG. 500.

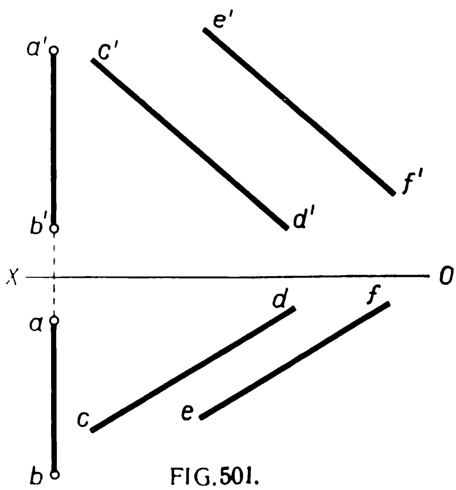



FIG. 501.

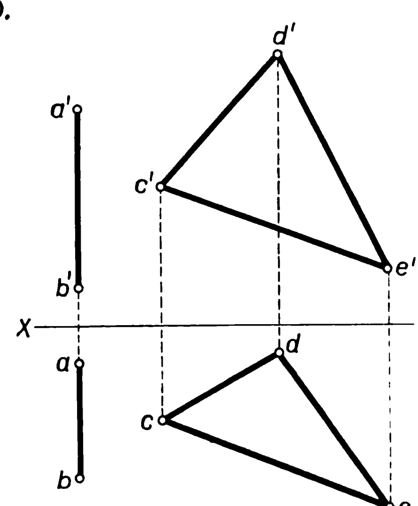



FIG. 502.

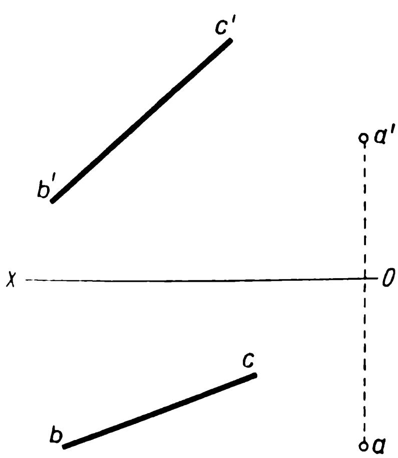



FIG. 503.

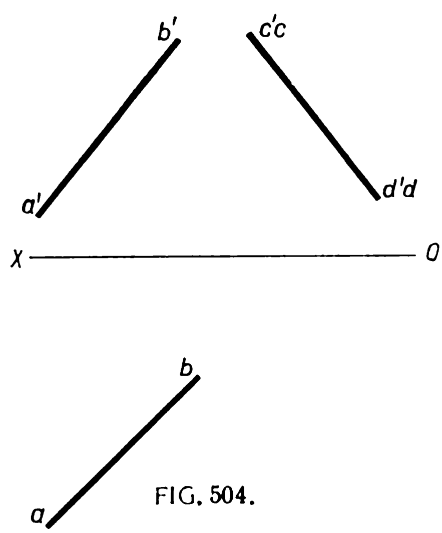



FIG. 504.

**Example 151**

Given: a point $A$ and a plane specified by the parallel lines $BC$ and $DE$. Required: pass through the point $A$ a plane parallel to the given plane (Fig. 491).

*Solution.* Draw an arbitrary line ($fk$, $f'k'$) in the given plane (what for?), and then, through the point ($a$, $a'$), draw lines ($am$, $a'm'$) and ($an$, $a'n'$) parallel to ($bc$, $b'c'$) or ($de$, $d'e'$) and ($fk$, $f'k'$).

**Example 152**

Given: a triangle $ABC$ and a plane $P$. Is the plane of the triangle parallel to $P$ (Fig. 492)?

*Solution.* The problem can be solved by either of the following two methods.

*First method.* Find the traces of the plane of the triangle and then determine the relative positions of the planes using the theorem on the position of the like traces of planes.

*Second method.* As is known, in parallel planes one can draw parallel lines. Construct arbitrary horizontal lines in both planes. If the horizontal lines are parallel, draw arbitrary frontal lines in the same planes. If they are also parallel, then the planes are parallel as well. If the first pair of the principal lines (either horizontal or frontal) are not parallel, discontinue the solution and conclude that the planes are not parallel. Here we have a case of non-parallelism.

*Note.* As is obvious from the drawing, it is not necessary to draw a horizontal (frontal) line for the plane $P$ (why?).

**Example 153**

Determine the parallelism of the planes specified by the parallel lines $AB$ and $CD$ and the intersecting lines $EF$ and $GK$ (Fig. 493).

*Solution.* We solve the problem without finding the traces of the given planes. Draw arbitrary horizontal lines in them. Since the horizontal lines are not parallel, the planes are not parallel either.

## PROBLEMS

223. Through the point $A$ draw a line parallel to the plane $P$ (Fig. 494).

224. Through the point $A$ draw a line parallel to the plane specified by the line $BC$ and the point $D$ (Fig. 495).

225. Through the point $A$ draw a line parallel to the plane specified by the parallel lines $BC$ and $DE$ (Fig. 496).

226. Through the point $A$ draw a line parallel to the plane of the triangle $BCD$ (Fig. 497).

227. Through the point $A$ draw a line parallel to the plane $P$ and equally inclined to the projection planes (Fig. 498).

228. Is the line $AB$ parallel to the plane $P$ (Fig. 499)?

229. Is the line $AB$ parallel to the plane specified by the line $CD$ and the point $K$ (Fig. 500)?

230. Is the line $AB$ parallel to the plane specified by the parallel lines $CD$ and $EF$ (Fig. 501)?

231. Is the line $AB$ parallel to the plane of the triangle $CDE$ (Fig. 502)?

232. Through the point $A$ draw a plane $Q$ perpendicular to the horizontal plane of projection and parallel to the line $BC$ (Fig. 503).

233. Through the line $AB$ draw a plane $P$ parallel to the line $CD$ (Fig. 504).

234. Pass parallel planes $P$ and $Q$ through the given lines $AB$ and $CD$ (Figs. 505, 506).

235. Construct the traces of the plane $Q$ which is parallel to the plane $P$, given the vanishing point $Q_x$ (Figs. 507 to 510).

236. Determine the parallelism of the planes $P$ and $Q$ (Figs. 511, 512):

(1) using the profile plane of projection;

(2) without using the profile plane of projection.

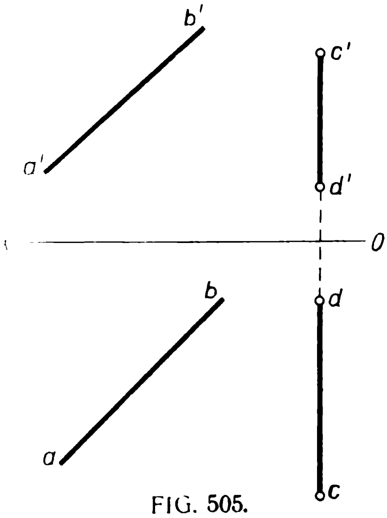



FIG. 505.

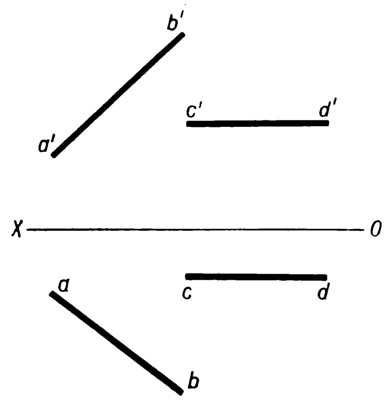



FIG. 506.

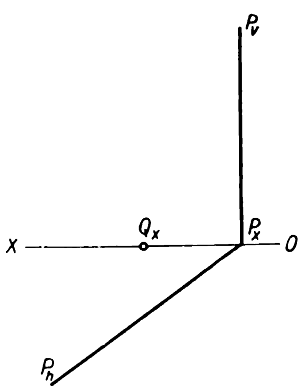



FIG. 507.

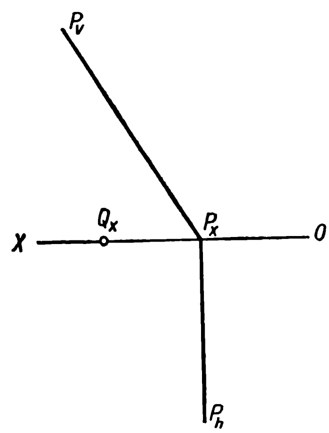



FIG. 508.

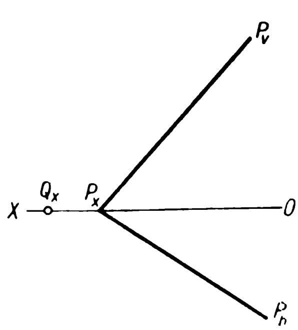



FIG. 509.

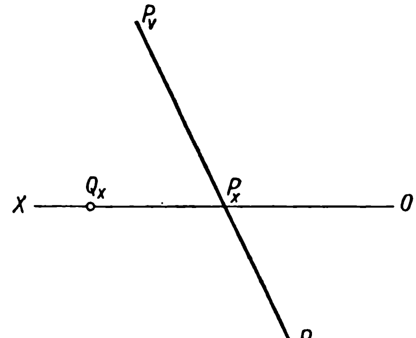



FIG. 510.

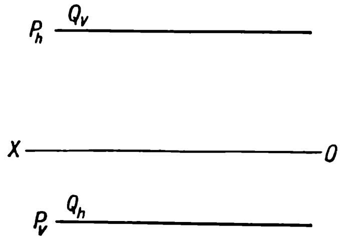


FIG. 511.

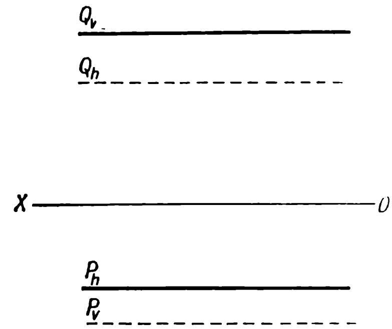


FIG. 512.

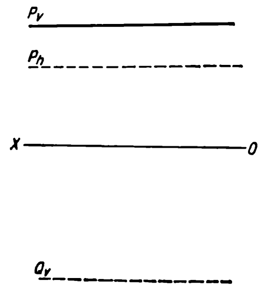


FIG. 513.

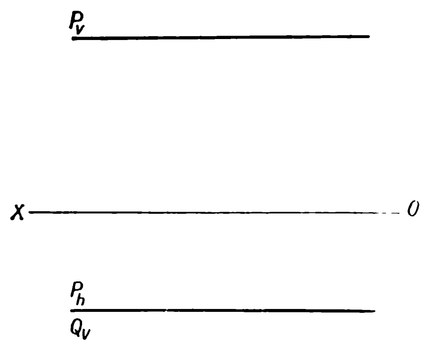


FIG. 514.

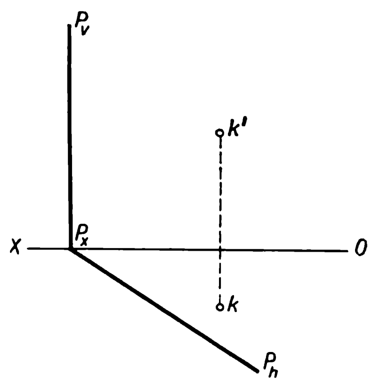


FIG. 515.

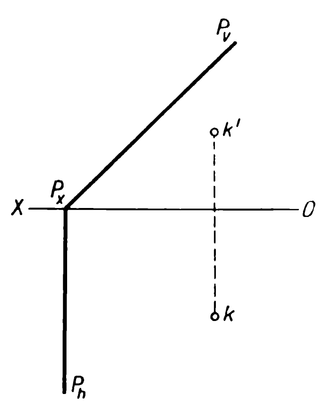


FIG. 516.

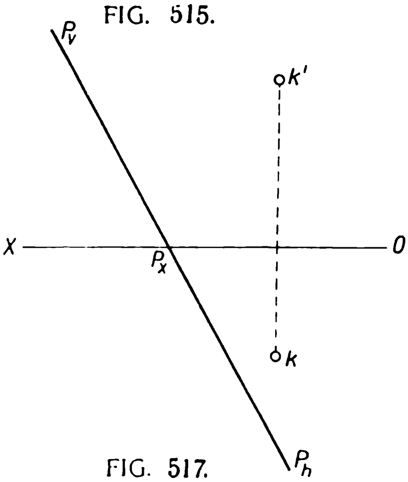


FIG. 517.

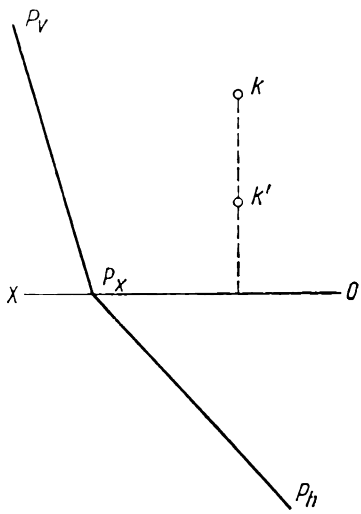



FIG. 518.

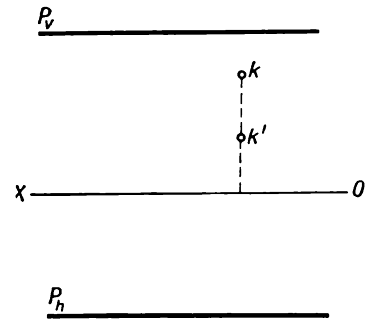



FIG. 519.

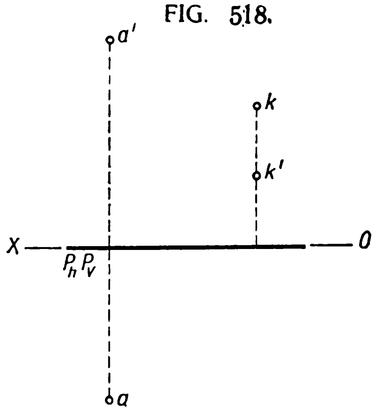



FIG. 520.

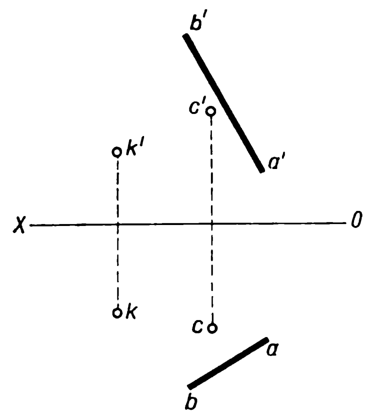



FIG. 521.

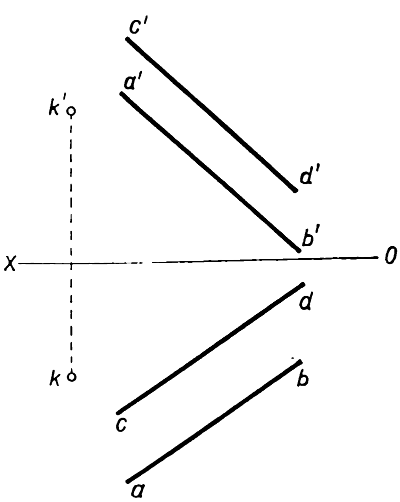



FIG. 522.

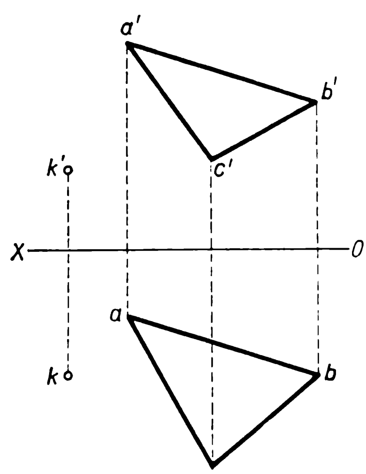



FIG. 523.

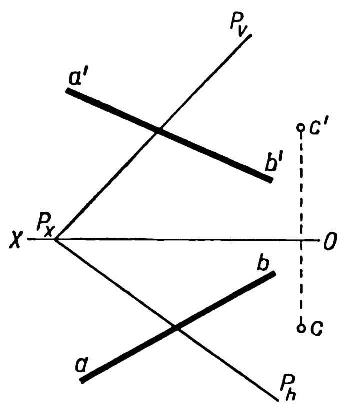



FIG. 524.

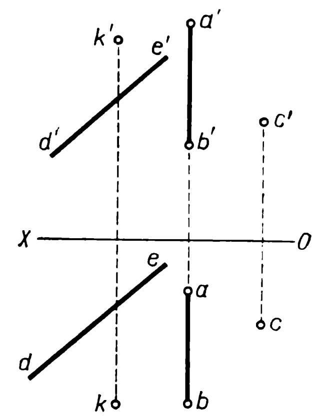



FIG. 525.

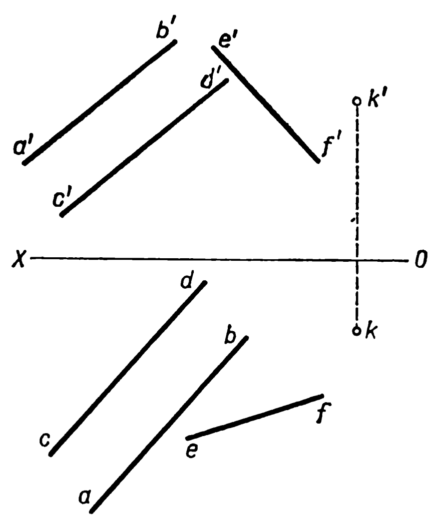



FIG. 526.

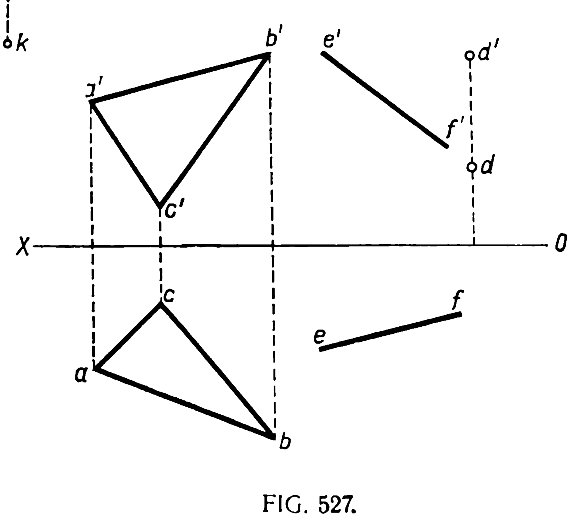



FIG. 527.

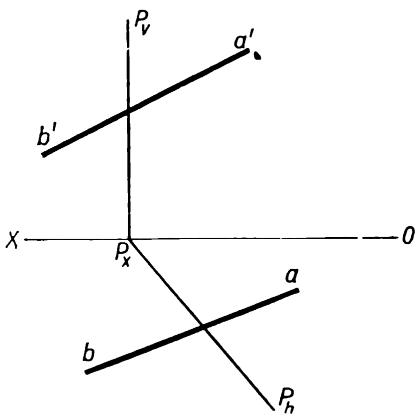



FIG. 528.

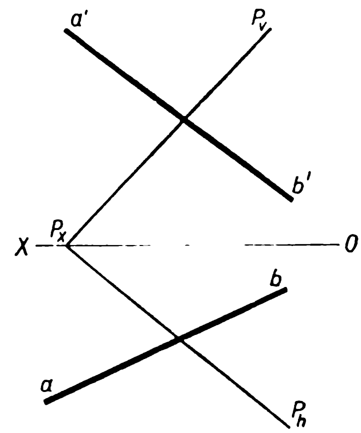



FIG. 529.

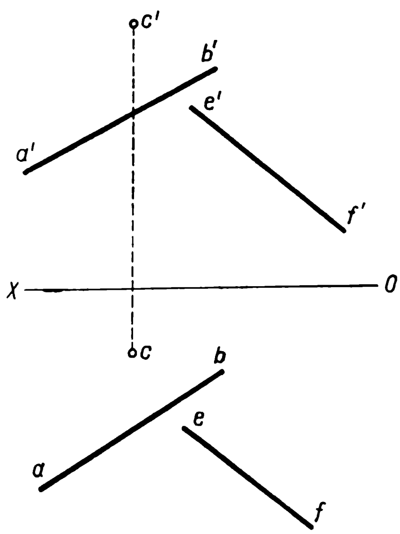



FIG. 530.

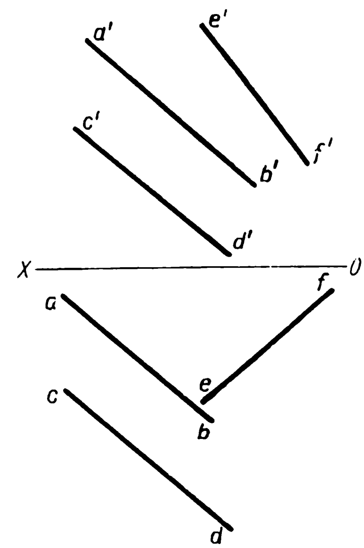



FIG. 531.

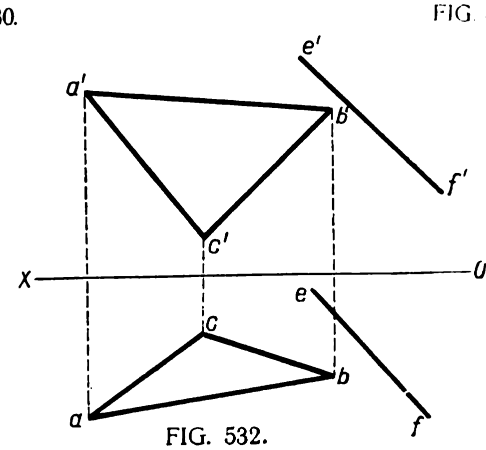



FIG. 532.

237. Find the lacking trace of the plane $Q$, given that the planes $P$ and $Q$ are parallel (Figs. 513, 514):

(1) using the profile plane of projection;

(2) without using the profile plane of projection.

238. Construct traces of a plane passing through the point $K$ and parallel to the plane $P$ (Figs. 515 to 518).

239. Construct traces of a plane passing through the point $K$ and parallel to the plane $P$ (Figs. 519, 520):

(1) using the profile plane of projection;

(2) without using the profile plane of projection.

240. Through the point $K$ pass a plane parallel to the plane specified by the line $AB$ and the point $C$ (Fig. 521).

241. Through the point $K$ pass a plane parallel to the plane specified by the parallel lines $AB$ and $CD$ (Fig. 522).

242. Through the point $K$ pass a plane parallel to the plane of the triangle $ABC$ (Fig. 523).

(In Problems 241 to 243 the desired plane should be specified both by traces and otherwise.)

243. Through the point $C$ draw a line intersecting the line $AB$ and parallel to the plane $P$ (Fig. 524).

244. Through the point $C$ draw a line intersecting the line $AB$ and parallel to the plane specified by the line $DE$ and the point $K$ (Fig. 525).

245. Through the point $K$ draw a line intersecting the line $EF$ and parallel to the plane specified by the parallel lines $AB$ and $CD$ (Fig. 526).

246. Through the point $D$ draw a line intersecting the line $EF$ and parallel to the plane of the triangle $ABC$ (Fig. 527).

247. Pass a plane $Q$ parallel to the plane $P$ so that the segment of the given line $AB$ lying between the planes has a length of 20 mm (Figs. 528, 529).

248. Pass a plane $P$ parallel to the plane specified by the line $AB$ and the point $C$ so that the segment of the given line $EF$ lying between the planes is 25 mm long (Fig. 530).

249. Pass a plane $P$ parallel to the plane specified by the parallel lines $AB$ and $CD$ so that the segment of the given line $EF$ lying between the planes is 30 mm long (Fig. 531).

250. Pass a plane $P$ parallel to the plane of the triangle $ABC$ so that the segment of the given line $EF$ lying between the planes is 30 mm long (Fig. 532).

CHAPTER XVI

# A LINE PERPENDICULAR TO A PLANE. MUTUALLY PERPENDICULAR PLANES

If a line is perpendicular to a plane specified by its traces, then the projections of the line are perpendicular to the like traces of the plane; furthermore, the horizontal projection of the line is perpendicular also to the horizontal projection of a horizontal line (why?), and the vertical projection of the line is perpendicular also to the vertical projection of a frontal line (why?).

This feature of the projections of principal lines of a plane perpendicular to a line should be used for

(1) establishing the perpendicularity of a line to a plane specified other than by its traces, without determining the traces of the plane;

(2) dropping a perpendicular from a given point to a plane specified other than by its traces (see examples below).

The converse theorem is not always true in the two-plane system ($H$ and $V$).

*Exception.* A line is perpendicular to a profile projecting plane if the profile projection of the line is perpendicular to the profile trace of the plane.

Planes $P$ and $Q$ are mutually perpendicular if the plane $P$ contains a line perpendicular to the plane $Q$.

EXAMPLES

**Example 154**

Given: a plane $P$ and a point $A$. Drop a perpendicular from the point $A$ to the plane $P$ (Fig. 533).

*Solution.* Through the projections of the given point $(a, a')$ draw like projections of the desired line perpendicular to the respective traces of the plane, i.e. $ab$ is perpendicular to $P_h$, and $a'b'$ to $P_v$.

**Example 155**

Given: a plane $P$ and a point $A$. Drop a perpendicular from the point $A$ to the plane $P$ (Fig. 534).

*Solution.* The desired line is a profile one. Its projections must pass through the like projections of the given point $(a, a')$ and be perpendicular to the corresponding traces of the plane. It should be noted, however, that the horizontal and vertical orthographic projections of any profile line (even if it is not perpendicular to a profile projecting plane) are always perpendicular to the traces of the plane. That is why in solving the problem one should begin with the profile plane of projection and ensure perpendicularity of the profile projection of the desired line to the profile trace of the plane; then find the other two projections from the profile one. Thus, we find the profile trace $(P_w)$ of the plane and the profile projection $(a'')$ of the point, and drop a perpendicular from $a''$ to $P_w$. Now cut from the profile projection of the line an arbitrary segment $a''b''$ and find its horizontal $(ab)$ and vertical $(a'b')$ projections.

**Example 156**

Given: a plane (specified by the parallel lines $AB$ and $CD$) and a point $K$. Drop a perpendicular from the given point to this plane (Fig. 535).

*Solution.* First draw arbitrarily a horizontal and a frontal line in the given plane; then draw the projections of the perpendicular: the horizontal one $(kl)$ through the point $k$ and perpendicular to the horizontal projection of the horizontal line (why?), and the vertical one $(k'l')$ through the point $k'$ and perpendicular to the vertical projection of the frontal line (why?).

**Example 157**

Given: a plane $P$ and a point $A$. Determine the distance from the point to the plane (Fig. 536).

*Solution.* As is known, the distance from a point to a plane is measured by a segment of a perpendicular from its foot on the plane to the given point. Drop a perpendicular from the point $(a, a')$ to the plane $P$ and find its foot $(b, b')$, i.e. the point of intersection of the perpendicular and the plane. Since the line-segment $(ab, a'b')$ is parallel to the horizontal plane of projection, its horizontal projection $(ab)$ yields the true distance required.

*Conclusion.* The distance from an arbitrary point to a horizontal projecting plane is orthographically measured by the distance from the horizontal projection of the point to the horizontal trace of the plane.

Likewise, we can draw the following conclusions:

(1) the distance from an arbitrary point to a vertical projecting plane is orthographically measured by the distance from the vertical projection of the point to the vertical trace of the plane;

(2) the distance from an arbitrary point to a profile projecting plane is orthographically measured by the distance from the profile projection of the point to the profile trace of the plane.

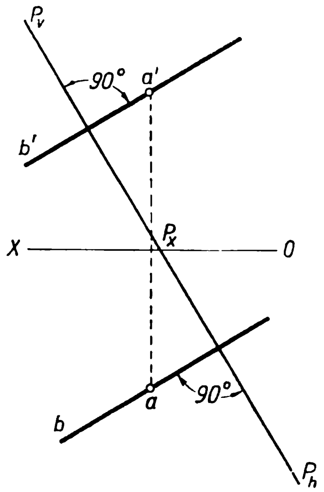


FIG. 533.

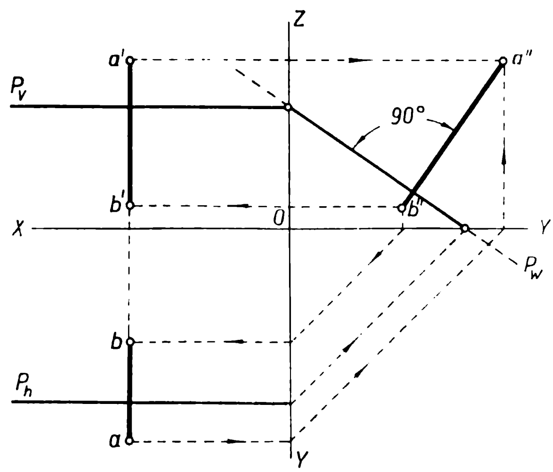


FIG. 534.

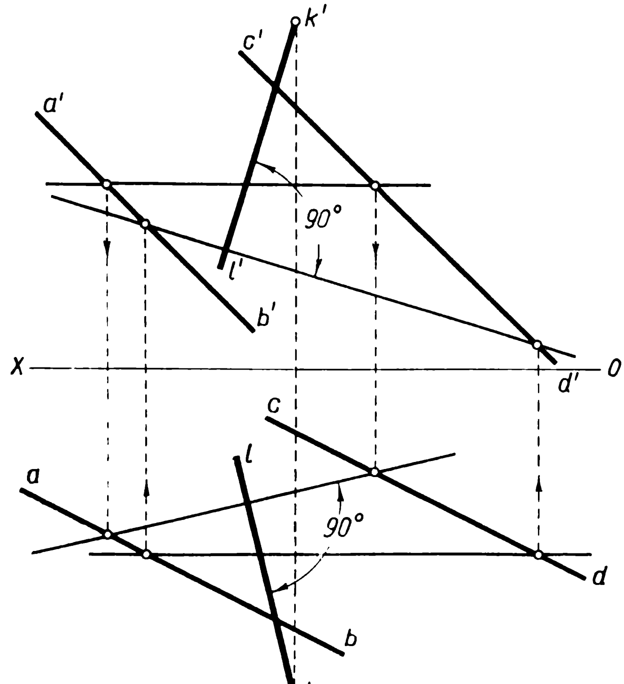


FIG. 535.

**Example 158**

Given: a plane $P$ and a point $A$. Determine the distance from the point to the plane (Fig. 537).

*Solution.* Drop a perpendicular from the point $(a, a')$ to the plane $P$ and find its foot in the plane, for which purpose determine the point $(b, b')$ of intersection of the perpendicular and the plane. Having the projections $(ab, a'b')$ of the segment of the perpendicular, determine its true length by constructing a triangle.

**Example 159**

Given: a triangle $ABC$ and a point $K$. Determine the distance from the point to the plane of the triangle (Fig. 538).

*Solution.* Drop a perpendicular from the given point $(k, k')$ to the plane of the triangle (see Example 156) and find point $(p, p')$, which is the foot of the perpendicular. Now determine the true length of the line-segment $(kp, k'p')$.

**Example 160**

Given: parallel planes $P$ and $Q$. Determine the distance between them (Fig. 539).

*Solution.* The general method of solution consists in taking an arbitrary point in either plane and finding the distance from it to the other plane (see Example 157).

*Conclusion.* The distance between parallel vertical projecting planes is orthographically measured by the distance between their vertical traces.

Similarly, the distance between horizontal projecting planes is orthographically measured by the distance between their horizontal traces; and the distance between parallel profile projecting planes is measured by the distance between their profile traces.

**Example 161**

Given: a plane $P$ and a point $A$ (in this plane) specified by its vertical projection. Erect at the point $P$ a perpendicular to the given plane and lay off on it a length of $l$ mm (Fig. 540).

*Solution.* Find the horizontal projection $(a)$ of the given point by means of, say, a horizontal line, and draw the projections of the required perpendicular to the plane through the point $(a, a')$. Cut from the perpendicular a hypothetical line-segment $(am, a'm')$, construct the latter in true length and lay off on it a line-segment $AN$, $l$ mm long. Then find its projections $(an, a'n')$.

**Example 162**

Given: a plane specified by the line $AB$ and the point $C$, and a point $K$ lying in the plane and specified by its horizontal projection. Erect at the point $K$ a perpendicular to the given plane and lay off on it a length of $l$ mm (Fig. 541).

*Solution.* First re-specify the given plane by two parallel lines $AB$ and $CD$. Find the vertical projection $(k')$ of the given point with the aid of a frontal line, and then draw a horizontal line of the plane through the point $(k, k')$. Construct the projections of the perpendicular to the plane: the horizontal one perpendicular to the horizontal projection of the horizontal line, and the vertical one perpendicular to the vertical projection of the frontal line. Cut from the perpendicular a hypothetical line-segment $(km, k'm')$, construct this segment in true length and on it lay off a segment $KN$, $l$ mm long. Then find its projections $(kn, k'n')$.

**Example 163**

Given: a line $AB$ and a point $P_x$. Construct the traces of the plane $P$, which is perpendicular to the line $AB$ (Fig. 542).

*Solution.* The desired plane is a horizontal projecting one. Draw its traces through the point $P_x$: $P_h$ perpendicular to $ab$ and $P_v$ to $a'b'$.

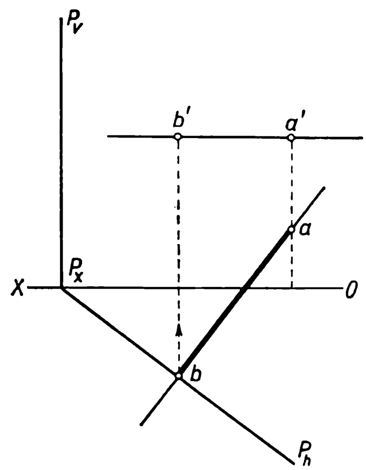



FIG. 536.

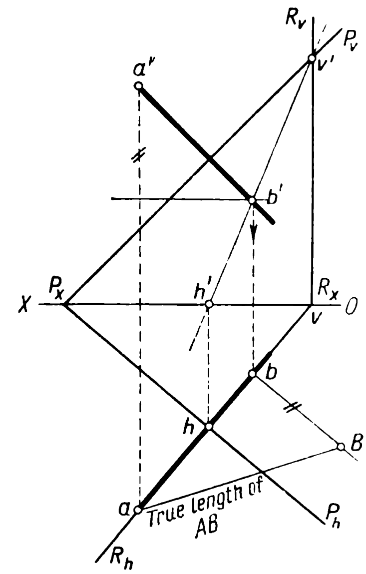



FIG. 537.

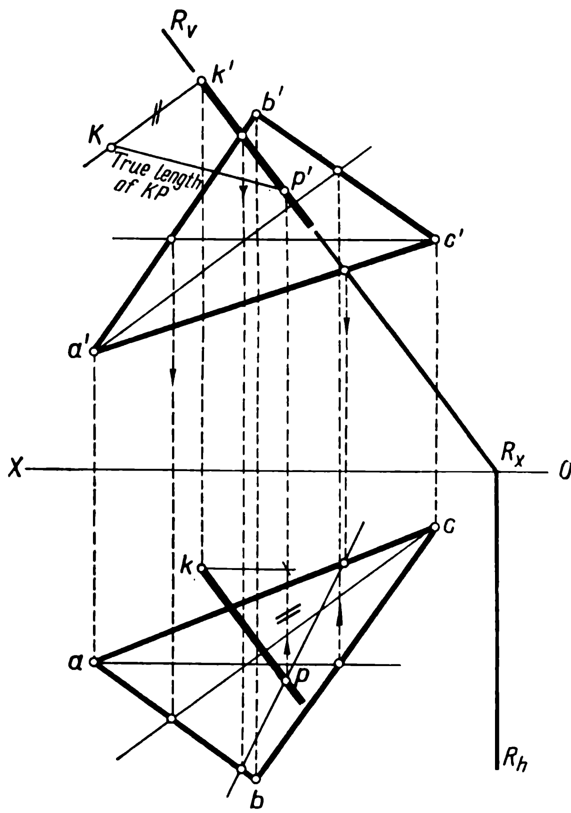



FIG.538.

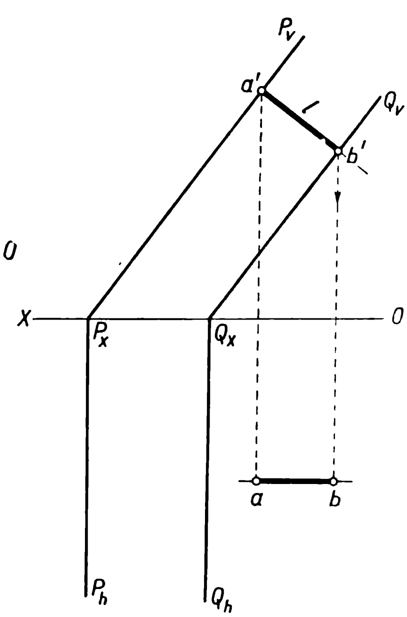



FIG.539.

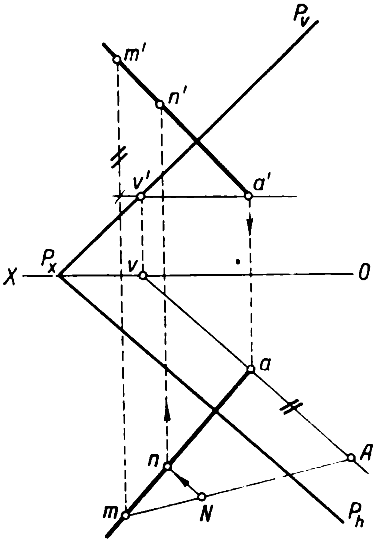



FIG. 540.

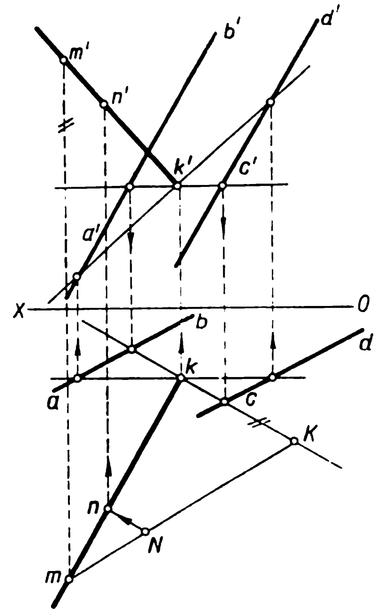



FIG. 541.

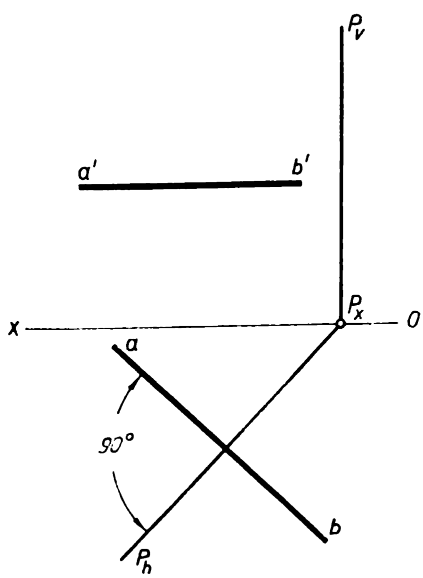



FIG. 542.

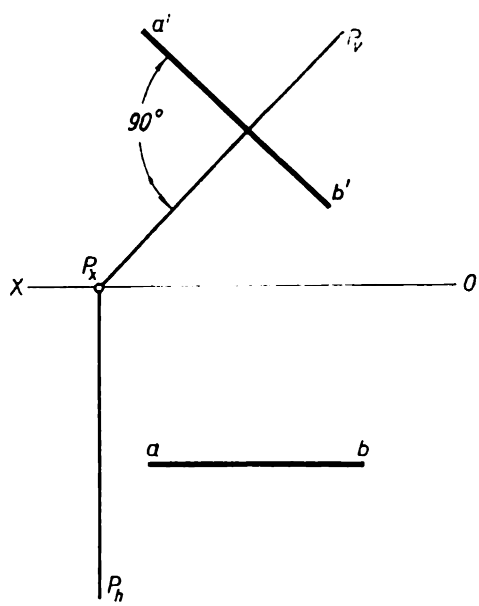



FIG. 543.

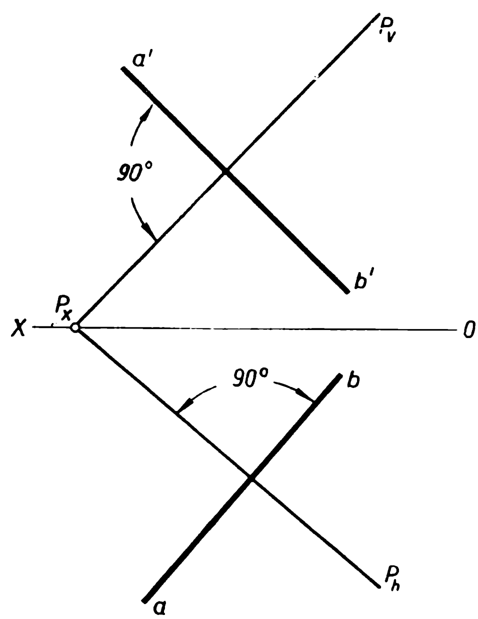



FIG. 544.

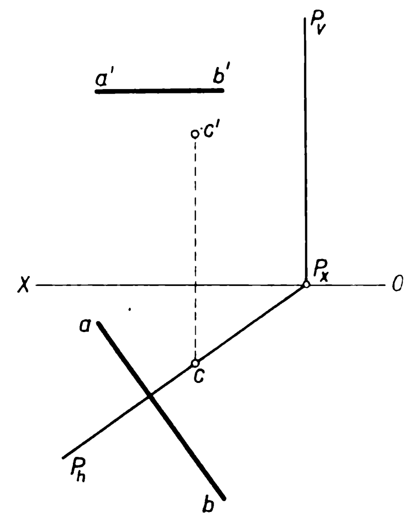



FIG. 545.

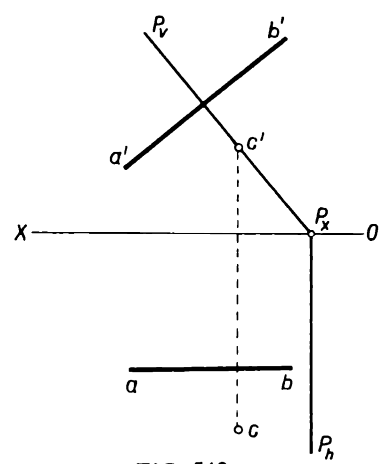



FIG. 546.

**Example 164**

Given: a line $AB$ and a point $P_x$. Construct the traces of the plane $P$, which is perpendicular to the line $AB$ (Fig. 543).

*Solution.* The desired plane is a vertical projecting one. Draw its traces through the point $P_x$ : $P_h$ perpendicular to $ab$, and $P_v$ to $a'b'$.

**Example 165**

Given: a line $AB$ and a point $P_x$. Construct the traces of the plane $P$, which is perpendicular to the line $AB$ (Fig. 544).

*Solution.* The desired plane is an oblique one. Draw its traces through the point $P_x$ : $P_h$ perpendicular to $ab$, and $P_v$ to $a'b'$.

**Example 166**

Given: a line $AB$ and a point $C$. Through $C$ pass a plane $P$ perpendicular to the line $AB$ (Fig. 545).

*Solution.* The desired plane is a horizontal projecting one. Since the plane must pass through the point ($c$, $c'$), first draw its horizontal trace ($P_h$) through the point $c$ and perpendicular to the line $ab$ to intersect the coordinate axis at the point $P_x$, and then the vertical trace ($P_v$) through this point and perpendicular to the line $a'b'$.

*Note.* If the point $P_x$ lies outside the limits of the drawing, it is not obligatory to draw the vertical trace of the plane (why?).

**Example 167**

Given: a line $AB$ and a point $C$. Through $C$ pass a plane $P$ perpendicular to the given line (Fig. 546).

*Solution.* The desired plane is a vertical projecting one. Since this plane must pass through the point ($c$, $c'$), first draw its vertical trace ($P_v$) through the point $c'$ and perpendicular to the line $a'b'$ to intersect the coordinate axis at the point $P_x$, and then the horizontal trace ($P_h$) through this point and perpendicular to the line $ab$.

*Note.* If the point $P_x$ lies outside the limits of the drawing, it is not obligatory to construct the horizontal trace of the plane (why?).

**Example 168**

Given: a line $AB$ and a point $C$. Through this point pass a plane $P$ perpendicular to line $AB$ (Fig. 547).

*Solution.* The required plane is a profile projecting one. Find the profile projection ($a''b''$) of the line and the profile projection ($c''$) of the point. Since this plane must pass through the point ($c$, $c'$), draw the profile trace ($P_w$) through the point $c''$ and perpendicular to $a''b''$, and then determine the other two traces ($P_h$ and $P_v$).

**Example 169**

Given: a line $AB$ and a point $C$. Pass, through $C$, a plane $P$ perpendicular to $AB$ (Figs. 548, 549).

*Solution.*

*First method.* Draw the projections of a horizontal line contained in the required plane through the like projections of the given point: the vertical one through the point $c'$ and parallel to the coordinate axis, and the horizontal one through the point $c$ and perpendicular to $ab$.

Find the trace ($v$, $v'$) of the horizontal line and draw the traces of the plane: first the vertical one ($P_v$) through the point $v'$ and perpendicular to $a'b'$ to intersect the coordinate axis at the point $P_x$, and then the horizontal one ($P_h$) through the latter point and perpendicular to $ab$.

*Second method.* Draw the projections of the frontal line contained in the desired plane through the like projections of the given point: the horizontal one through the point $c$ and parallel to the coordinate axis, and the vertical one through the point $c'$ and perpendicular to $a'b'$. Find the trace ($h$, $h'$) of the frontal line and draw the traces of the plane: first the horizontal one ($P_h$) through the point $h$ and perpendicular to $ab$ to intersect the coordinate axis at the point $P_x$, and then the vertical one ($P_v$) through this point and perpendicular to $a'b'$.

*Note.* Sometimes, when a horizontal or a frontal line is used, the point $P_x$ lies outside the limits of the drawing; in such cases the traces $P_h$ and $P_v$ of the required plane are constructed independently of each other, for which purpose both frontal and horizontal lines are used.

**Example 170**

Given: a line $AB$ and a point $C$. Drop a perpendicular from $C$ to $AB$ (Fig. 550).

A perpendicular can be dropped directly from a point to a line only when the given line is parallel to one of the projection planes (according to what theorem?).

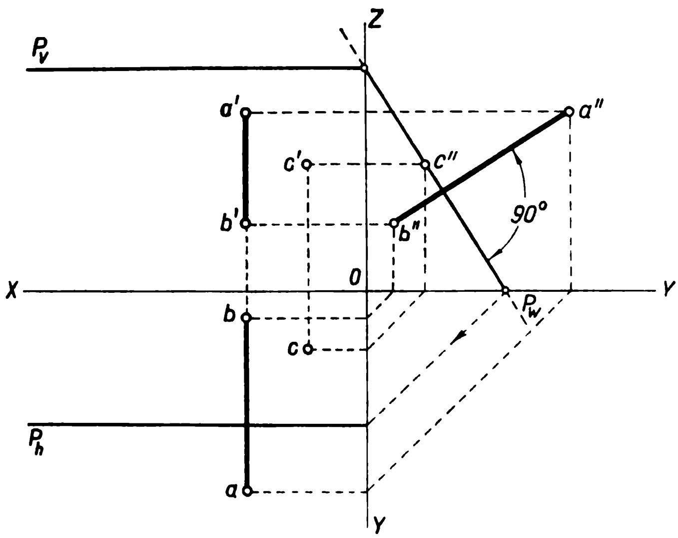



FIG. 547.

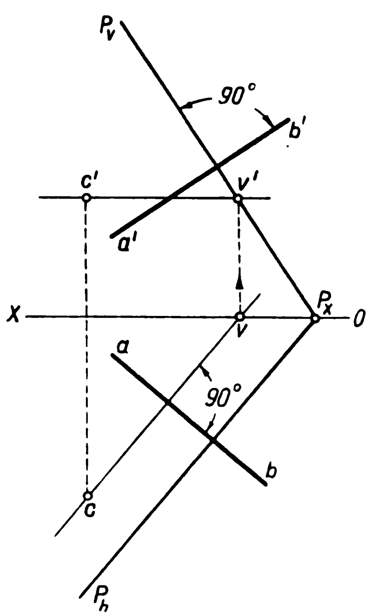



FIG. 548.

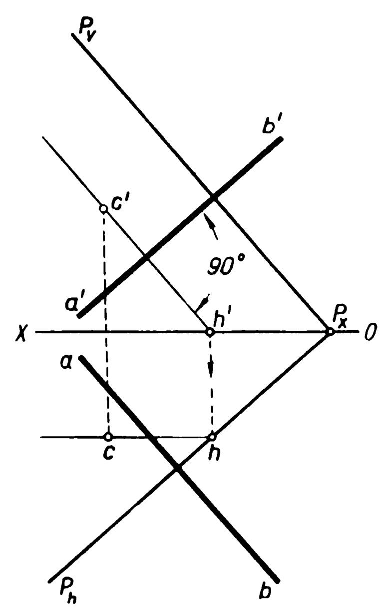



FIG. 549.

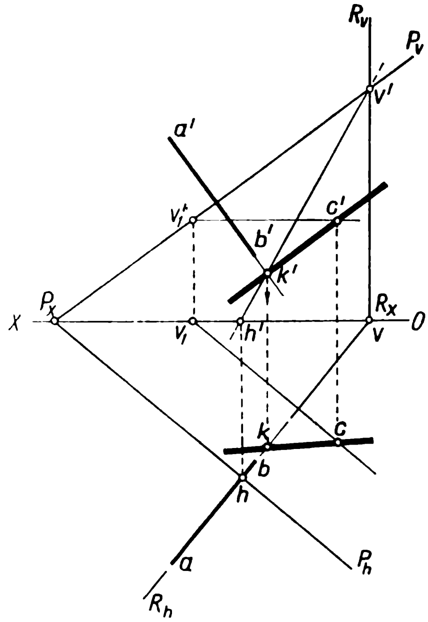



FIG. 550.

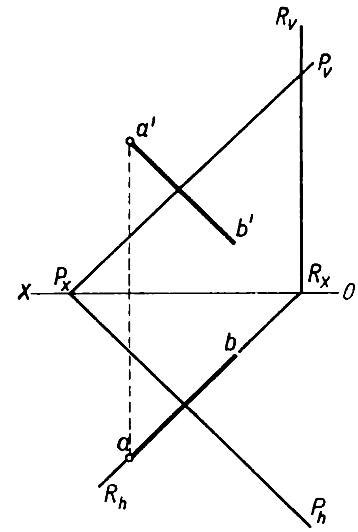



FIG. 551.

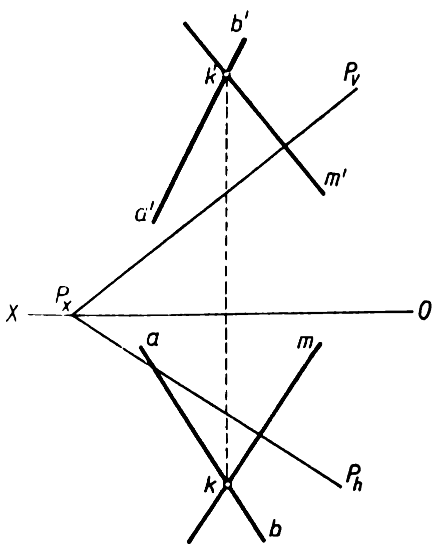



FIG. 552.

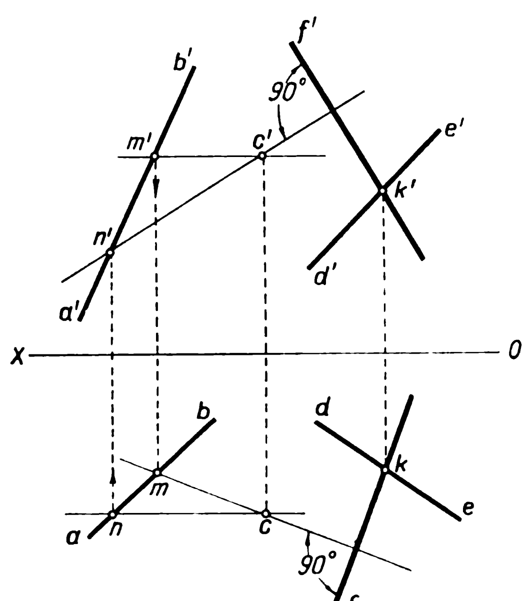



FIG. 553.

In the general case the problem is solved in the following way.

*Solution.* Through the given point ($c$, $c'$) pass a plane $P$ perpendicular to the line ($ab$, $a'b'$) and find the point ($k$, $k'$) of their intersection. A line passing through the points ($c$, $c'$) and ($k$, $k'$) is the desired one.

(To determine the distance from $C$ to $AB$, find the length of $CK$.)

**Example 171**

Given: a plane $P$ and a point $A$. Through $A$ pass a plane $R$ perpendicular to the given plane (Fig. 551).

*Solution.* To be perpendicular to the plane $P$ the plane $R$ must contain a line perpendicular to this plane. Through the point ($a$, $a'$) draw a line perpendicular to the plane $P$. Putting the line into the plane $R$, we obtain a plane perpendicular to the given plane (the problem is indeterminate).

*Conclusion.* Only one plane can be drawn through a point perpendicular to another plane, if the former plane is a projecting or oblique one but with coincident traces (why?).

**Example 172**

Given: a line $AB$ and a plane $P$. Through $AB$ pass a plane perpendicular to the given plane $P$ (Fig. 552).

*Solution.* From an arbitrary point ($k$, $k'$) on the given line drop a perpendicular to the plane $P$. The intersecting lines ($ak$, $a'k'$) and ($km$, $k'm'$) specify the desired plane.

**Example 173**

Given: a line $DE$ and a plane specified by the line $AB$ and the point $C$. Through $DE$ pass a plane perpendicular to the given plane (Fig. 553).

*Solution.* Through the point $C$ draw a horizontal and a frontal line and from an arbitrary point on the line, say ($k$, $k'$), drop a perpendicular to the plane, for which purpose draw, through the point $k$, a line $kf$ perpendicular to the horizontal projection of the horizontal line, and through the point $k'$, a line $k'f'$ perpendicular to the vertical projection of the frontal line. The desired plane is specified by the two intersecting lines ($ed$, $e'd'$) and ($kf$, $k'f'$).

**Example 174**

Given: a plane $P$. Construct the locus of points in space 40 mm distant from the given plane (Fig. 554).

*Solution.* The desired locus is a plane parallel to the given one and passing at a distance of 40 mm from it. Hence, proceed in the following manner:

(1) take an arbitrary point on the given plane;

(2) erect a perpendicular to the plane at this point;

(3) lay off on this perpendicular a segment 40 mm long (one solution is sufficient);

(4) pass through the end point of the perpendicular a plane parallel to the given one.

**Example 175**

Construct the locus of points lying in the plane $P$ and 40 mm distant from the plane $Q$ (Fig. 555).

*Solution.* The desired locus is the line of intersection of the plane $P$ and a plane $R$ parallel to the given plane $Q$ and at the given distance from it. Hence, make the following constructions:

(1) take an arbitrary point on the plane $Q$;

(2) erect a perpendicular to the plane $Q$ at this point;

(3) lay off on the perpendicular a segment 40 mm long (one solution is sufficient);

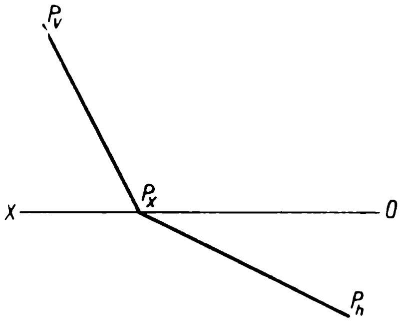



FIG. 554.

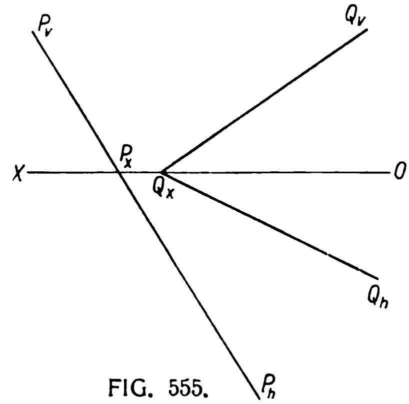



FIG. 555.

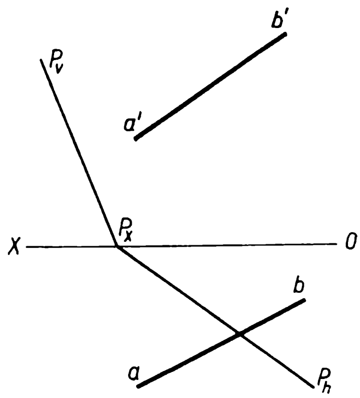



FIG. 556.

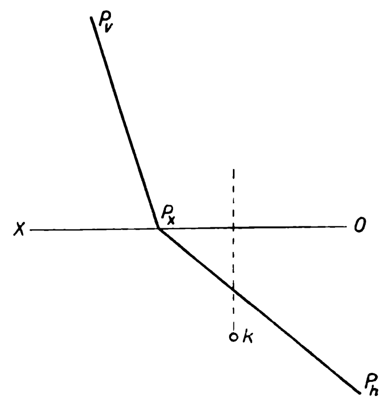



FIG. 557.

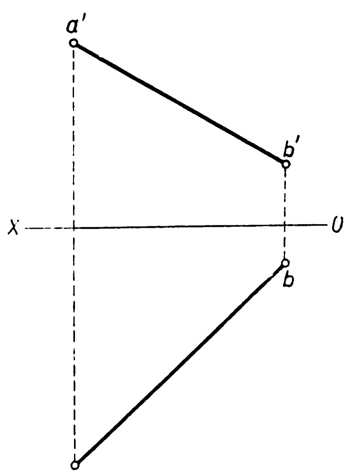



FIG. 558.

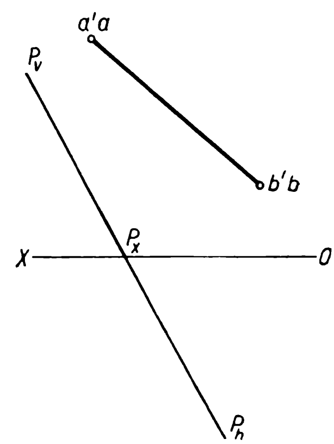



FIG. 559.

(4) pass through the end point of the perpendicular a plane *R* parallel to the plane *Q*;

(5) find the line of intersection of the planes *P* and *R*; this line is the required locus.

**Example 176**

On the line *AB* find a point 40 mm distant from the plane *P* (Fig. 556).

*Solution.* The desired point is the point of intersection of the line *AB* and a plane *R* parallel to the given plane *P* and at a distance of 40 mm from it. Hence, make the following constructions:

(1) take an arbitrary point on the given plane;

(2) erect a perpendicular to the plane at this point;

(3) lay off on the perpendicular a segment 40 mm long (one solution is sufficient);

(4) pass through the end point of the perpendicular a plane *R* parallel to the plane *P*;

(5) find the point of intersection of the line *AB* and the plane *R*.

**Example 177**

Find the lacking projection of the point *K*, which is 40 mm distant from the plane *P* (Fig. 557).

*Solution.* The desired projection of the point *K* is to be found as the lacking projection (with one projection given) of an arbitrary point lying in a plane *R* parallel to the given plane and at a distance of 40 mm from it. Hence, proceed as follows:

(1) take an arbitrary point on the given plane;

(2) erect a perpendicular to the plane at this point;

(3) lay off on the perpendicular a segment 40 mm long (one solution is sufficient);

(4) pass through the end point of the perpendicular a plane *R* parallel to the plane *P*;

(5) find the lacking projection of the point *K* (lying in the plane *R*) from the given projection.

**Example 178**

Construct the locus of points equidistant from the endpoints of the line-segment *AB* (Fig. 558).

*Solution.* The desired locus is a plane perpendicular to the given segment and passing through its mid-point. Hence, proceed as follows:

(1) bisect the given line-segment at point *K*;

(2) pass through this point a plane perpendicular to the given segment.

**Example 179**

In the plane *P* construct the locus of points equidistant from the endpoints of the line-segment *AB* (Fig. 559).

*Solution.* The desired locus is the line of intersection of the plane *P* and a plane *R* perpendicular to the line-segment *AB* and passing through its mid-point. Hence, proceed as follows:

(1) bisect the given line-segment at point *K*;

(2) pass through this point a plane *R* perpendicular to the given segment;

(3) find the line of intersection of the planes *P* and *R*.

**Example 180**

On the line-segment *CD* find a point equidistant from the endpoints of the line-segment *AB* (Fig. 560).

*Solution.* The desired point is the point of intersection of the line-segment *CD* and a plane *R* perpendicular to *AB* and passing through its mid-point. Hence, proceed as follows:

(1) bisect the given line-segment at point $K$;
(2) pass through this point a plane $R$ perpendicular to the given line-segment;
(3) find the point of intersection of the segment $CD$ and the plane $R$.

**Example 181**

Find the lacking projection of the point $K$ equidistant from the endpoints of the line-segment $AB$ (Fig. 561).

*Solution.* The desired projection of the point $K$ is found as the lacking projection (with one of its projections given) of an arbitrary point lying in a plane $R$, which is perpendicular to the line-segment $AB$ and passes through its mid-point. Hence, proceed as follows:

(1) bisect the given line-segment at point $M$;
(2) through this point pass a plane $R$ perpendicular to the given line-segment;
(3) from the given projection of the point $K$ (lying in the plane $R$) find its other projection.

**Example 182**

On the line-segment $AB$ find a point 40 mm distant from the point $K$ (Fig. 562).

*Solution.* In the general case there are two such points, say, $M$ and $N$, which are the vertices of an isosceles triangle $KMN$ with the base $MN$ lying on the line-segment $AB$. Hence proceed as follows:

(1) drop a perpendicular from the point $K$ to the line $AB$ and find the point $D$ (the foot of the perpendicular) belonging to the altitude $KD$;
(2) determine the true length of the altitude $KD$ and make a separate drawing of the auxiliary triangle $KMN$ true size with the side 40 mm long;
(3) lay off on the line $AB$ (from point $D$) line-segments $DM$ and $DN$. The points $M$ and $N$ thus obtained are the desired ones.

What other cases are possible?

**Example 183**

Through the point $K$ draw a line intersecting the line-segment $AB$ at a given angle $\varphi$ (Fig. 562).

*Solution.* There are two such lines, say $KM$ and $KN$, which are the sides of an isosceles triangle $KMN$ with the base $MN$ lying on $AB$ and with angles $\varphi$ at the base. Hence, proceed as follows:

(1) drop a perpendicular from the point $K$ to the line $AB$ to find the point $D$ (the foot of the perpendicular) belonging to the altitude $KD$;
(2) determine the true length of the altitude $KD$ and make a separate drawing, in true size, of the auxiliary triangle $KMN$ with an angle $\varphi$ at its base;
(3) lay off on the line $AB$ (from the point $D$) line-segments $DM$ and $DN$ and join $K$ to $M$ and to $N$.

**Example 184**

Find the lacking projection of the line-segment $CD$, which intersects the line-segment $AB$, given that these lines are mutually perpendicular (Fig. 563).

*Solution.* The locus of straight lines in space perpendicular to an arbitrary line and intersecting it is a plane $R$ perpendicular to this line and passing through the point of intersection of the lines.

The desired projection of the line $CD$ is found as the lacking projection of an arbitrary line contained in the plane $R$. Hence, proceed as follows:

(1) determine the projections of the point of intersection of the lines;
(2) pass through this point a plane $R$ perpendicular to the line-segment $AB$;
(3) find the lacking projection of the line lying in the plane $R$.

**Example 185**

Determine the distance between the skew lines $AB$ and $CD$ (Fig. 564).

*Solution.* The distance between skew lines is measured by the distance between the planes of parallelism, or, in other words, by the distance from one of the lines

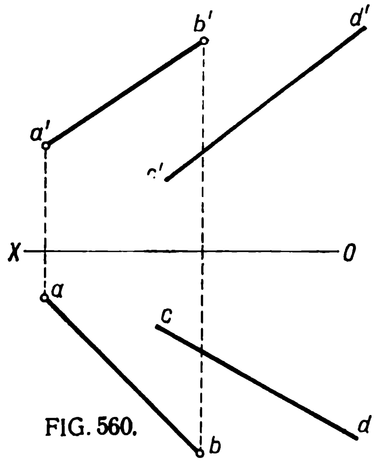



FIG. 560.

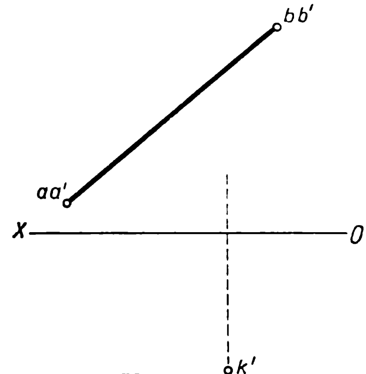



FIG. 561.

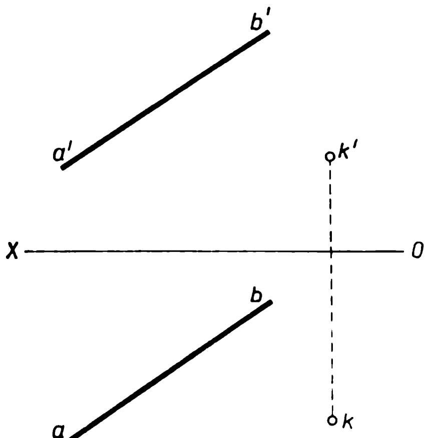



FIG. 562.

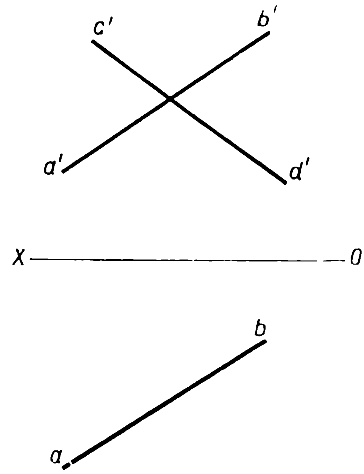



FIG. 563.

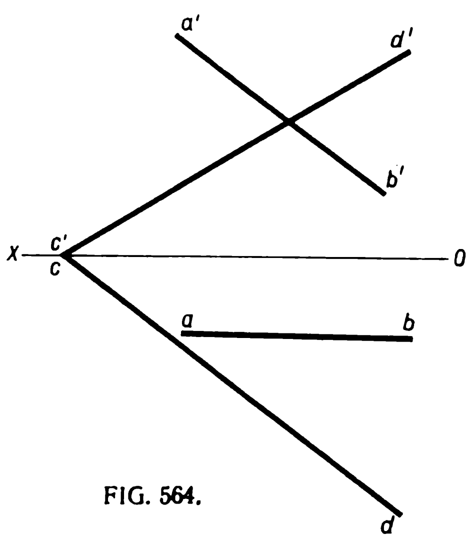



FIG. 564.

to a plane passing through the other line and parallel to the first one. Hence, proceed as follows:

(1) pass through the line-segment $AB$ a plane parallel to the line $CD$ (one of the planes of parallelism);

(2) take an arbitrary point $K$ on the line-segment $CD$;

(3) determine the distance from this point to the plane.

*Note*. The above method makes it possible to determine only the distance between the given skew lines, but not the position ensuring the *shortest* distance.

## PROBLEMS

251. Drop a perpendicular from the point $K$ to the plane $P$ (Figs. 560 to 569).

252. Drop a perpendicular from the point $K$ to the plane of the triangle $ABC$ (Figs. 570, 571).

253. Drop a perpendicular from the point $K$ to the plane specified by the parallel lines $AB$ and $CD$ (Fig. 572).

254. Drop a perpendicular from the point $K$ to the plane specified by the line $AB$ and the point $C$ (Fig. 573).

255. Determine the distance from the point $K$ to the plane $P$ (Figs. 565 to 569).

256. Determine the distance from the point $K$ to the plane of the triangle $ABC$ (Figs. 570, 571).

257. Determine the distance from the point $K$ to the plane specified by the parallel lines $AB$ and $CD$ (Fig. 572).

258. Determine the distance from the point $K$ to the plane specified by the line $AB$ and the point $C$ (Fig. 573).

259. Determine the distance between the parallel planes $P$ and $Q$ (Figs. 574 to 577).

260. Determine the distance between the skew lines $AB$ and $CD$ (Fig. 578).

261. Find the lacking trace of the plane $P$, given that it is 15 mm distant from the point $K$ (Figs. 579 to 581).

262. From the centre $C$ describe a sphere tangent to the plane $P$ (Figs. 582 to 584).

263. From the centre $C$ describe a sphere tangent to the plane specified by the line $MN$ and the point $K$ (Fig. 585).

264. From the centre $C$ describe a sphere tangent to the plane specified by the parallel lines $KL$ and $MN$ (Fig. 586).

265. From the centre $C$ describe a sphere tangent to the plane of the triangle $KMN$ (Fig. 587).

266. At point $K$ erect a perpendicular ($l = 40$ mm) to the plane $P$, given the horizontal projection of the point $K$ (Fig. 587).

267. At point $K$ erect a perpendicular ($l = 40$ mm) to the plane specified by the line $AB$ and the point $C$, given the horizontal projection of the point $K$ (Fig. 588).

268. At point $K$ erect a perpendicular ($l = 40$ mm) to the plane specified by the parallel lines $AB$ and $CD$, given the vertical projection of the point $K$ (Fig. 589).

269. At point $K$ erect a perpendicular ($l = 40$ mm) to the plane of the triangle $ABC$, given the vertical projection of the point $K$ (Fig. 590).

270. Construct the projections of a right trihedral prism with its base $ABC$ lying in the plane $P$, given the horizontal projection of the base and the altitude $h = 40$ mm (Fig. 591).

271. Construct the locus of points in space 20 mm distant from the plane $P$ (Figs. 592 to 597).

272. Construct the locus of points in space 30 mm distant from the plane specified by the line $AB$ and the point $C$ (Fig. 598).

273. Construct the locus of points in space 30 mm distant from the plane specified by the parallel lines $AB$ and $CD$ (Fig. 599).

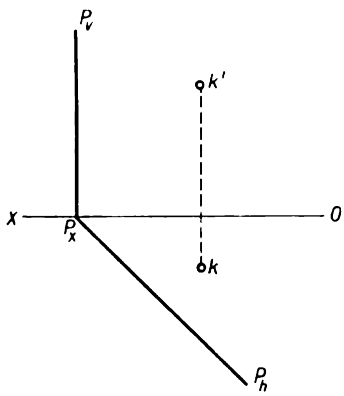



FIG. 565.

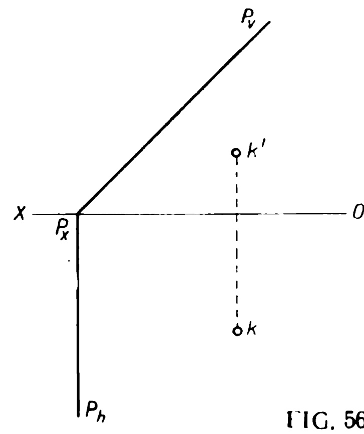



FIG. 566.

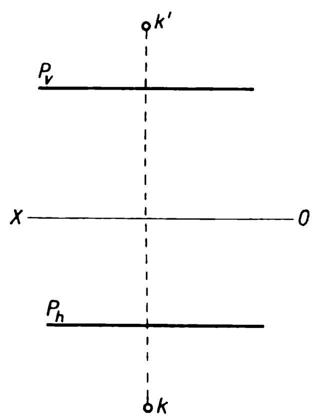



FIG. 567.

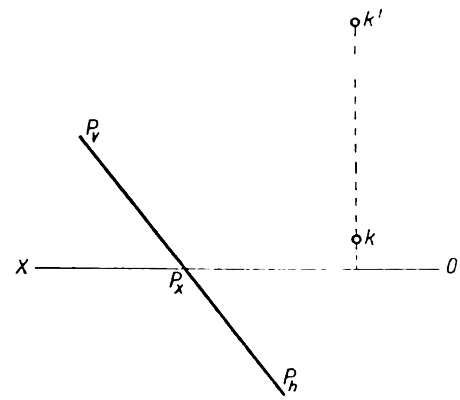



FIG. 568.

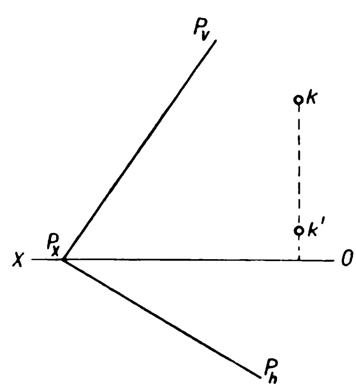



FIG. 569.

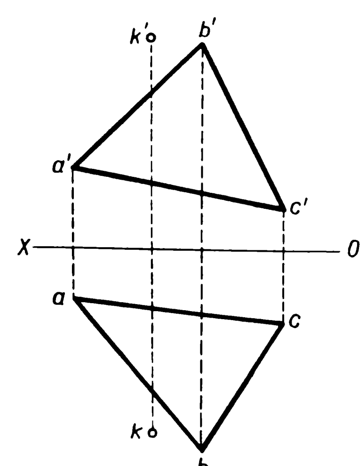



FIG. 570.

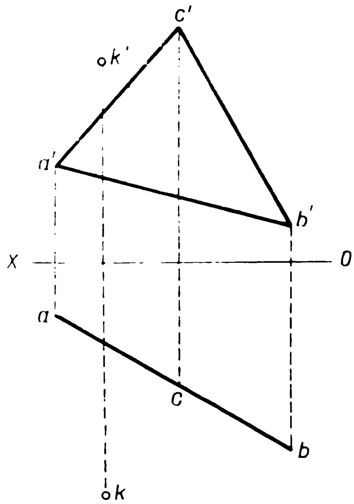



FIG. 571.

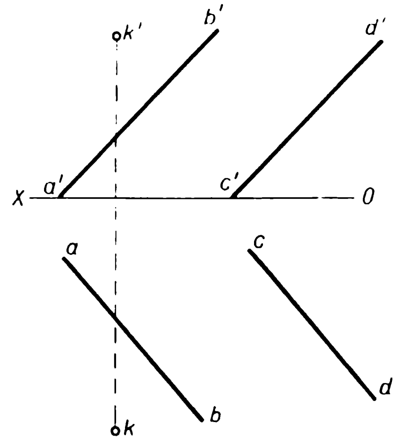



FIG. 572.

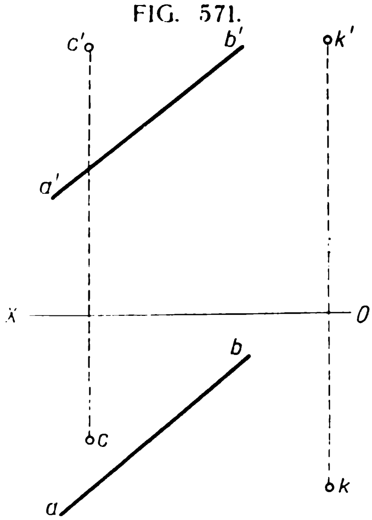



FIG 573.

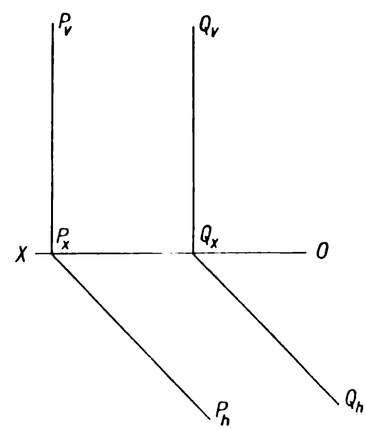



FIG. 574.

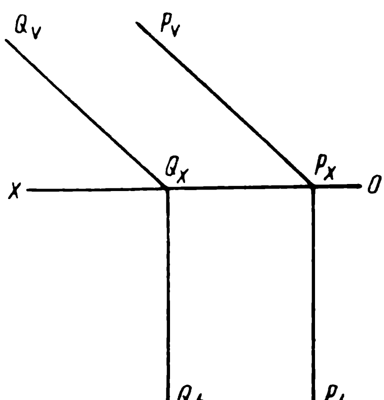



FIG. 575.

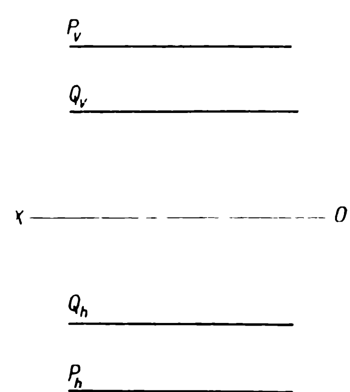



FIG. 576.

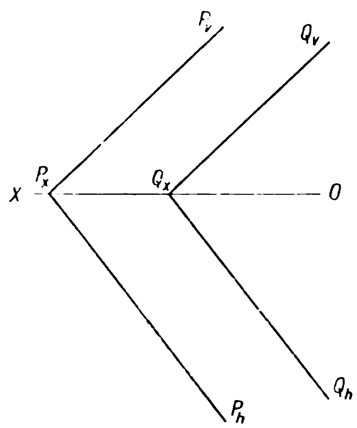



FIG. 577.

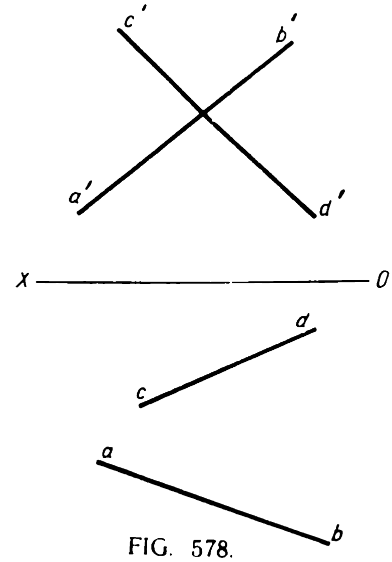



FIG. 578.

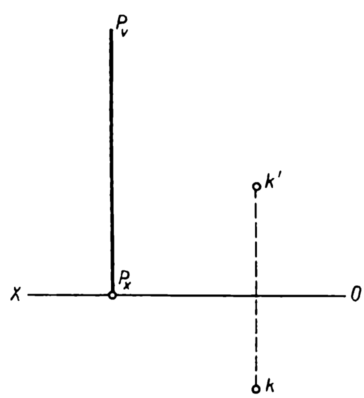



FIG. 579.

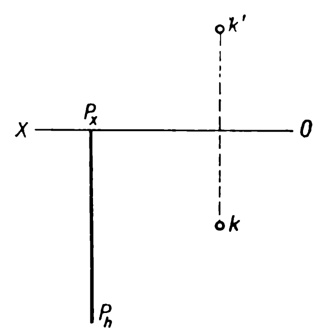



FIG. 580.

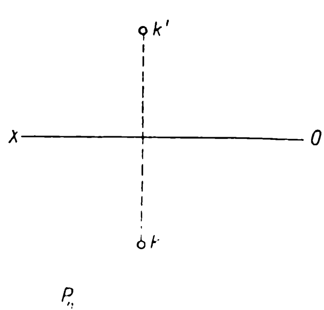



FIG. 581.

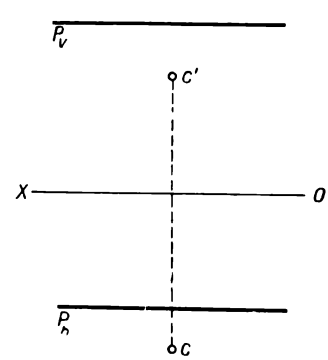



FIG. 582.

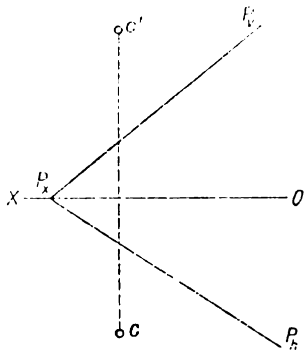



FIG. 583.

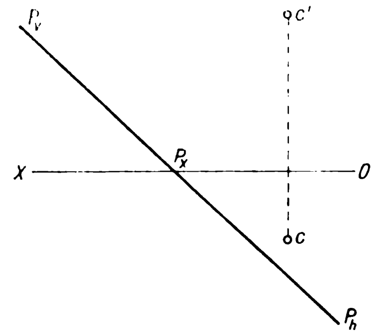



FIG. 584.

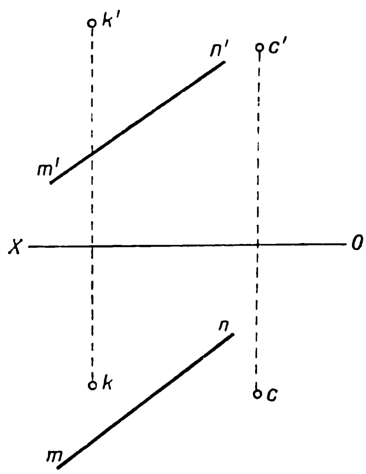



FIG. 585.

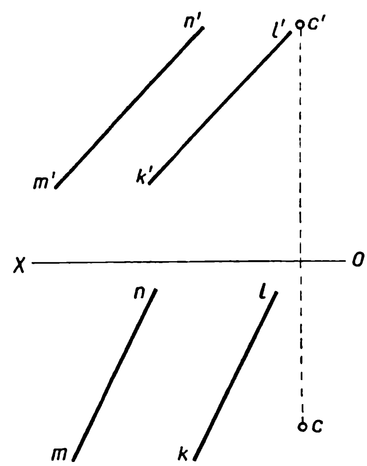



FIG. 586.

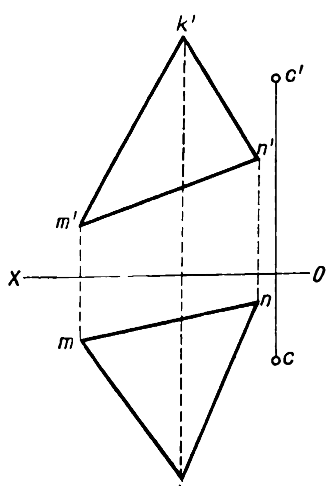



FIG. 587.

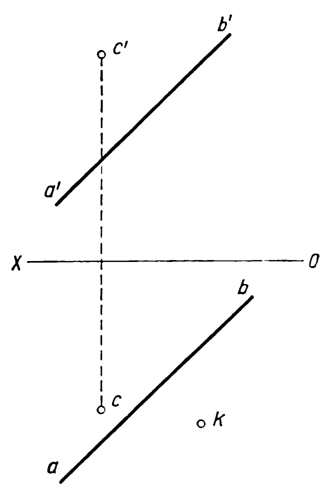



FIG. 588.

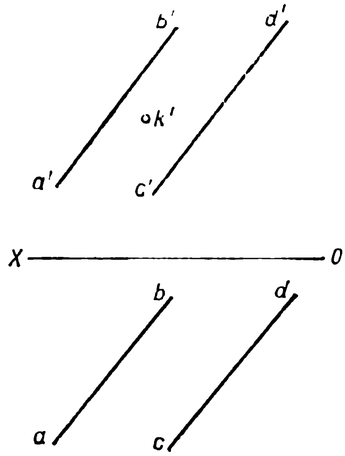



FIG. 589.

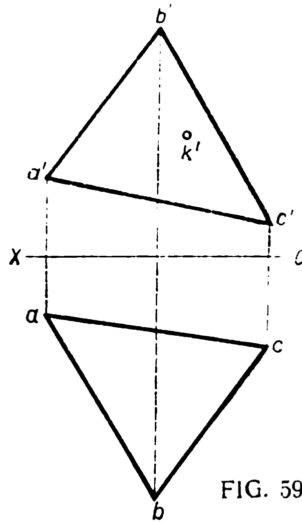



FIG. 590.

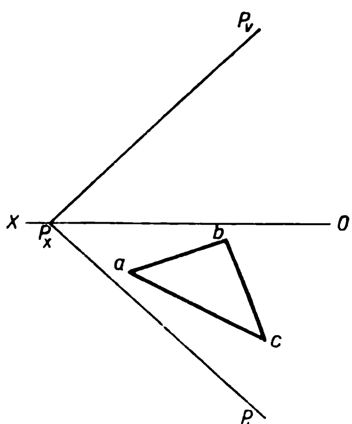



FIG. 591.

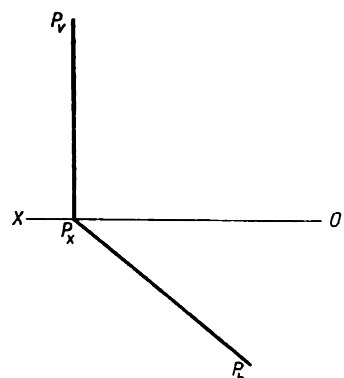



FIG. 592.

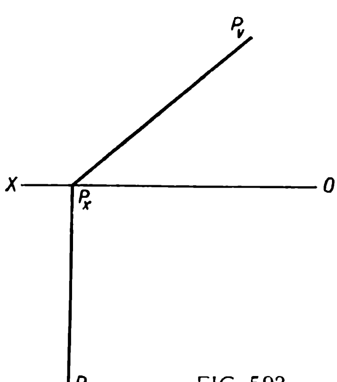



FIG. 593.

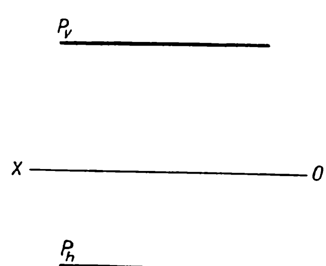



FIG. 594.

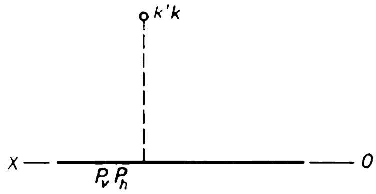



FIG. 595.

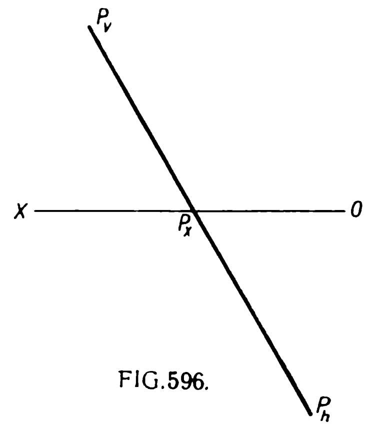



FIG. 596.

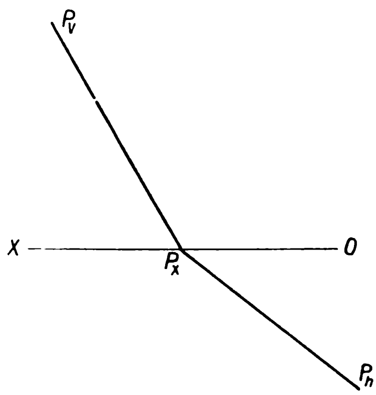



FIG. 597.

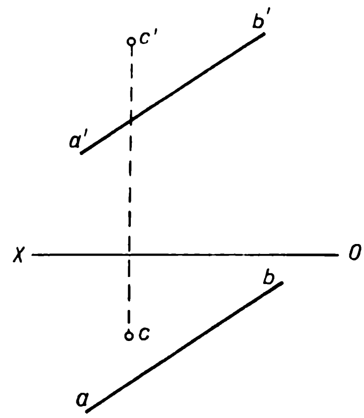



FIG. 598.

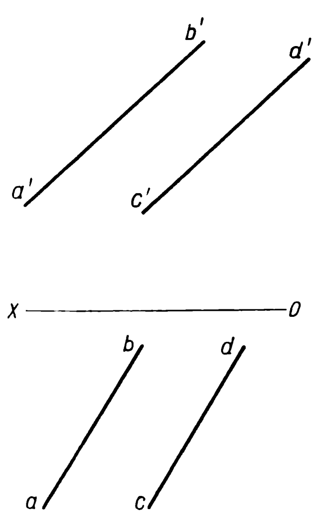



FIG. 599.

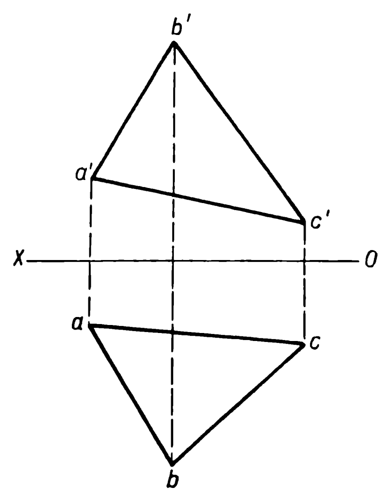



FIG. 600.

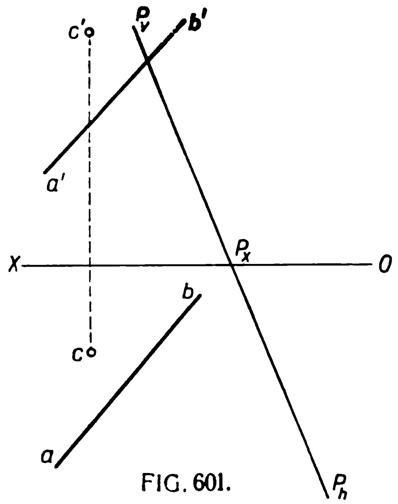



FIG. 601.

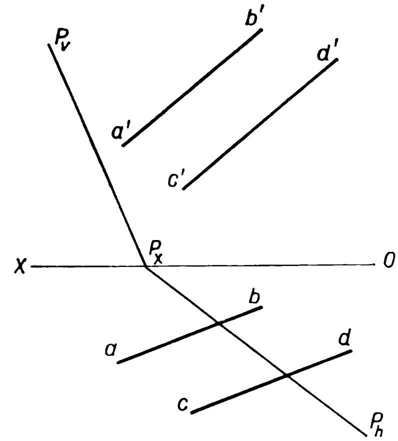



FIG. 602.

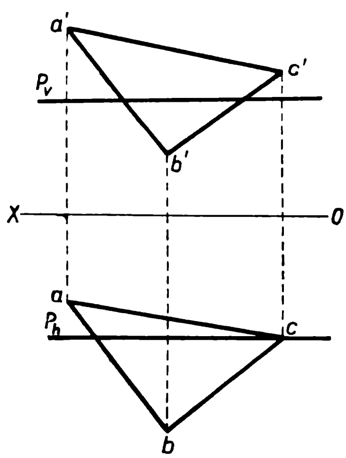



FIG. 603.

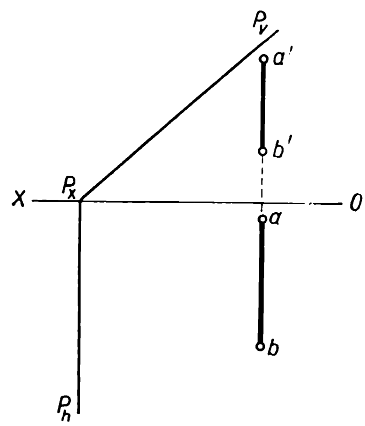



FIG. 604.

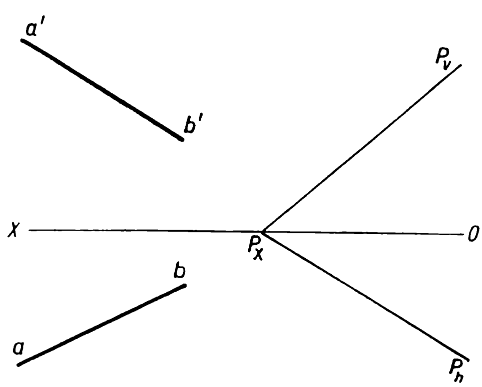



FIG. 605.

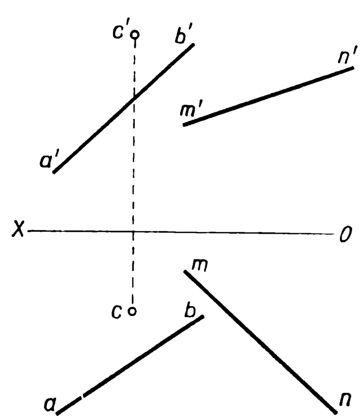



FIG. 606.

274. Construct the locus of points in space 30 mm distant from the plane of the triangle $ABC$ (Fig. 600).

275. Construct in the plane $Q$ the locus of points 30 mm distant from the plane $P$ (Figs. 393, 394).

276. Construct in the plane specified by the line $AB$ and the point $C$ the locus of points 25 mm distant from the plane $P$ (Fig. 601).

277. Construct in the plane specified by the parallel lines $AB$ and $CD$ the locus of points 25 mm distant from the plane $P$ (Fig. 602).

278. Construct in the plane of the triangle $ABC$ the locus of points 25 mm distant from the plane $P$ (Fig. 603).

279. Construct in the plane $P$ the locus of points 25 mm distant from the plane specified by the line $AB$ and the point $C$ (Fig. 601).

280. Construct in the plane $P$ the locus of points 30 mm distant from the plane specified by the parallel lines $AB$ and $CD$ (Fig. 602).

281. Construct in the plane $P$ the locus of points 20 mm distant from the plane of the triangle $ABC$ (Fig. 603).

282. Find on the line-segment $AB$ a point 25 mm distant from the plane $P$ (Figs. 604, 605).

283. Find on the line-segment $MN$ a point 20 mm distant from the plane specified by the line $AB$ and the point $C$ (Fig. 606).

284. Find on the line-segment $MN$ a point 25 mm distant from the plane specified by the parallel lines $AB$ and $CD$ (Fig. 607).

285. Find on the line-segment $MN$ a point 25 mm distant from the plane of the triangle $ABC$ (Fig. 608).

286. Determine the lacking projection of the point $A$ 25 mm distant from the plane $P$ (Figs. 609 to 612).

287. Determine the lacking projection of the point $K$ 20 mm distant from the plane specified by the line $AB$ and the point $C$ (Fig. 613).

288. Determine the lacking projection of the point $K$ 25 mm distant from the plane specified by the parallel lines $AB$ and $CD$ (Fig. 614).

289. Determine the lacking projection of the point $K$ 25 mm distant from the plane of the triangle $ABC$ (Fig. 615).

290. Construct the traces of the plane $P$ perpendicular to the line-segment $AB$, given the vanishing point of the plane (Figs. 616 to 619).

291. Construct the lacking trace (element) of the plane $P$ perpendicular to the line-segment $AB$ (Figs. 620, 621).

292. Construct the traces of a plane passing through the point $C$ and perpendicular to the line-segment $AB$ (Figs. 622 to 624).

293. Are the lines $AB$ and $CD$ perpendicular (Fig. 625)? (You need not find the angle between the lines.)

294. Determine the lacking projection of the line $CD$, given that it intersects the line $AB$ at right angles (Fig. 563).

295. Through the point $A$ pass a plane perpendicular to the planes $P$ and $Q$ (Fig. 626).

296. Construct the locus of points in space equidistant from the endpoints of the line-segment $AB$ (Figs. 558, 627).

297. Construct in the plane $P$ the locus of points equidistant from the endpoints of the line-segment $AB$ (Fig. 628).

298. Construct in the plane specified by the line $AB$ and the point $C$ the locus of points equidistant from the endpoints of the line-segment $MN$ (Fig. 629).

299. Construct in the plane specified by the parallel lines $AB$ and $CD$ the locus of points equidistant from the endpoints of the line-segment $MN$ (Fig. 630).

300. Construct in the plane of the triangle $ABC$ the locus of points equidistant from the endpoints of the line-segment $MN$ (Fig. 631).

301. Find the lacking projection of the point $K$ equidistant from the endpoints of the line-segment $AB$ (Figs. 632, 633).

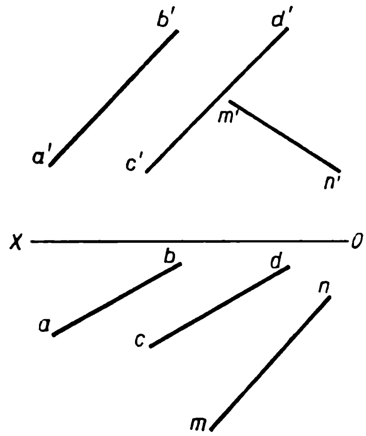



FIG. 607.

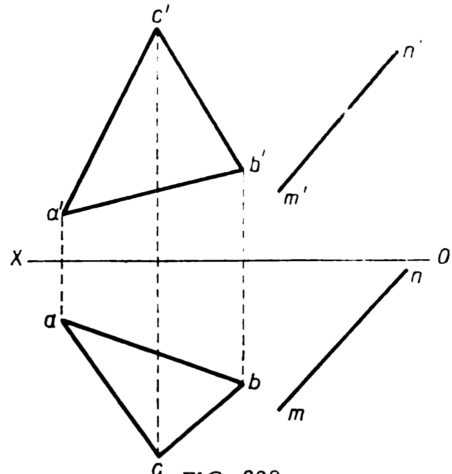



FIG. 608.

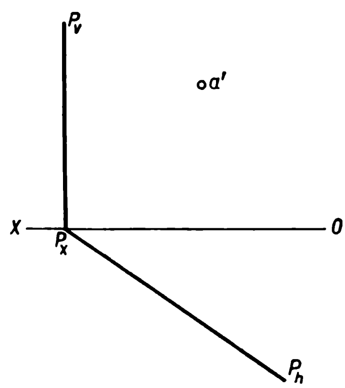



FIG. 609.

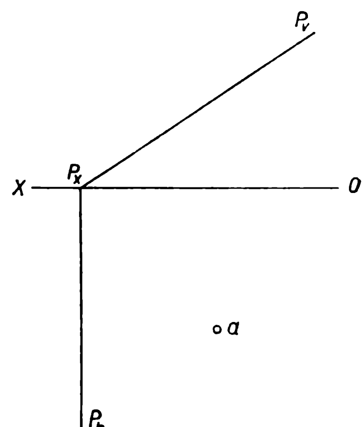



FIG. 610.

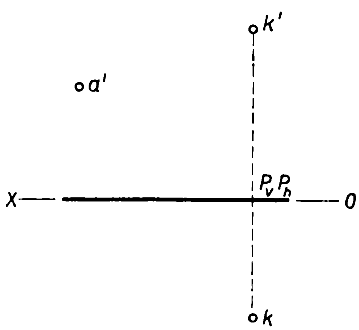



FIG. 611.

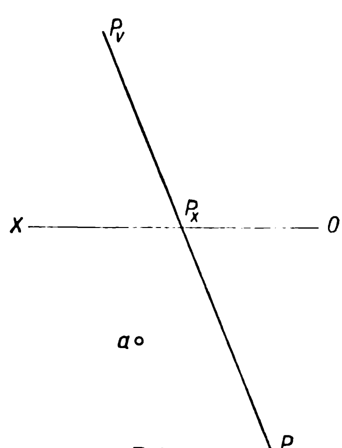



FIG. 612.

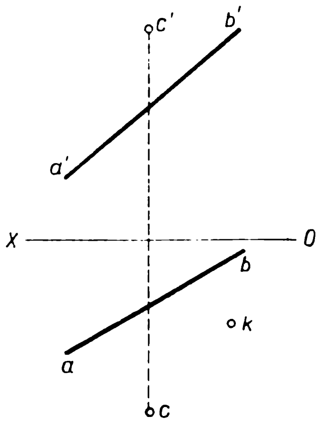



FIG. 613.

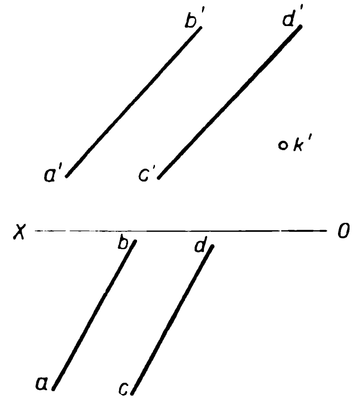



FIG. 614.

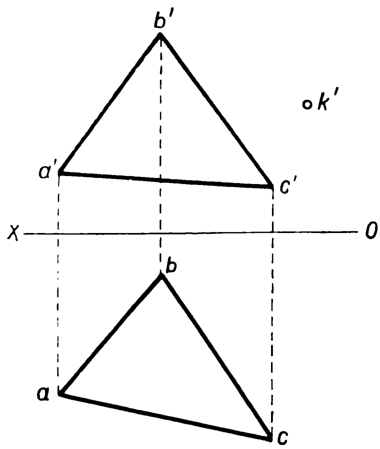



FIG. 615.

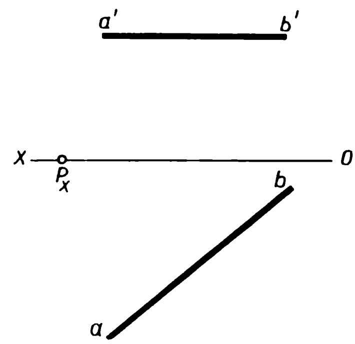



FIG. 616.

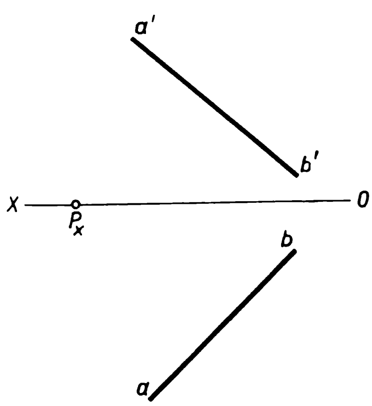



FIG. 617.

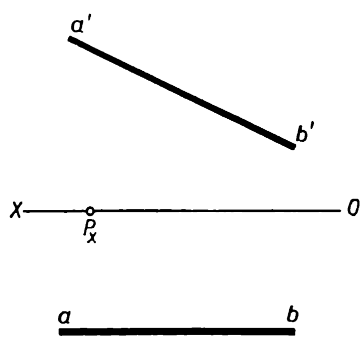



FIG. 618.

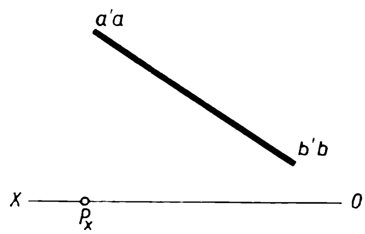



FIG. 619.

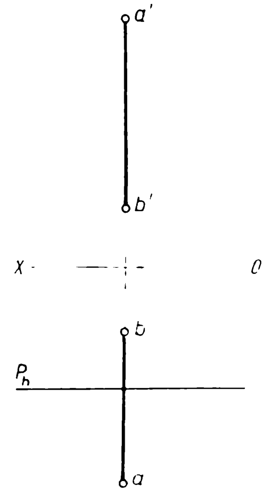



FIG. 620.

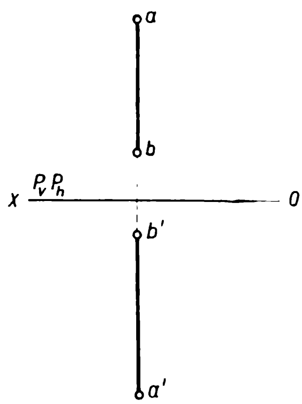



FIG. 621.

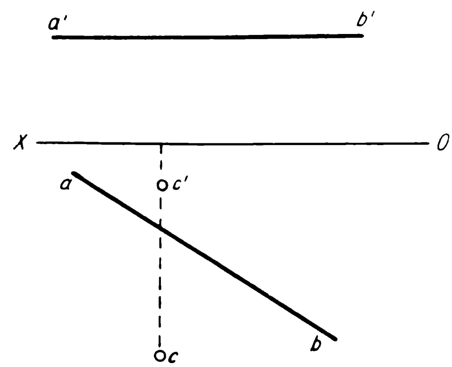



FIG. 622.

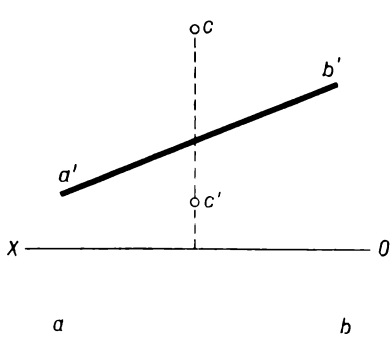



FIG. 623.

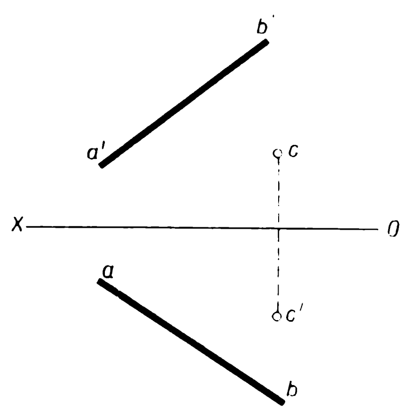



FIG. 624.

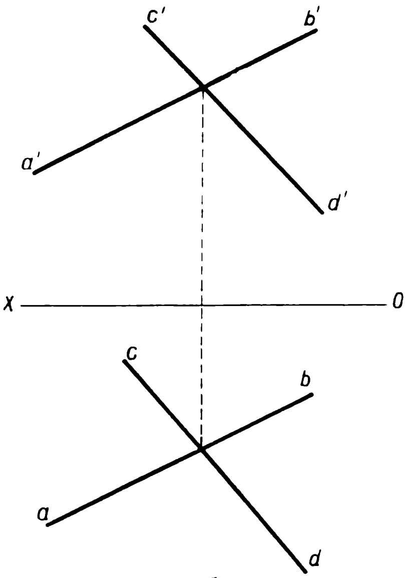



FIG. 625.

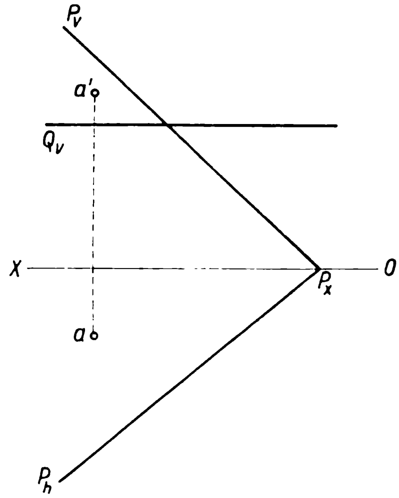



FIG. 626.

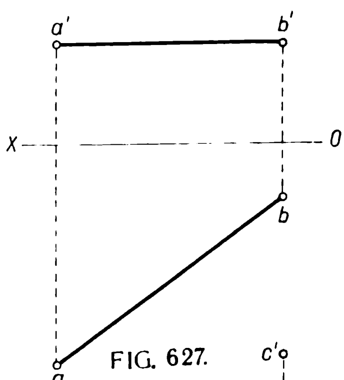



FIG. 627.

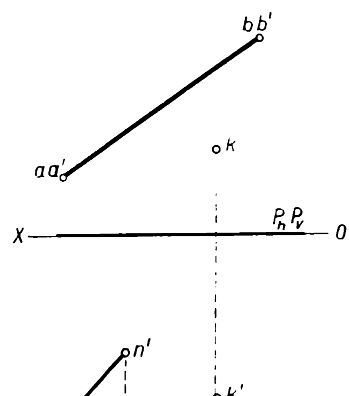



FIG. 628.

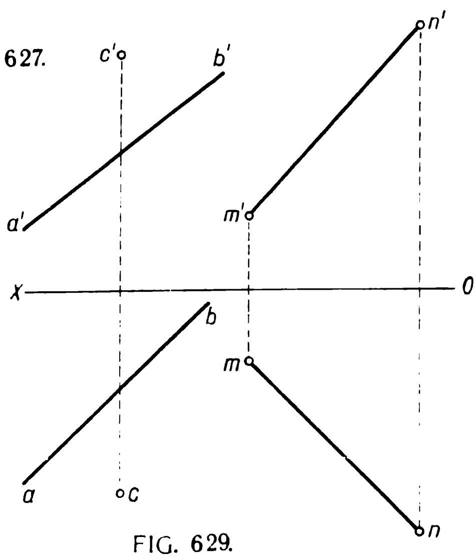



FIG. 629.

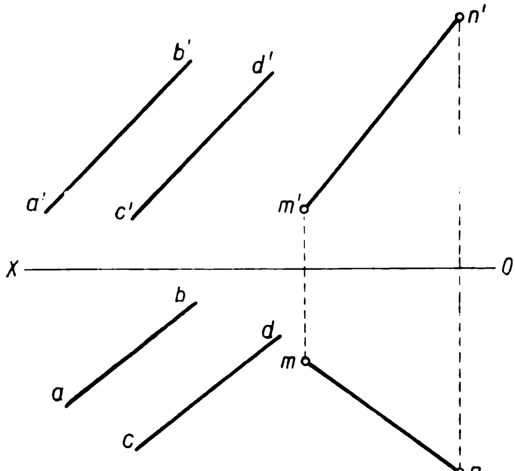



FIG. 630.

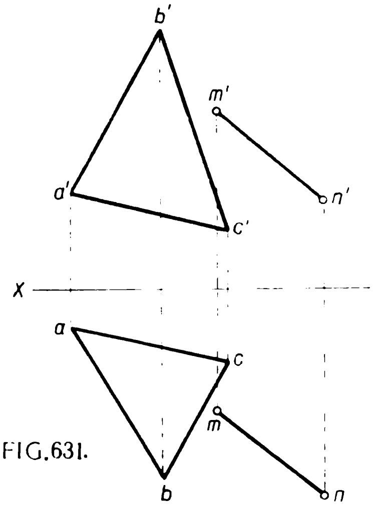



FIG. 631.

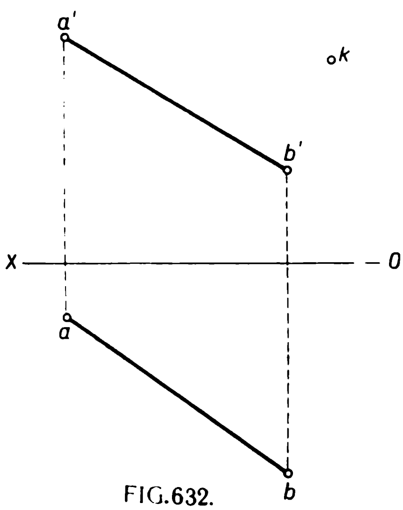



FIG. 632.

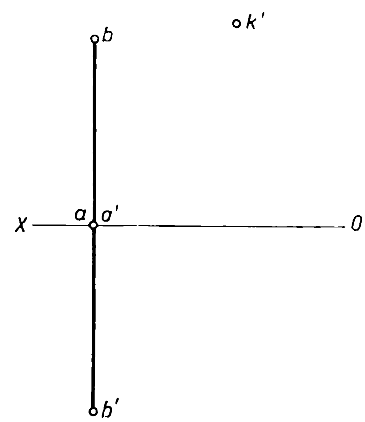



FIG. 633.

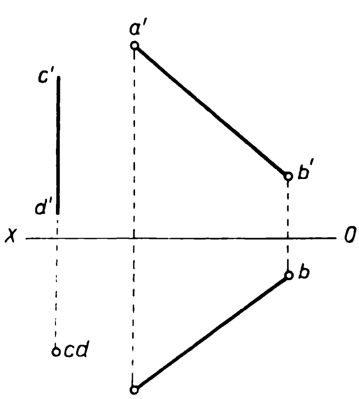



FIG. 634.

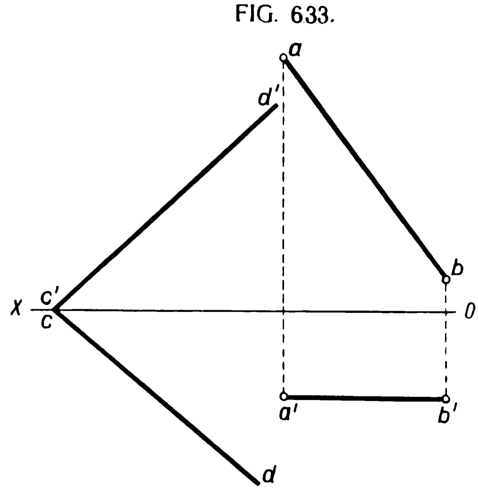



FIG. 635.

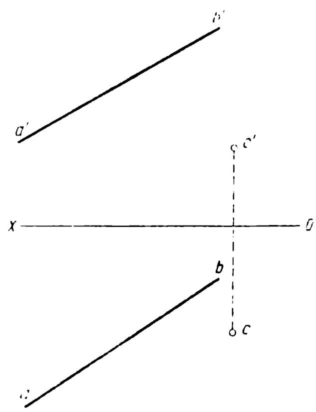



FIG. 636.

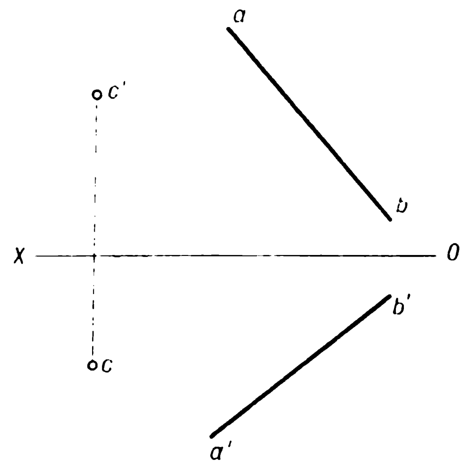



FIG. 637.

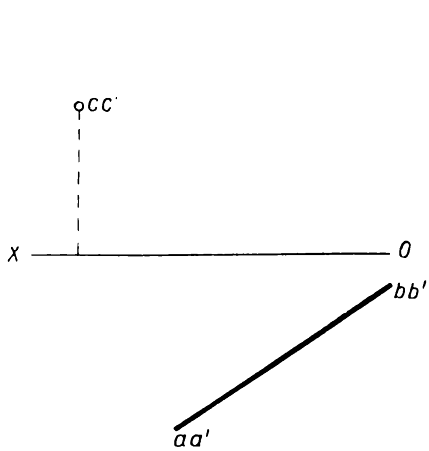



FIG. 638.

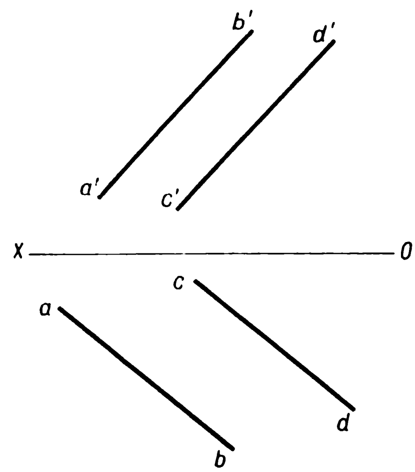



FIG. 639.

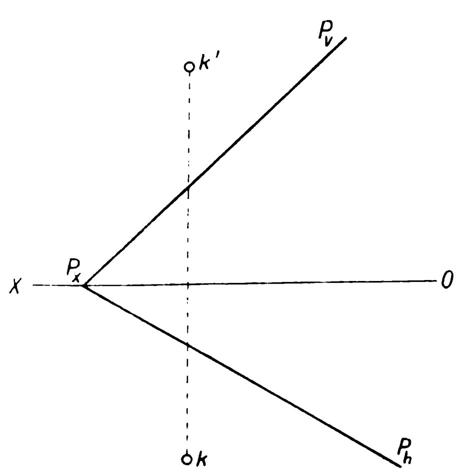



FIG. 640.

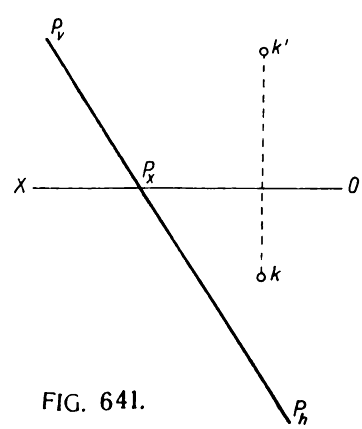



FIG. 641.

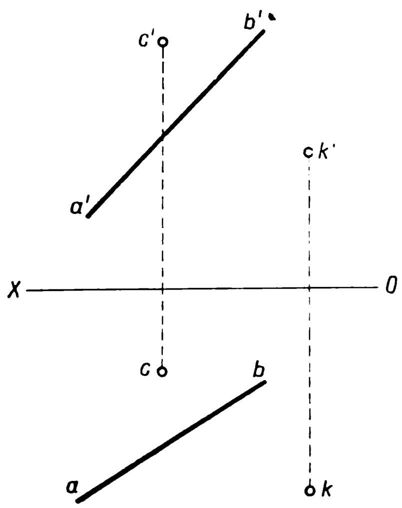



FIG. 642.

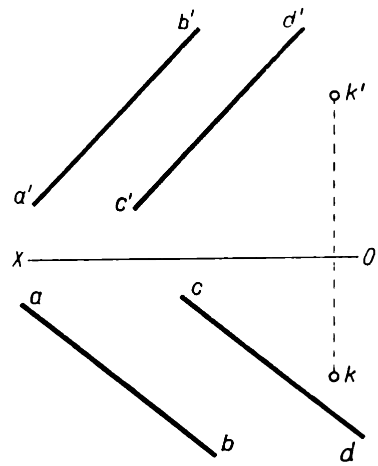



FIG. 643.

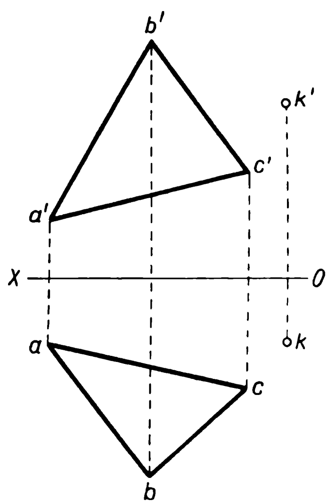



FIG. 644.

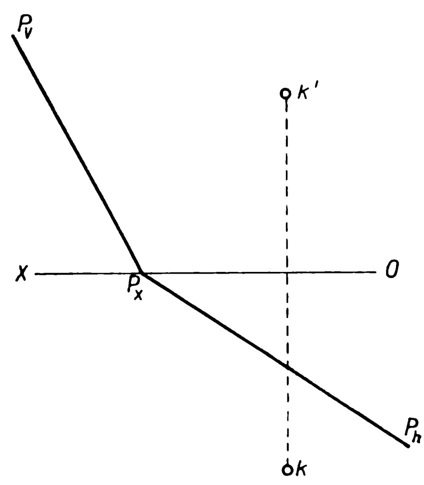



FIG. 645.

302. Find on the line-segment $CD$ a point equidistant from the endpoints of the line-segment $AB$ (Figs. 634, 635).

303. Drop a perpendicular from the point $C$ to the line $AB$ (Figs. 636 to 638).

304. Determine the distance from the point $C$ to the line $AB$ (Figs. 636 to 638).

305. From $C$ as centre construct a sphere tangent to the line-segment $AB$ (Figs. 636 to 638).

306. Determine the distance between the parallel lines $AB$ and $CD$ (Fig. 639).

307. Find on the line-segment $AB$ a point 30 mm distant from the point $C$ (Fig. 636). What variants are possible?

308. Through the point $C$ draw a line intersecting the line $AB$ at an angle of 45° (Fig. 637). How many lines can be drawn?

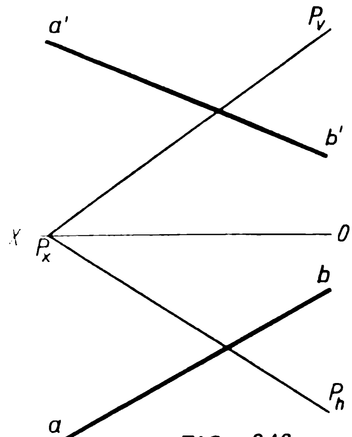


FIG. 646.

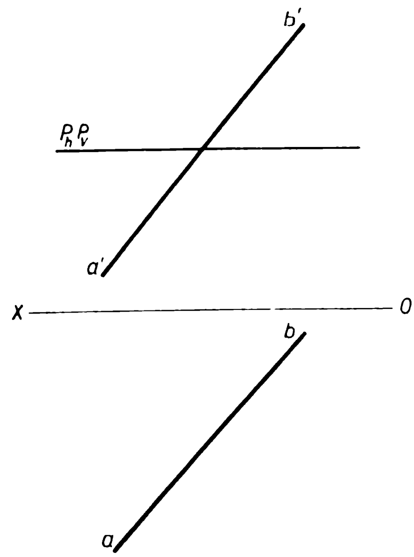


FIG. 647.

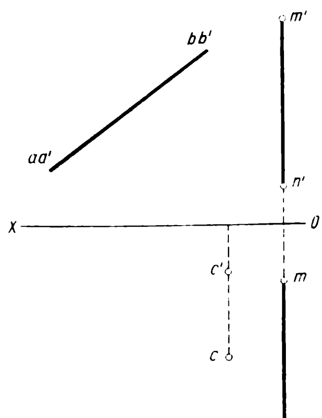


FIG. 648.

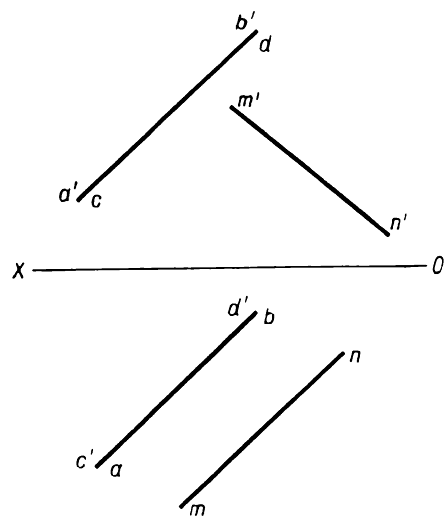


FIG. 649.

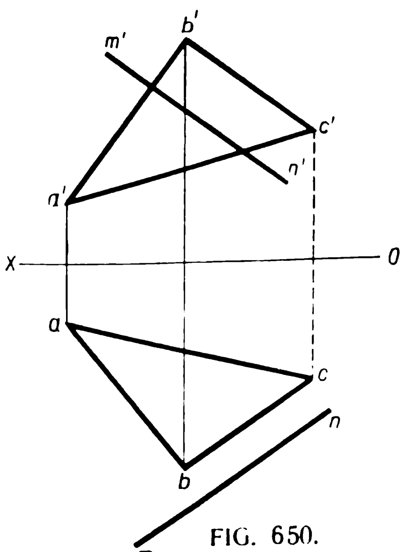


FIG. 650.

309. Pass through the point $K$ and perpendicular to the plane $P$:

(1) a horizontal projecting plane $R$ (Fig. 640);

(2) a vertical projecting plane $S$ (Fig. 641).

310. Pass a horizontal projecting plane $R$ through the point $K$ and perpendicular to the plane specified by the line $AB$ and the point $C$ (Fig. 642).

311. Pass a vertical projecting plane $S$ through the point $K$ and perpendicular to the plane specified by the parallel lines $AB$ and $CD$ (Fig. 643).

312. Pass a profile projecting plane $S$ through the point $K$ and perpendicular to the plane of the triangle $ABC$ (Fig. 644).

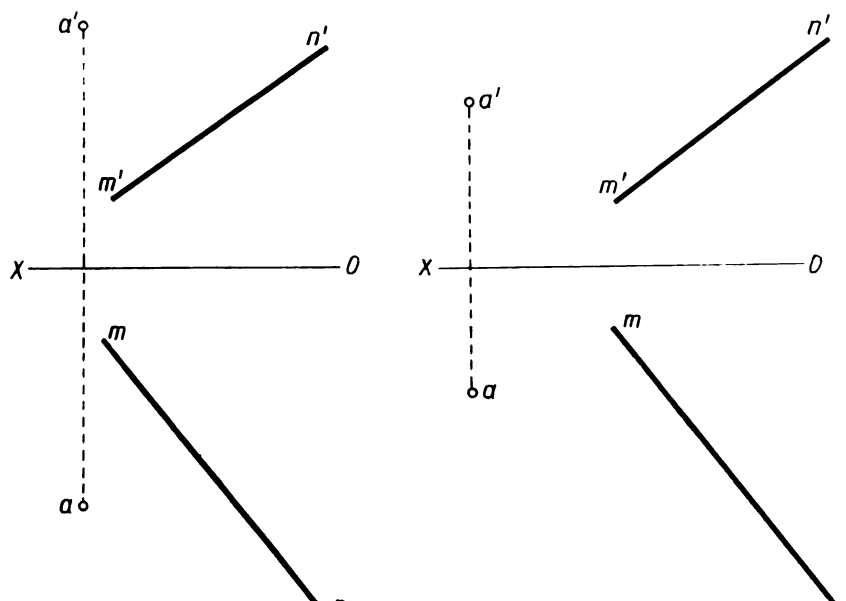



FIG. 651. FIG. 652.

313. Pass an oblique plane $Q$ with coincident traces through the point $K$ and perpendicular to the plane $P$ (Fig. 645).

314. Pass a plane through the line $AB$ and perpendicular to the plane $P$ (Figs. 646, 647).

315. Pass a plane through the line $MN$ and perpendicular to the plane specified by the line $AB$ and the point $C$ (Fig. 648). (The desired plane should be specified by traces.)

316. Pass a plane through the line $MN$ and perpendicular to the plane specified by the parallel lines $AB$ and $CD$ (Fig. 649). (The desired plane should be specified by traces.)

317. Pass a plane through the line $MN$ and perpendicular to the plane of the triangle $ABC$ (Fig. 650). (The desired plane should be specified by traces.)

318. Construct an isosceles triangle $ABC$ with the base $BC$ lying on the line-segment $MN$, given that the length of the side is 1.25 times the altitude (Fig. 651).

319. Construct an isosceles triangle $ABC$ with the base $BC$ lying on the line-segment $MN$, given that the length of the base is 1.5 times the altitude (Fig. 651).

320. Construct an isosceles triangle $ABC$ with the base $BC$ lying on the line $MN$, given that the angle at the base is equal to 30° (Fig. 651).

321. Construct an equilateral triangle $ABC$ with the base $BC$ on the line $MN$ (Fig. 651).

322. Construct a right triangle $ABC$ with the leg $BC$ on the line $MN$, given that the length of the hypotenuse is equal to 1.25 times the altitude (Fig. 652).

323. Construct a right triangle $ABC$ with the leg $BC$ on the line $MN$, given that the acute angle $C$ is equal to 30° (Fig. 652).

324. Construct a right-angled isosceles triangle $ABC$ with the hypotenuse $BC$ lying on the line $MN$ (Fig. 651).

325. Construct a right-angled isosceles triangle $ABC$ with the leg $BC$ lying on the line $MN$ (Fig. 652).

326. Construct a rectangle $ABCD$ with the longer base $BC$ lying on the line $MN$, given that its area equals 1.5 $AB^2$ (Fig. 652).

327. Construct a rectangle $ABCD$ with the longer base $BC$ lying on the line $MN$, given that the length ratio of its sides is equal to 1.5 (Fig. 652).

328. Construct a square $ABCD$ with the side $BC$ lying on the line $MN$ (Fig. 652).

329. Construct a square $ABCD$ with the diagonal $BD$ lying on the line $MN$ (Fig. 651).

330. Construct a parallelogram $ABCD$ with the base $BC$ lying on the line $MN$, given that the acute angle $B$ equals 60° and the diagonal $AC$ is 5 mm longer than the lateral side (Fig. 651).

331. Construct a parallelogram $ABCD$ with the base $BC$ lying on the line $MN$, given that the length of the lateral side is equal to 1.25 h and the length ratio of the sides equals 2 (Fig. 651).

332. Construct a rhombus $ABCD$ with the side $BC$ lying on the line $MN$, given that the length of the side equals 1.2 times the altitude (Fig. 651).

333. Construct a rhombus $ABCD$ with the side $BC$ lying on the line $MN$, given that the acute angle $B$ is equal to 60° (Fig. 651).

334. Construct a rhombus $ABCD$ with the longer diagonal $BD$ on the line $MN$, given that the length ratio of the diagonals is equal to 2 (Fig. 651).

335. Construct a right trapezoid $ABCD$ with the longer base $BC$ on the line $MN$, given $AD = AB$ and $DC = 1.15\ AB$ (Fig. 652).

336. Construct a right trapezoid $ABCD$ with the longer base $BC$ on the line $MN$, given $AD = AB = \frac{2}{3} BC$ (Fig. 652).

337. Construct a right trapezoid $ABCD$ with the longer base $BC$ on the line $MN$, given $AD = AB$ and angle $C = 45°$ (Fig. 652).

338. Construct an isosceles trapezoid $ABCD$ with the longer base $BC$ on the line $MN$, given $AB = AD = DC = 40$ mm (Fig. 651).

339. Construct an isosceles trapezoid $ABCD$ with the longer base $BC$ on the line $MN$, given that the acute angle is equal to 45° and the shorter base equals the side (Fig. 651).

340. Construct an isosceles triangle $ABC$ with the base $BC$ on the line $MN$, given that the side is 10 mm longer than the altitude $AD$ (Fig. 651).

341. Construct an equilateral triangle $ABC$ with the base $BC$ on the line $MN$, given that the point $K$ is the foot of the altitude (Fig. 653).

342. Construct a right-angled triangle $ABC$ with the leg $BC$ on the line $MN$, given that the radius of a circle described about the triangle is equal to 0.75 $AB$ (Fig. 651).

343. Construct a rectangle $ABCD$ with the longer side $BC$ on the line $BM$, given that the length ratio of the sides is equal to 2 (Fig. 654).

344. Construct a square $ABCD$ with the side $BC$ on the line $MN$ (Fig. 652).

345. Construct a square $ABCD$ with the side $BC$ on the line $BM$ (Fig. 654).

346. Construct a parallelogram $ABCD$ with the longer side $BC$ on the line $MN$, given that the point $K$ (the foot of the altitude) divides it in the ratio 1 : 2 from $B$ to $C$ and the angle $B = 60°$ (Fig. 653).

347. Construct a rhombus $ABCD$ with the side $BC$ on the line $MN$, given that the length of the side is 1.2 times the altitude (Fig. 651).

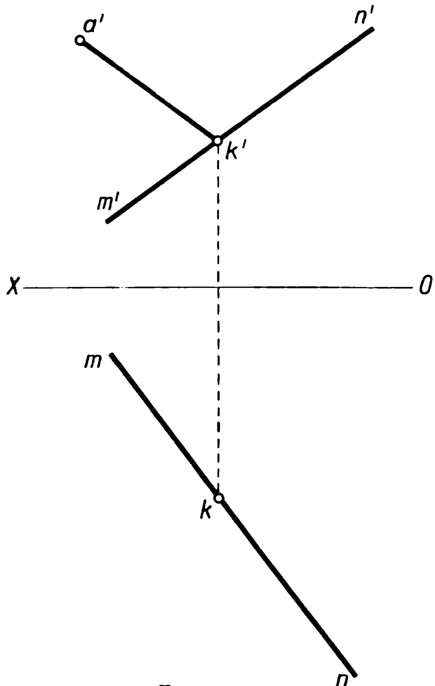



FIG. 653.

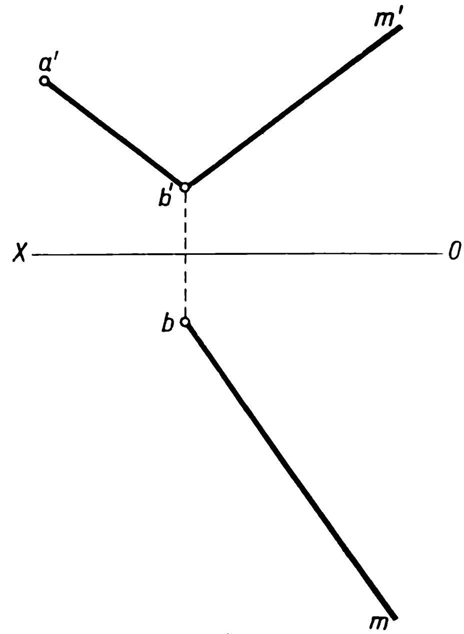



FIG. 654.

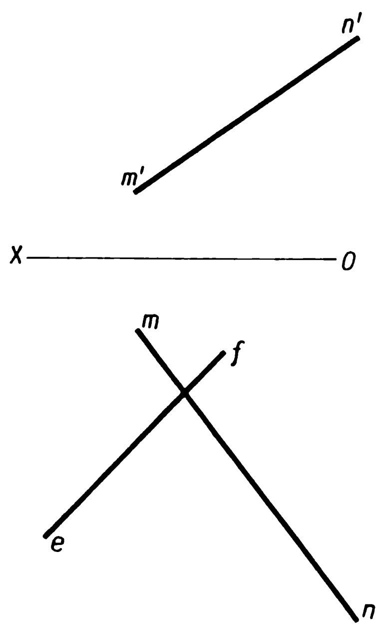



FIG. 655.

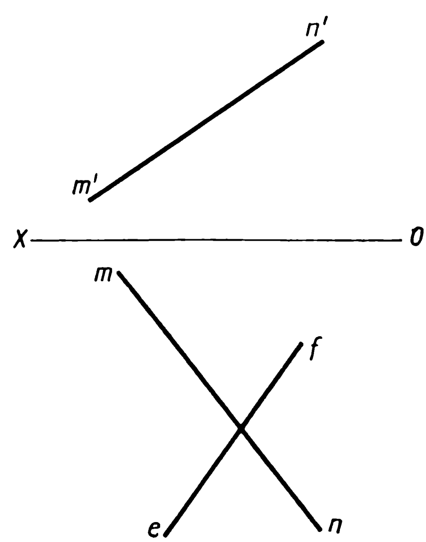



FIG. 656.

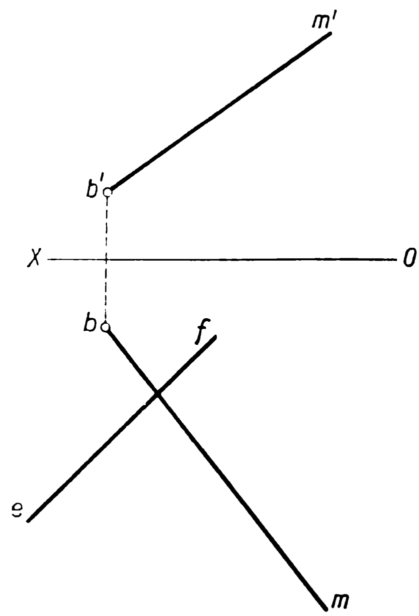


FIG. 657.

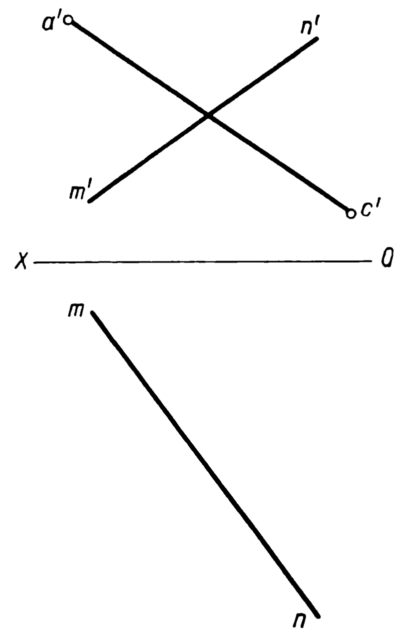


FIG. 658.

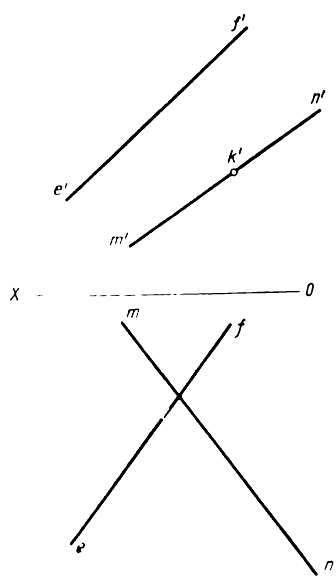


FIG. 659.

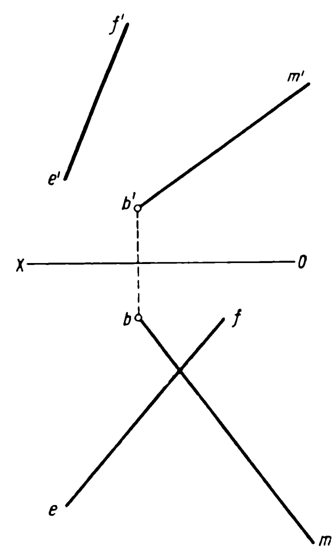


FIG. 660.

348. Construct a right trapezoid $ABCD$ with the longer base $BC$ on the line $BM$, given $B = 90°$, $AD = AB$, $CD = 1.2\,AB$ (Fig. 654).

349. Construct a right trapezoid $ABCD$ with the longer base $BC$ on the line $MN$ and the side $AB$ on the line $EF$, given $B = 90°$, $AD = AB = 40$ mm, and angle $C = 45°$ (Fig. 655).

350. Construct a rhombus $ABCD$ with the longer diagonal $BD$ on the line $MN$, given that the shorter diagonal is 40 mm long and lies on the line $EF$, and the area of the rhombus is equal to $AC^2$ (Fig 656).

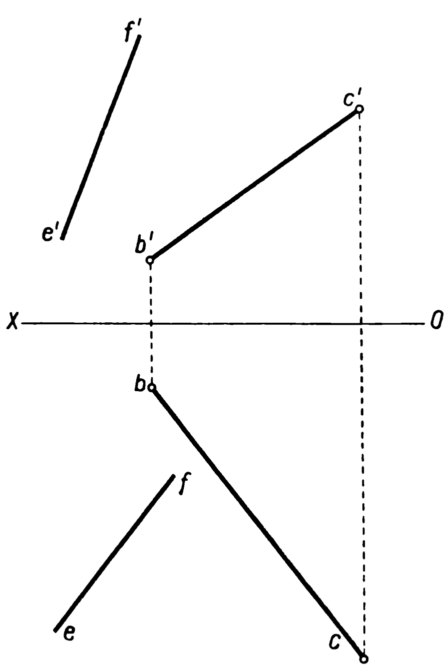



FIG. 661.

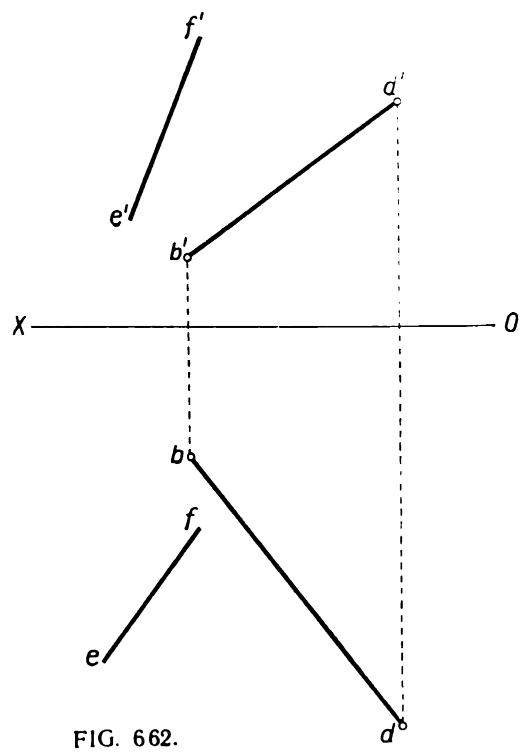



FIG. 662.

351. Construct a parallelogram $ABCD$ with the base $BC$ (60 mm long) on the line $BM$, given that the altitude $AK$ of the parallelogram lies on the line $EF$, and the lateral side is 40 mm long (Fig. 657).

352. Construct a square $ABCD$ with the diagonal $BD$ on the line $MN$ (Fig. 658).

353. Construct a rectangle $ABCD$ with the longer side $BC$ on the line $MN$, given that the side $AB$ is 40 mm long and lies on the line $EF$, and the length ratio of the sides is 1.5 (Fig. 655).

354. Construct a right-angled triangle $ABC$ with the base $BC$ lying on the line $MN$, given that the leg $AB$ is 30 mm long and lies on the line $EF$, and the area of the triangle is $0.75\,AB^2$ (Fig. 655).

355. Construct an equilateral triangle $ABC$ with the base $BC$ (50 mm long) on the line $MN$, given that the vertex $A$ lies on the line $EF$, which is perpendicular to $MN$ (Fig. 656).

356. Construct an equilateral triangle $ABC$ with the base $BC$ on the line $MN$, given that the altitude $AD$ is 40 mm long and lies on the line $EF$ (Fig. 656).

357. Construct an isosceles triangle $ABC$ with the base $BC$ on the line $MN$, given that the altitude is 40 mm long and lies on the line $EF$, and the angle at the base is $30°$ (Fig. 656).

358. Construct an isosceles triangle $ABC$ with the base $BC$ (60 mm long) on the line $MN$, given that the vertex $A$ lies on the line $EF$, which is perpendicular to $MN$, and the altitude is 40 mm long (Fig. 656).

359. Construct an isosceles triangle $ABC$ with the base $BC$ on the line $MN$ and with the vertex $A$ on the line $EF$, given that the point $K$ is the foot of the altitude $AK$, and the side is equal to $1.15\,AK$ (Fig. 659).

360. Construct an equilateral triangle $ABC$ with the base $BC$ on the line $MN$ and the vertex $A$ on the line $EF$, given that the point $K$ is the foot of the altitude $AK$ (Fig. 659).

361. Construct a right isosceles triangle $ABC$ with the leg $BC$ on the line $BM$ and the vertex $A$ on the line $EF$ (Fig. 660).

362. Construct a rectangle $ABCD$ with the longer side $BC$ on the line $BM$ and the vertex $A$ on the line $EF$, given that the diagonal of the rectangle is equal to $2\,AB$ (Fig. 660).

363. Construct a square $ABCD$ with the side $BC$ on the line $BM$, given that the vertex $A$ lies on the line $EF$ (Fig. 660).

364. Construct a square $ABCD$ with the diagonal $BD$ on the line $MN$, given that the vertex $A$ lies on the line $EF$, and $K$ is the point of intersection of the diagonals (Fig. 659).

365. Construct a parallelogram $ABCD$ with the longer side $BC$ on the line $MN$ and the vertex $A$ on the line $EF$, given that the side $AB$ is 5 mm longer than the altitude $AK$, and the side $BC$ is equal to $1.5\,AK$ (Fig. 659).

366. Construct a rhombus $ABCD$ with the longer diagonal $BD$ on the line $MN$ and the vertex $A$ on the line $EF$, given that $K$ is the point of intersection of the diagonals, and the length ratio of the diagonals is 2 (Fig. 659).

367. Construct a right trapezoid $ABCD$ with the longer base $BC$ on the line $BM$, given that the vertex $A$ lies on the line $EF$, $AD = AB$; $B = 90°$; $C = \varphi$ (Fig. 660).

368. Construct an isosceles triangle $ABC$ with the vertex $A$ lying on the line $EF$ (Fig. 661).

369. Construct a rectangle $ABCD$ with the vertex $A$ lying on the line $EF$, and calculate its area (Fig. 661).

370. Construct a rhombus $ABCD$ with the vertex $A$ lying on the line $EF$ (Fig. 662).

# PART THREE

## CHAPTER XVII

## THE METHOD OF REVOLUTION. DISPLACEMENT PARALLEL TO THE PROJECTION PLANES

EXAMPLES

**Example 186**

Revolve the point $A$ clockwise through an angle of 120° about axis $I$, which is perpendicular to the plane $H$ (Fig. 663).

*Solution.* Revolving about the axis $(i, i')$, the point $(a, a')$ describes a circle of radius $ai$ in the plane $R$, which is perpendicular to the axis of revolution, i.e. parallel to the plane $H$. The horizontal projection $(a)$ of the point describes an arc $aa_1$

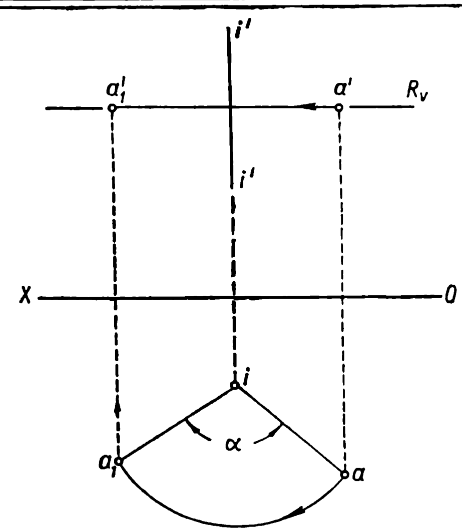



FIG. 663.

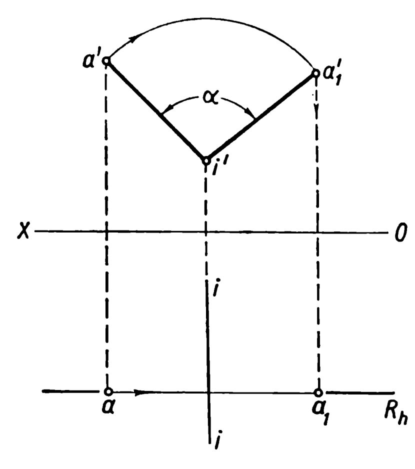



FIG. 664.

of radius $ai$ with a central angle $\alpha = 120°$. The vertical projection $(a')$ of the point moves along a straight line parallel to the coordinate axis ($z$ = const). Knowing the new position of the horizontal projection $(a_1)$ of the point, find its vertical projection $(a'_1)$.

**Example 187**

Revolve the point $A$ clockwise through an angle of 120° about axis $I$, which is perpendicular to the plane $V$ (Fig. 664).

*Solution.* Revolving about the axis $(i, i')$, the point $(a, a')$ describes a circle of radius $a'i'$ in the plane $R$, which is perpendicular to the axis of revolution, i.e. parallel to the plane $V$. The vertical projection ($a'$) of the point describes an arc $a'a_1'$ of radius $a'i'$ with a central angle $\alpha = 120°$. The horizontal projection ($a$) of the point will move along a straight line parallel to the coordinate axis ($y = \text{const}$). Knowing the new position of the vertical projection ($a_1'$) of the point, find its horizontal projection ($a_1$).

**Example 188**

Revolve the line-segment $AB$ clockwise through an angle of 120° about the axis $I$, which is perpendicular to the plane $H$ (Fig. 665).

*Solution.* To revolve the line-segment through the given angle, it is sufficient to revolve the points $A$ and $B$ through this angle *in one and the same direction.*

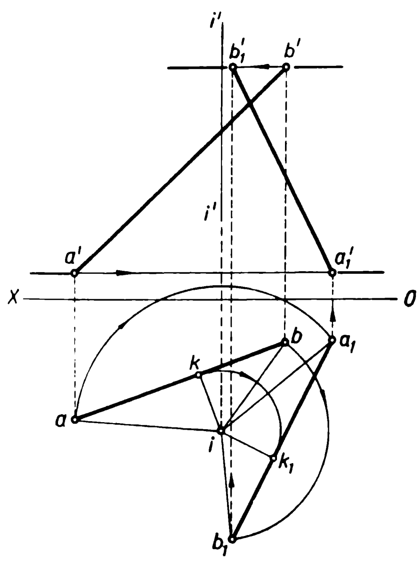



FIG. 665.

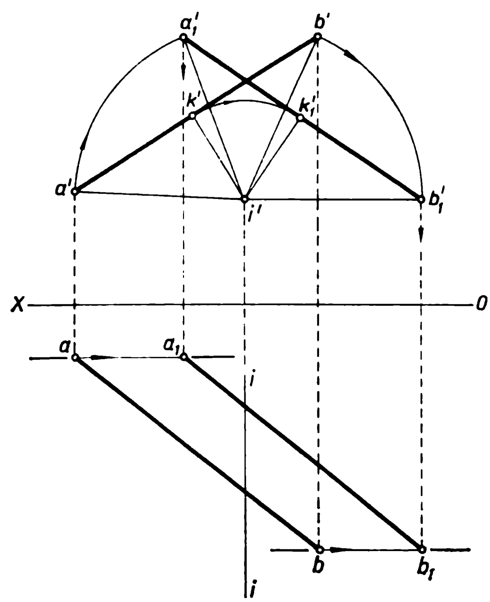



FIG. 666.

Hence, proceed as indicated above. As is obvious from the construction, the length of the horizontal projection ($a_1b_1$) of the line-segment remains unchanged; the only thing that changes is its position relative to the coordinate axis. This fact enables us to restrict ourselves to revolving through the given angle *one* point ($k$, $k'$) of the line-segment, i.e. the point nearest to the axis of revolution. Revolve the point $k$ through an angle of 120°, draw through the newly obtained point $k_1$ a straight line perpendicular to the radius $ik_1$, and lay off on it line-segments $a_1k_1 = ak$ and $b_1k_1 = bk$. Knowing the horizontal projection ($a_1b_1$) of the line-segment, find the vertical projection ($a_1'b_1'$). The construction is obvious from the drawing.

**Example 189**

Revolve the line-segment $AB$ clockwise through an angle of 60° about the axis $I$, which is perpendicular to the plane $V$ (Fig. 666).

*Solution.* To revolve the line-segment through the given angle, it is sufficient to revolve the points $A$ and $B$ through this angle *in one and the same direction.* Hence, proceed as indicated above. As is obvious from the construction, the length of the vertical projection ($a_1'b_1'$) of the line-segment remains unchanged; the only thing that changes is its position relative to the coordinate axis. This enables us to restrict ourselves to revolving through the given angle *one* point ($k$, $k'$) of the line-segment, i.e. the point nearest to the axis of revolution. Revolve the point $k'$ through an angle of 60°, draw through the newly obtained point $k'_1$ a straight line perpendicular to the radius $i'k'_1$, and lay off on it line-segments $a'_1k'_1 = a'k'$ and $b'_1k'_1 = b'k'$. Knowing the vertical projection $(a'_1b'_1)$ of the line-segment, determine the horizontal projection $(a_1b_1)$. The construction is obvious from the drawing.

**Example 190**

Revolve the triangle $ABC$ clockwise through an angle of 120° about the axis $I$, which is perpendicular to the plane $H$ (Fig. 667).

*Solution.* Revolve the vertices $A$, $B$, and $C$ of the triangle through an angle of 120°. Knowing the horizontal projections $(a_1)$, $(b_1)$, and $(c_1)$ of the points, find

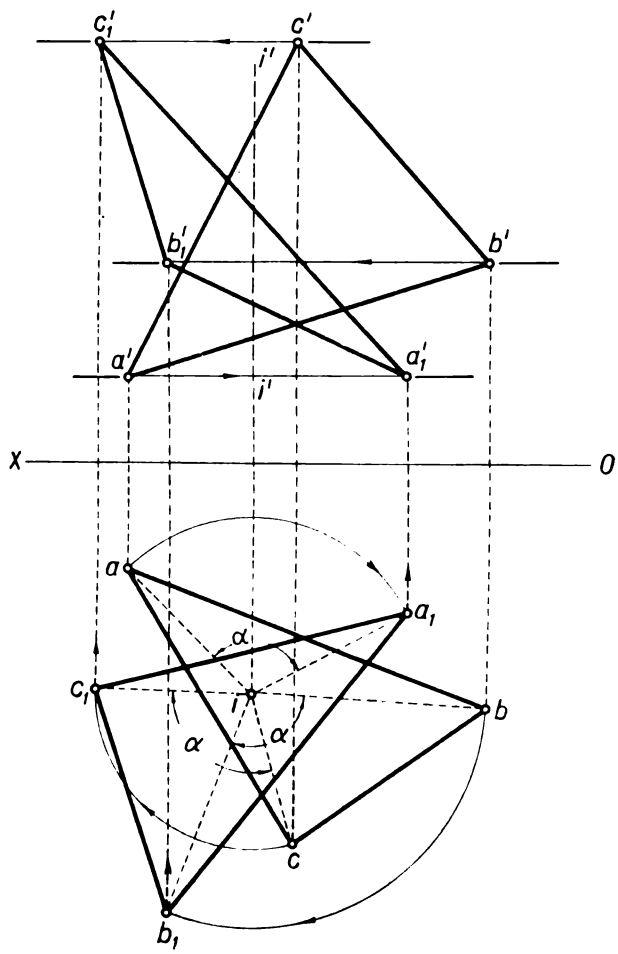


FIG. 667.

their vertical projections $(a'_1)$, $(b'_1)$, and $(c'_1)$. Join the like projections of the vertices to obtain new projections of the triangle: the horizontal one $(a_1b_1c_1)$ and the vertical one $(a'_1b'_1c'_1)$. It should be noted that the triangles $abc$ and $a_1b_1c_1$ are equal.

*Note.* If the triangle is revolved about the axis $I$ which is perpendicular to the plane $V$, then the triangles $a'b'c'$ and $a'_1b'_1c'_1$ will be equal.

*Conclusions*:

1. If a point is revolved about an axis $I$ which is perpendicular to the $H$ plane, then the *vertical* projection of the point moves along a straight line parallel to the coordinate axis.

2. If a point is revolved about an axis $I$ which is perpendicular to the $V$ plane, then the *horizontal* projection of the point moves in a straight line parallel to the coordinate axis.

3. If a line-segment is revolved about an axis $I$ which is perpendicular to the $H$ plane, then the length of the *horizontal* projection of the line-segment remains unchanged and, consequently, the angle at which the line-segment is inclined to the $H$ plane also remains unchanged.

4. If a line-segment is revolved about an axis $I$ which is perpendicular to the $V$ plane, then the length of the *vertical* projection of the line-segment remains unchanged and, consequently, the angle at which the line is inclined to the $V$ plane also remains unchanged.

5. If a plane figure (say, a triangle) is revolved about an axis $I$ which is perpendicular to the $H$ plane, the *horizontal* projection of the figure remains unchanged and, consequently, the angle at which the figure is inclined to the $H$ plane also remains unchanged.

6. If a plane figure is revolved about an axis $I$ which is perpendicular to the $V$ plane, the *vertical* projection of the figure remains unchanged and, consequently, the angle at which the figure is inclined to the $V$ plane also remains unchanged.

7. If a system of geometrical elements (points, straight lines, or combinations thereof) is revolved about an axis which is perpendicular to the $H$ (or $V$) plane, the relative positions of the *horizontal* (*vertical*) projections of the given elements remain unchanged, the only thing that changes being their position relative to the coordinate axis.

In some cases, when revolving a system of geometrical elements about an axis which is perpendicular to a projection plane, the obtained projection coincides with the initial one. To avoid such a coincidence it is advisable to displace the system parallel to the projection plane instead of revolving it.

**Example 191**

Given a point $A$ and a plane $R$ parallel to the $H$ plane and passing through this point. Follow the movement of the point in the plane (Fig. 668).

*Solution*. No matter what line is traced by the point ($a$, $a'$) in the plane $R$, the vertical projection of this line (trajectory) is orthographically represented as a straight line coinciding with the trace $R_v$ of the given plane. The vertical projection ($a'$) of the point moves in a straight line parallel to the coordinate axis, the horizontal projection ($a$) moving along a line identical to the one along which the point $A$ moves in the plane $R$ (why?). Suppose the point $a$ moves along line $I$ to a new position $a_1$. Knowing this, it is easy to find the position of the vertical projection ($a_1'$) of the point.

If the point $a$ moves to the same new position $a_1$ along line $II$, the vertical projection ($a_1'$) of the point remains unchanged. Hence, to determine precisely the position ($a_1'$) of the vertical projection of the point, it is sufficient to indicate the new position of its horizontal projection ($a_1$). In this case the point does not revolve about an axis, but is displaced arbitrarily in a plane parallel to the $H$ plane. We will call such displacement the "movement of a point parallel to the $H$ plane". As is seen from the drawing, the distance from the vertical projection ($a_1'$) of the point to the coordinate axis remains constant, as in revolving about axis $I$.

**Example 192**

Given: a point $A$ and a plane $R$ parallel to the $V$ plane and passing through the given point. Follow the movement of the point in the plane (Fig. 669).

*Solution*. No matter what line is traced by the point ($a$, $a'$) in the plane $R$, the horizontal projection of this line (trajectory) is orthographically represented as a straight line coinciding with the trace $R_h$ of the given plane. The horizontal projection ($a$) of the point moves along a straight line parallel to the coordinate axis, its vertical projection ($a'$) moving in a line identical to the one along which the point $A$ moves in the plane $R$ (why?). Suppose the point $a'$ is displaced along line $I$ to a new position $a_1'$. Knowing this it is easy to find the position of the horizontal projection ($a_1$) of the point. If the point $a'$ moves to the same new position $a_1'$ by being displaced along line $II$, then the horizontal projection ($a_1$) of the point remains unchanged. Hence, to determine precisely the position ($a_1$) of the horizontal projection of the point, it is sufficient to indicate the new position of its vertical projection ($a_1'$). In this case the point again does not revolve about an axis, but is displaced arbitrarily

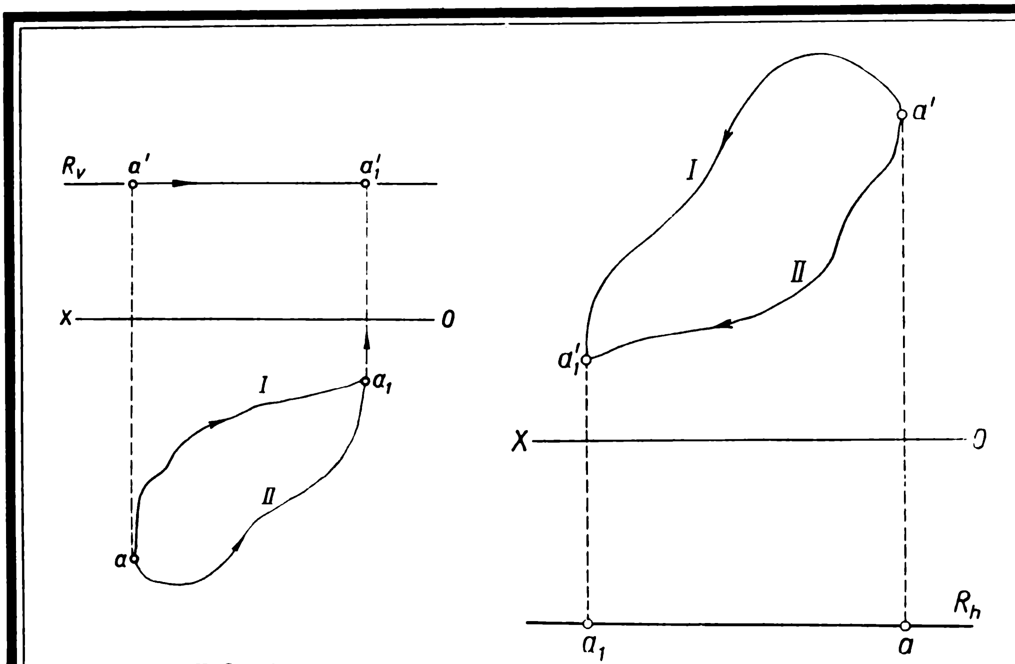



FIG. 668. FIG. 669.

in a plane parallel to the $V$ plane. We will term such displacement the "movement of a point parallel to the $V$ plane". As is seen from the drawing, the distance from the horizontal projection ($a_1$) of the point to the coordinate axis remains constant, as in revolving about an axis perpendicular to the $V$ plane.

**Example 193**

Given: a point $A$ located in the third quadrant. Required: to transfer it to the first quadrant (Fig. 670).

*Solution.* To transfer the point ($a$, $a'$) to the first quadrant it should be displaced twice: first parallel to the vertical plane of projection in order to bring the point to the second quadrant, and then parallel to the horizontal plane of projection in order to bring the point to the first quadrant (the order of displacements may be reversed: first parallel to the $H$ plane, and then parallel to the $V$ plane). Hence, on transferring the point to the second quarter both of its projections must be found *above* the coordinate axis; the horizontal projection of the point moves along a straight line parallel to the coordinate axis. Taking the vertical projection ($a_1'$) of the point arbitrarily, we find its horizontal projection ($a_1$). During the second displacement the vertical projection ($a_1'$) of the point moves along a straight line parallel to the coordinate axis. Arbitrarily taking the horizontal projection ($a_2$) of the point *below* the coordinate axis (why?), we find the vertical projection ($a_2'$) of the point. The point ($a_2a_2'$) is the desired point.

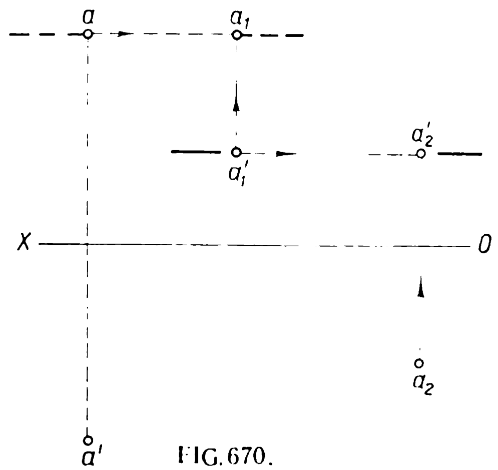



FIG. 670.

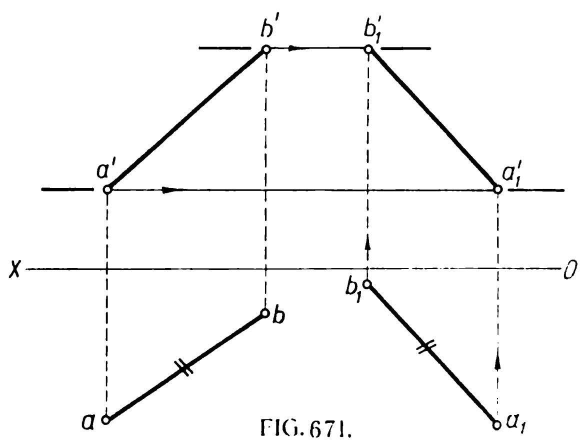



FIG. 671.

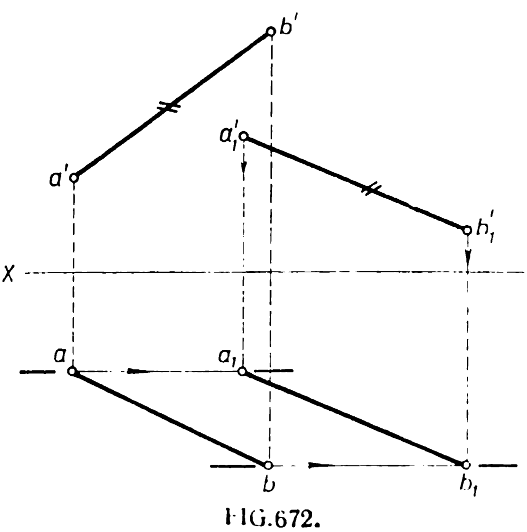



FIG. 672.

**Example 194**

Bring the line-segment $AB$ to an arbitrary position by displacing it parallel to the horizontal plane of projection (Fig. 671).

*Solution.* In displacing a line-segment parallel to the horizontal plane of projection the length of its horizontal projection remains unchanged, as in revolving about an axis perpendicular to the $H$ plane. Take a line-segment $a_1b_1$ (equal to $ab$) in an arbitrary position *below* the coordinate axis (in order to keep it in the first quadrant) and then find its vertical projection ($a'_1b'_1$) from the horizontal projection ($a_1b_1$).

*Note.* When solving such problems it is advisable to keep the given elements in the first quadrant, or transfer them to the first quadrant if they are given in other quadrants.

**Example 195**

Bring the line-segment $AB$ to an arbitrary position by displacing it parallel to the vertical plane of projection (Fig. 672).

*Solution.* In displacing a line-segment parallel to the vertical plane of projection the length of its vertical projection remains unchanged, as in revolving about an axis perpendicular to the $V$ plane. Take a line-segment $a'_1b'_1$ (equal to $a'b'$) in an arbitrary position and then find its horizontal projection ($a_1b_1$) from the vertical projection ($a'_1b'_1$).

**Example 196**

Bring the line-segment $AB$ to a position parallel to the vertical plane of projection (Fig. 673).

*Solution.* Displace the line-segment parallel to the $H$ plane. Since it must occupy a position parallel to the $V$ plane, its horizontal projection must be parallel to the coordinate axis. Take, *below* the coordinate axis, a line segment $a_1b_1$ (equal to $ab$) in a position parallel to the axis. Then find its vertical projection ($a'_1b'_1$) from the horizontal projection ($a_1b_1$).

**Example 197**

Bring the line-segment $AB$ to a position parallel to the horizontal plane of projection (Fig. 674).

*Solution.* Displace the line-segment parallel to the $V$ plane. Since it must occupy a position parallel to the $H$ plane, its vertical projection must be parallel to the coordinate axis. Take, *above* the coordinate axis, a line-segment $a'_1b'_1$ (equal to $a'b'$) in a position parallel to the axis. Then find its horizontal projection ($a_1b_1$) from the vertical projection ($a'_1b'_1$).

**Example 198**

Bring the line-segment $AB$ to a position perpendicular to the $H$ plane (Fig. 675).

*Solution.* Since the line-segment $AB$ is given in a position parallel to the $V$ plane, it is sufficient to displace it parallel to the $V$ plane. Take a line-segment $a'_1b'_1$ (equal to $a'b'$) in a position perpendicular to the coordinate axis (why?) and then find its horizontal projection ($a_1b_1$)—a point—from ($a'_1b'_1$).

**Example 199**

Bring the line-segment $AB$ to a position perpendicular to the $V$ plane (Fig. 676).

*Solution.* Since the line-segment $AB$ is given in a position parallel to the $H$ plane, it is sufficient to displace it parallel to the $H$ plane. Take a line-segment $a_1b_1$ (equal to $ab$) in a position perpendicular to the coordinate axis (why?) and then find its vertical projection ($a'_1b'_1$)—a point—from the horizontal projection ($a_1b_1$).

**Example 200**

Bring the line-segment $AB$ to a position perpendicular to the $H$ plane (Fig. 677).

*Solution.* Since the given line-segment ($ab$, $a'b'$) is an oblique one, two consecutive displacements must be made to bring it to a position perpendicular to the $H$ pla-

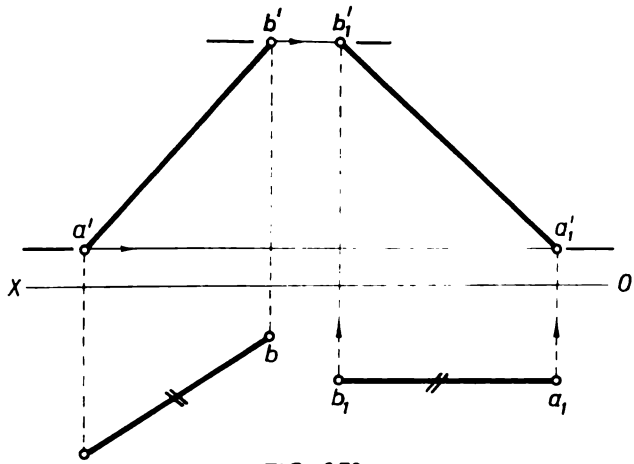


FIG. 673.

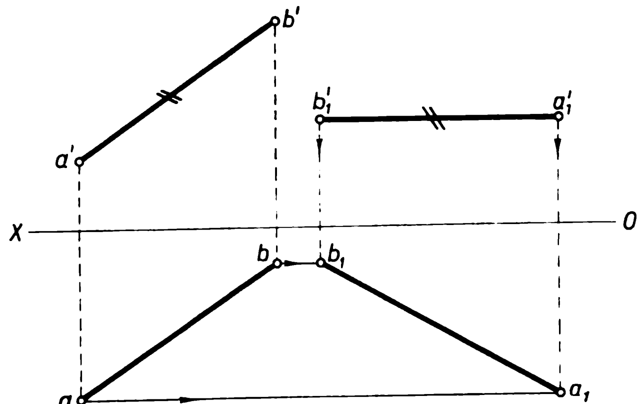


FIG. 674.

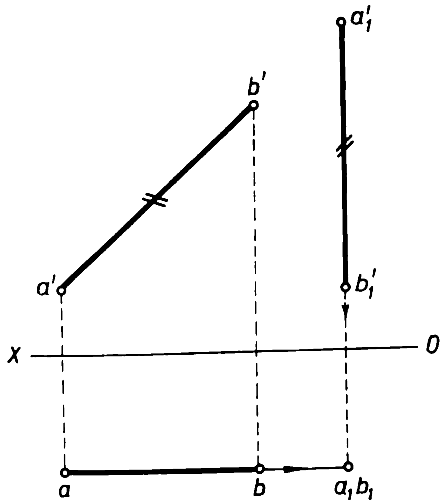


FIG. 675.

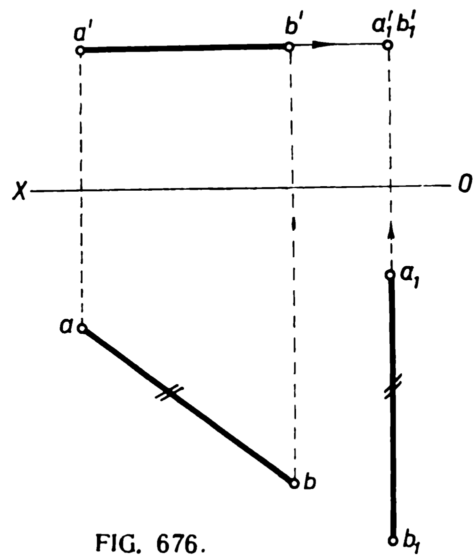


FIG. 676.

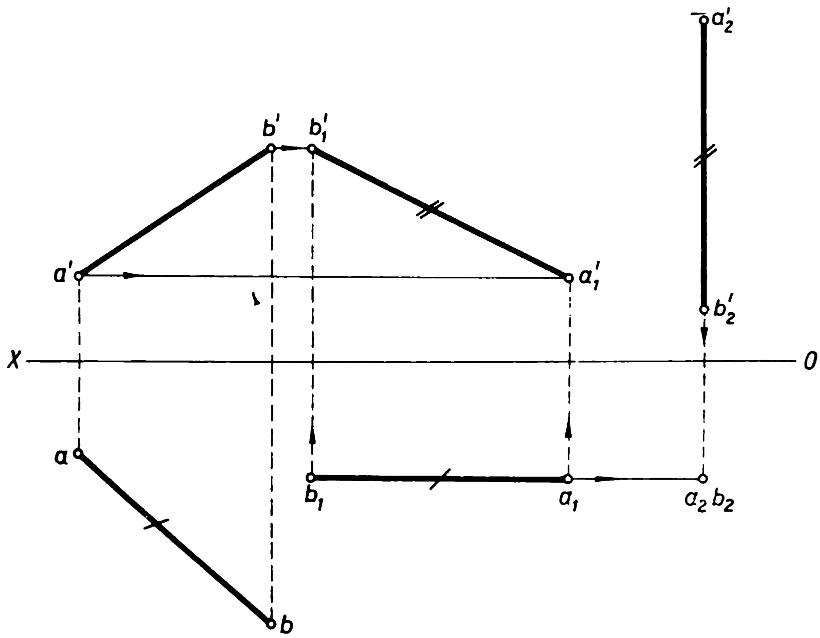



FIG. 677.

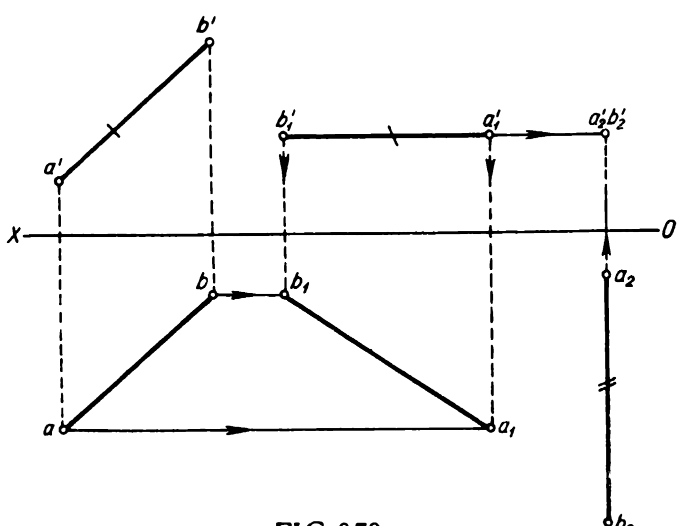



FIG.678.

ne: first parallel to the $H$ plane to bring it to a position parallel to the $V$ plane, and then parallel to the $V$ plane to bring it to a position perpendicular to the $H$ plane. Hence, place the line-segment $a_1b_1$ (equal to $ab$) in any position parallel to the coordinate axis and determine the vertical projection ($a_1'b_1'$). Then place a line-segment $a_2'b_2'$ (equal to $a_1'b_1'$) in a position perpendicular to the coordinate axis and find the horizontal projection ($a_2b_2$)—a point. The line-segment ($a_2b_2$, $a_2'b_2'$) thus obtained is the desired one.

**Example 201**

Bring the line-segment $AB$ to a position perpendicular to the $V$ plane (Fig. 678).

*Solution.* Since the given line-segment ($ab$, $a'b'$) is an oblique one, two consecutive displacements must be made to bring it to a position perpendicular to the $V$ plane: first parallel to the $V$ plane to bring it to a position parallel to the $H$ plane, then parallel to the $H$ plane to bring the line-segment ($a_1b_1$, $a'b'$) to a position perpendicular to the $V$ plane. Hence, place the line-segment $a_1'b_1'$ (equal to $a'b'$) in any position parallel to the coordinate axis and determine the horizontal projection ($a_1b_1$). Then place a line-segment $a_2b_2$ (equal to $a_1b_1$) in a position perpendicular to the coordinate axis and find the vertical projection ($a_2'b_2'$)—a point. The line-segment ($a_2b_2$, $a_2'b_2'$) thus obtained is the desired one.

**Example 202**

Bring the plane $P$ to a position perpendicular to the $H$ plane (Fig. 679).

*Solution.* To bring the given plane to a position perpendicular to the $H$ plane, draw a frontal line in it and place it in a position perpendicular to the $H$ plane. Then the given plane will also occupy a position perpendicular to the $H$ plane. Draw an arbitrary frontal line in the $P$ plane and displace this line parallel to the $V$ plane until it occupies a position perpendicular to the $H$ plane. The vertical projection of the frontal line and, consequently, the vertical trace ($P_{v_1}$) of the plane are perpendicular to the coordinate axis, and the horizontal projection of the vertical line shrinks into a point. Since during the displacement of a plane parallel to the $V$ plane its angle of inclination to this projection plane remains unchanged (as in revolving about an axis perpendicular to the $V$ plane), it is also necessary that the distance between its vertical trace and the vertical projection of the frontal line remain constant for any position of the plane.

Assume on the coordinate axis an *arbitrary* vanishing point $P_{x_1}$ and through it draw the vertical trace ($P_{v_1}$) of the plane perpendicular to the coordinate axis. Then draw the vertical projection of the vertical line parallel to $P_{v_1}$ and at a distance $l$ from it. Find the horizontal trace of the frontal line, i.e. point $h_1$, and draw the horizontal trace ($P_{h_1}$) of the plane through the points $P_{x_1}$ and $h_1$.

**Example 203**

Bring the plane $P$ to a position perpendicular to the $V$ plane (Fig. 680).

*Solution.* To bring the given plane to a position perpendicular to the $V$ plane, draw a horizontal line in it and place it in a position perpendicular to the $V$ plane. As a result the given plane will also occupy a position perpendicular to the $V$ plane. Draw an arbitrary horizontal line in the $P$ plane and displace this line parallel to the $H$ plane until it occupies a position perpendicular to the $V$ plane. The horizontal projection of the horizontal line and, consequently, the horizontal trace ($P_{h_1}$) of the plane are perpendicular to the coordinate axis. The vertical projection of the horizontal line shrinks into a point. Since in displacing a plane parallel to the $H$ plane its angle of inclination to this plane remains unchanged, it is necessary that the distance between its horizontal trace and the horizontal projection of the horizontal line also remain constant for any position of the plane.

Therefore we assume an arbitrary vanishing point ($P_{x_1}$) on the coordinate axis and draw through it the horizontal trace ($P_{h_1}$) of the plane perpendicular to the coordinate axis. Then draw the horizontal projection of the horizontal line parallel

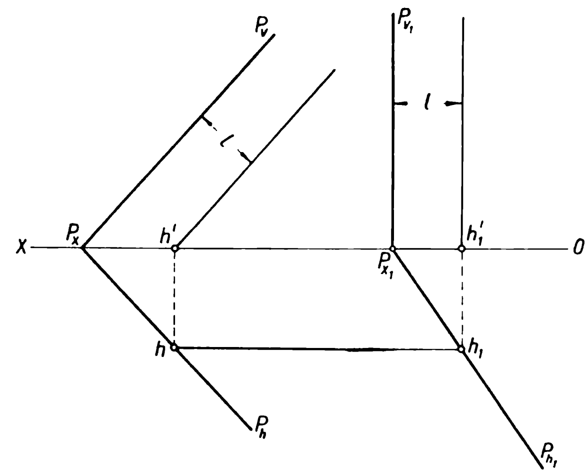



FIG. 679.

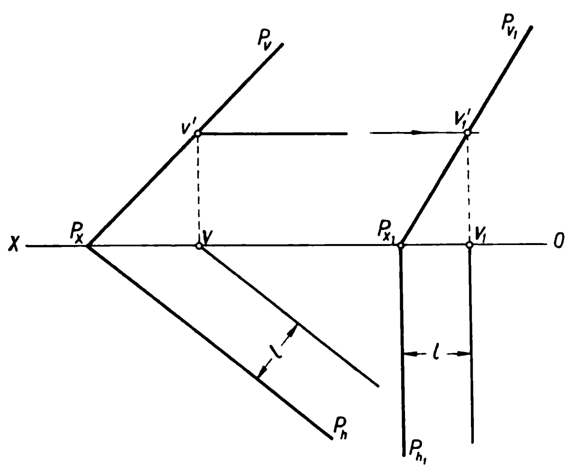



FIG. 680.

to $P_{h_1}$ and at a distance $l$ from it. Find the vertical trace of the horizontal line, i.e. point $v'_1$, and draw the vertical trace ($P_{v_1}$) of the plane through the points $v'_1$ and $P_{x_1}$.

**Example 204**

Determine the centre of a circle described about the triangle $ABC$ (Fig. 681).

*Solution.* The centre of a circle described about a triangle is found at the point of intersection of perpendiculars erected at the mid-points of the sides. To be able to draw these perpendiculars, one should know the true size of the triangle. To obtain it, bring the plane of the triangle to a position parallel to a projection plane, say, the $H$ plane. This is achieved by two consecutive displacements: first parallel to the $H$ plane, and then parallel to the $V$ plane.

Hence, draw a horizontal line through the point ($a$, $a'$) in the plane of the triangle and displace it parallel to the $H$ plane until it occupies a position perpendicular to the $V$ plane. As is known, the horizontal projection of a triangle moving parallel to the $H$ plane remains unchanged. Thus, bring the horizontal projection of the triangle to the position $a_1b_1c_1$ so that the horizontal projection of the horizontal line is perpendicular to the coordinate axis. From $a_1b_1c_1$ find the vertical projection ($a'_1b'_1c'_1$) of the triangle, which is a straight line (why?). Then displace the triangle

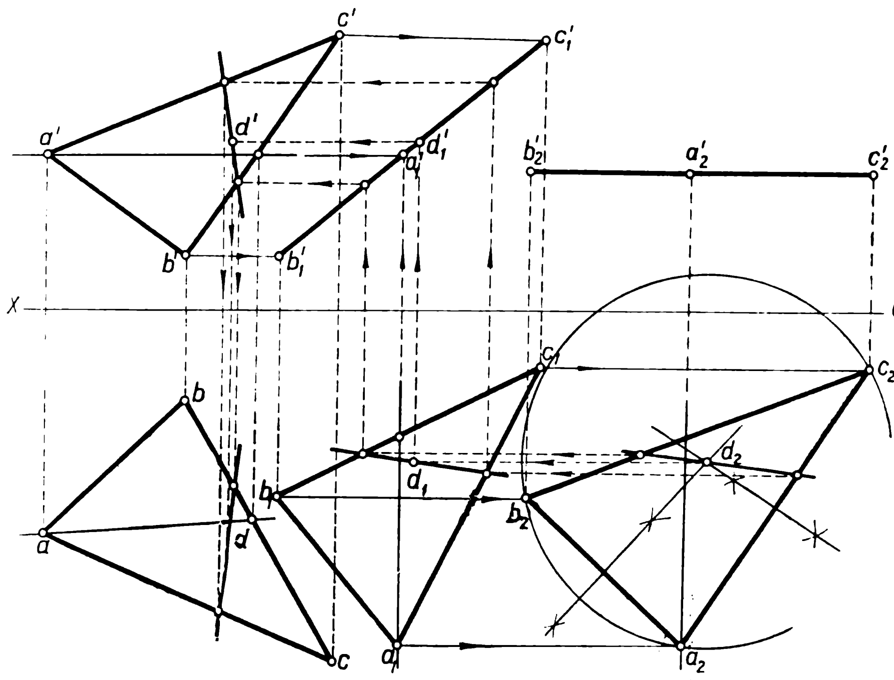



FIG. 681.

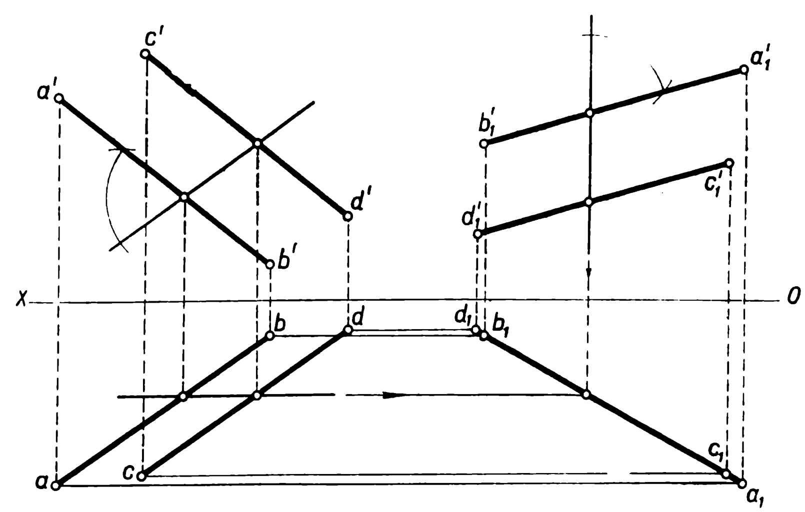



FIG. 682.

($a_1b_1c_1$, $a_1'b_1'c_1'$) parallel to the $V$ plane and bring the vertical projection ($a_2'b_2'c_2'$) of the triangle to a position parallel to the coordinate axis. From $a_2'b_2'c_2'$ find $a_2b_2c_2$, which is the true size of the triangle. Draw perpendiculars through the mid-points of the sides $a_2b_2$ and $a_2c_2$, the perpendiculars will intersect at point ($d_2d_2'$) to yield the desired centre of the circle.

Then transfer this point to the initial position. The construction is given in the drawing.

**Example 205**

Displace the parallel lines $AB$ and $CD$ to a position in which their horizontal projections merge into one straight line (Fig. 682).

*Solution.* The parallel lines specify a plane; their horizontal projections merge into one line when this plane is perpendicular to the $H$ plane. Hence, draw a frontal line in this plane and, by displacing the whole system parallel to the $V$ plane, place the frontal line in a position perpendicular to the $H$ plane. As is known, the relative positions of the vertical projections of the elements in this case remain the same. Having the vertical projections $(a_1'b_1')$ and $(c_1'd_1')$ of the lines, determine their horizontal projections $(a_1b_1)$ and $(c_1d_1)$ merged into one line.

**Example 206**

Drop a perpendicular from the point $C$ to the line $AB$ (Fig. 683).

*Solution.* It is possible to drop a perpendicular from a point to a line on an orthographic drawing only when the given line is parallel to a projection plane (according to what theorem?). Hence, displace the given system parallel to the $V$ plane to occupy

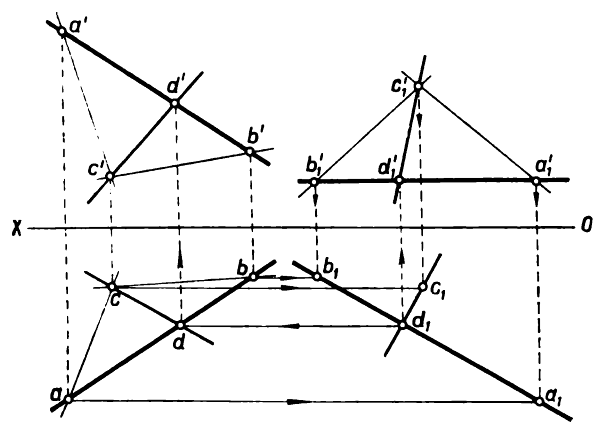



FIG. 683.

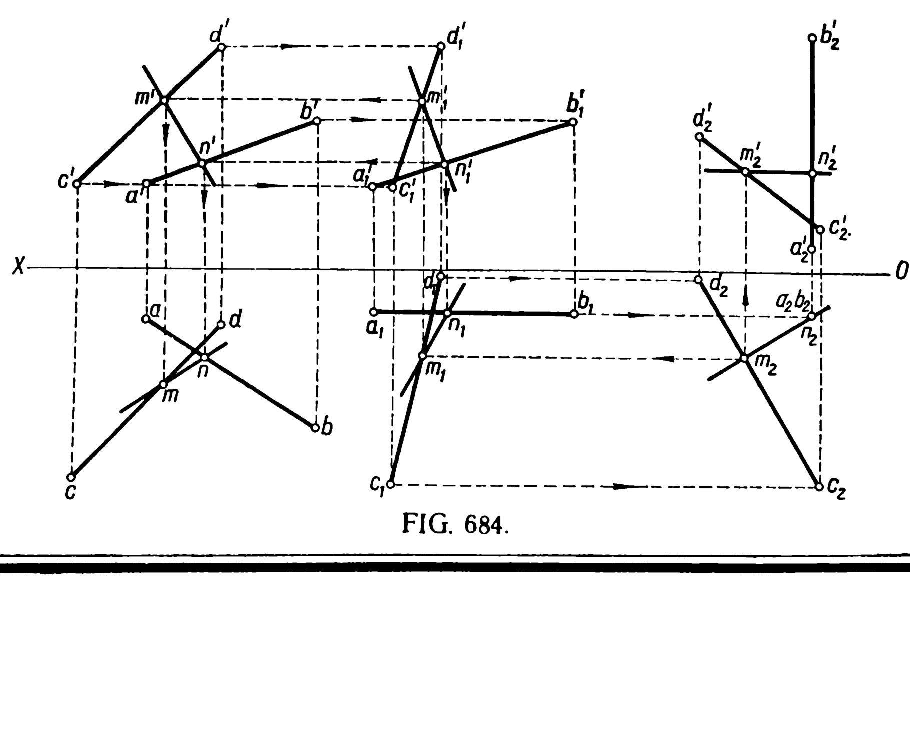



FIG. 684.

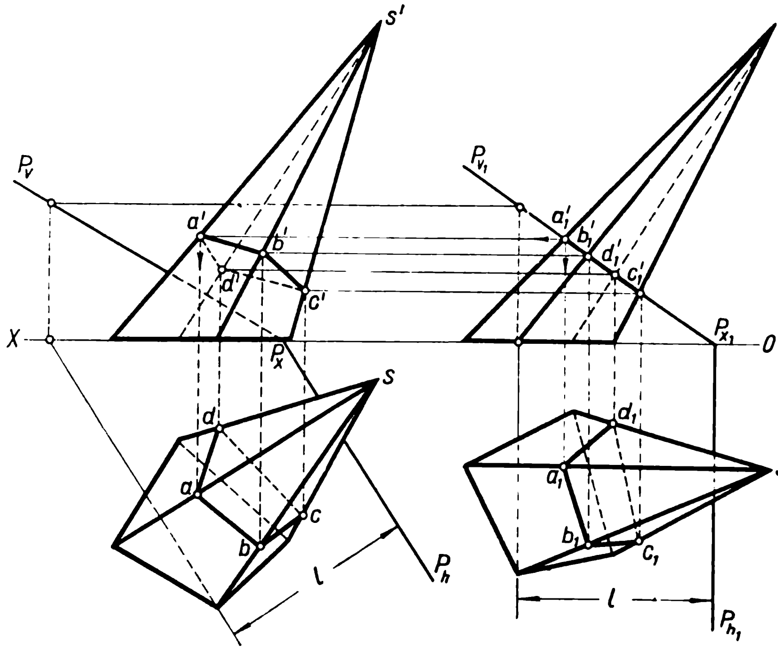



FIG. 685.

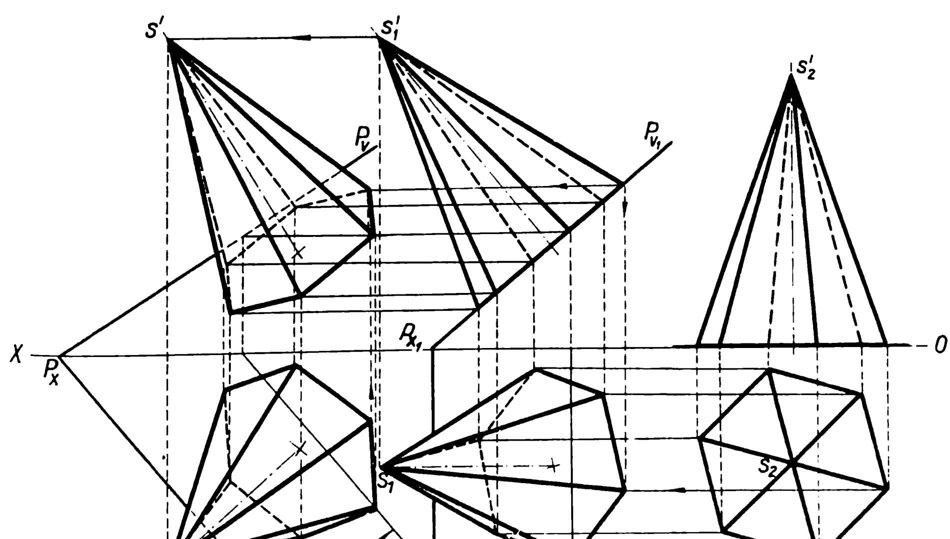



FIG. 686.

a position in which the line ($a_1b_1$, $a'_1b'_1$) becomes parallel to the $H$ plane. Having the projections of point $C_1$ and straight line $A_1B_1$, drop a perpendicular from $c_1$ to $a_1b_1$. Their intersection will yield the horizontal projection ($d_1$) of the foot of the perpendicular. Find its projections ($d$, $d'$) in the initial system and draw the projections of the desired perpendicular: the horizontal one through points $c$ and $d$, and the vertical one through points $c'$ and $d'$.

**Example 207**

Intersect the straight lines $AB$ and $CD$ with the line $MN$ perpendicular to them (Fig. 684).

*Solution.* Displace the lines ($ab$, $a'b'$) and ($cd$, $c'd'$) so that one of them, say ($ab$, $a'b'$), occupies a position perpendicular to the $H$ plane. Since this is an oblique line, make two consecutive displacements of the given system: first parallel to the $H$ plane with the line ($a_1b_1$, $a'_1b'_1$) parallel to the $V$ plane, and then parallel to the $V$ plane with the line ($a_2b_2$, $a'_2b'_2$) perpendicular to the $H$ plane. Intersect the lines ($a_2b_2$, $a'_2b'_2$) and ($c_2d_2$, $c'_2d'_2$) with a perpendicular line ($m_2n_2$, $m'_2n'_2$) and then find the line ($mn$, $m'n'$) in the initial position. The construction is obvious from the accompanying drawing.

**Example 208**

Construct the line of intersection of the pyramid with the plane $P$ (Fig. 685).

*Solution.* To construct the line of intersection, it is necessary to find the points of intersection of the edges of the pyramid with the given plane. Since the edges and the plane $P$ are oblique, it is advisable to change the position of the given system so that the plane $P_1$ becomes a vertical projecting one. Draw a horizontal line in the plane and, displacing the system parallel to the $H$ plane, bring the horizontal line and the plane $P_1$ to a position perpendicular to the $V$ plane. Construct all elements of the given system in the new position and find (according to the relevant rule) the projections ($a_1b_1c_1d_1$, $a'_1b'_1c'_1d'_1$) of the line of intersection of the pyramid with the plane $P_1$; then determine the respective projections in the initial position. The construction is evident from the drawing.

**Example 209**

Construct the projections of a regular hexahedron pyramid with the base lying in the plane $P$, given that the side of the base is 10 mm long and the altitude $h =$ $= 30$ mm (Fig. 686).

*Solution.* First construct the projections of the pyramid with the base on the $H$ plane. Then displace the plane $P$ so as to make it a vertical projecting plane $P_1$ and construct new projections of the pyramid with the base lying in the plane $P_1$. Finally, transfer these projections to the initial position. The construction is clear from the accompanying drawing.

**Example 210**

Bring the planes $P$ and $Q$ to a position perpendicular to the horizontal plane of projection (Fig. 687).

*Solution.* For the given planes to be perpendicular to the $H$ plane the line of their intersection should be perpendicular to this plane. Hence, find the line of intersection of the given planes, and by two consecutive displacements (as in Example 200) bring it, *together with the whole system*, to a position perpendicular to the $H$ plane. In this position the line of intersection of the planes $P_2$ and $Q_2$ is also perpendicular to the $H$ plane. The construction is obvious from the drawing.

*Note.* If two planes intersect along a horizontal or a frontal line, *one* displacement is sufficient to solve the problem.

*Conclusion.* An analysis of the problems solved by the method of displacement shows that the main task here is to find such a position of the given elements which

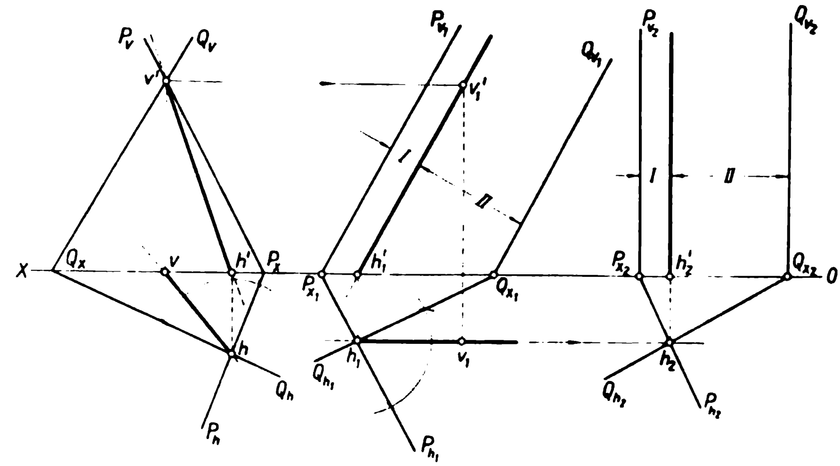


FIG. 687.

leads to a simple solution of the problem. We will call the position of geometrical elements relative to the projection planes at which the problem is solved without any auxiliary constructions "the most favourable" position of the given elements.

## PROBLEMS

(The following problems should be solved using the methods of revolution and displacement.)

371. Bring the line-segment *AB* to a position parallel to the horizontal plane of projection (Figs. 688, 689).

372. Bring the line-segment *AB* to a position parallel to the vertical plane of projection (Figs. 688, 689).

373. Bring the line-segment *AB* to a position perpendicular to the horizontal plane of projection (Figs. 690, 691).

374. Place the line-segment *AB* in a position perpendicular to the *V* plane (Figs. 691, 692).

375. Place the plane *P* in a position perpendicular to the *H* plane (Figs. 693, 694).

376. Place the plane *P* in a position perpendicular to the *V* plane (Figs. 693, 694).

377. Place the triangle *ABC* in a position at which its horizontal (vertical) projection will merge into a straight line (Fig. 600).

378. Place the parallel lines *AB* and *CD* in a position at which their vertical (horizontal) projections will merge into a straight line (Fig. 639).

379. Place the line-segments *AB* and *CD* in a position at which their horizontal (vertical) projections will be parallel (Fig. 578).

380. Place the line-segments *AB* and *CD* in a position at which *AB* will be perpendicular to the *H* plane (Fig. 578); to the *V* plane (Fig. 144).

381. Place the planes *P* and *Q* in a position perpendicular to the *H* plane (Figs. 393, 394); to the *V* plane (Figs. 400, 408).

382. Intersect the line-segments *AB* and *CD* with a line *MN*, which is perpendicular to *AB*, so that the segment of *MN* between the given lines is 200 mm long (Fig. 578).

383. Find the point at which the line *AB* pierces the plane *P* (Figs. 466, 467).

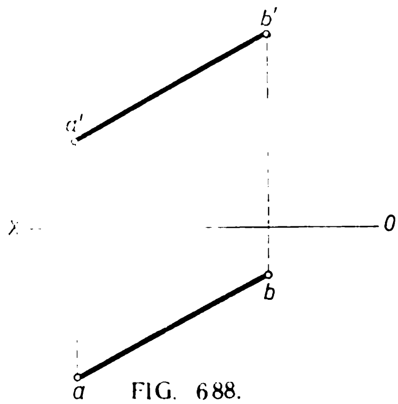



FIG. 688.

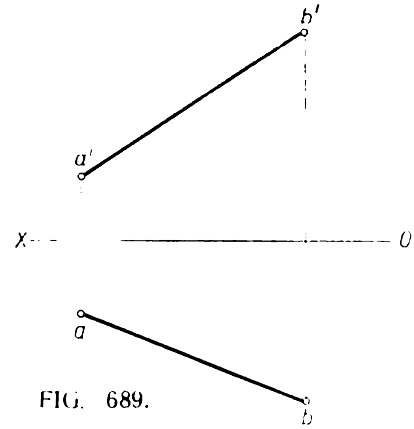



FIG. 689.

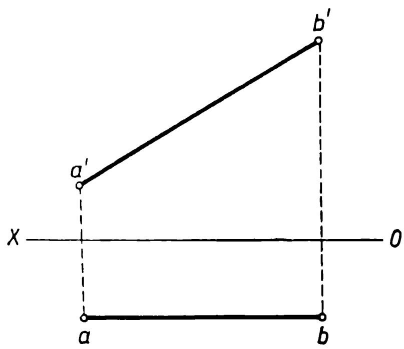



FIG. 690.

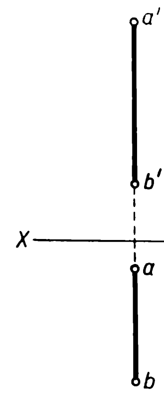



FIG. 691.

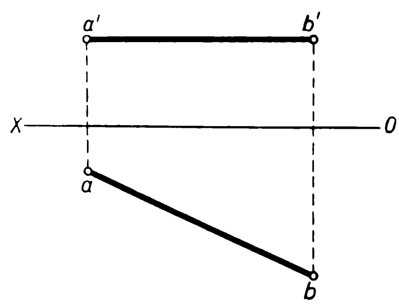



FIG. 692.

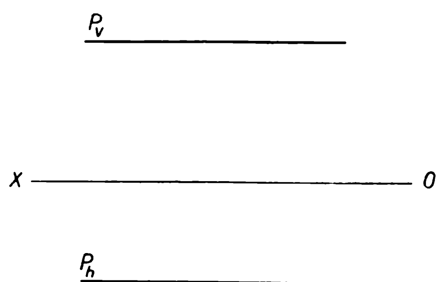



FIG. 693.

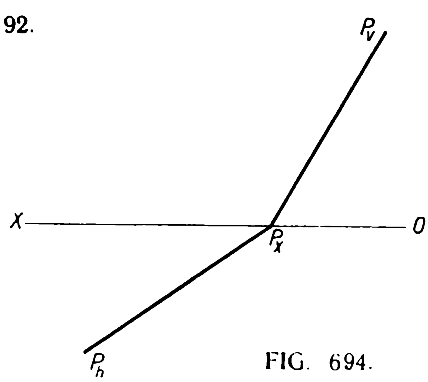



FIG. 694.

# CHAPTER XVIII

# THE COINCIDENCE METHOD. REVOLVING ABOUT HORIZONTAL AND FRONTAL LINES

## EXAMPLES

### Example 211

Given: the horizontal trace ($P_h$) of the plane $P$ and the point $A$ in this plane. Required: to find the coincident position of the point $A$ in the $H$ plane without determining the vertical trace of the plane (Fig. 695).

*Solution.* Through the point ($a$, $a'$) pass a plane $R$ perpendicular to the axis of revolution $P_h$ and find the centre of revolution ($\alpha$, $\alpha'$) at the intersection of the plane $R$ and the trace $P_h$. Determine the true length of the radius of revolution ($a\alpha$,

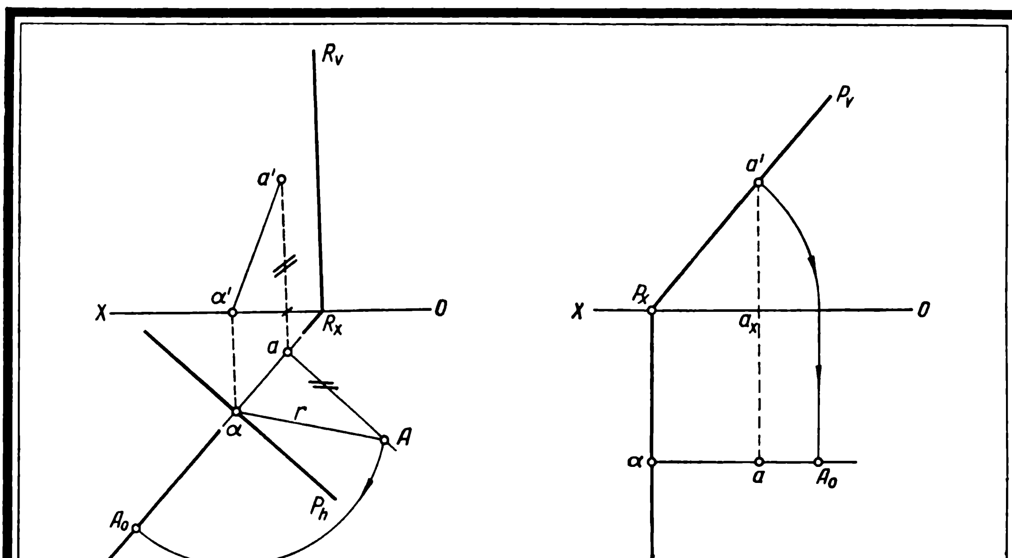



FIG. 695. FIG. 696.

$a'\alpha'$); then from the point $\alpha$ as centre describe an arc of radius $r$ to intersect the trace $P_h$ at the desired point $A_0$. The drawing shows only one alternative solution.

*Conclusions.* In bringing a plane to coincidence with the $H$ plane:

(1) the radius of revolution is the hypotenuse of a right-angle triangle one of whose legs is the distance from the horizontal projection of the point to the horizontal trace of the plane, the other leg being the $z$-coordinate of the point;

(2) the position of any point in the plane is found on the horizontal trace ($R_h$) of the plane of revolution at a distance equal to the radius of revolution;

(3) in a particular case when the plane is a horizontal projecting one, $a\alpha = 0$ and $r = z$.

### Example 212

Given: the vertical projecting plane $P$ and the point $A$ in it. Required: to determine the coincident position of the point $A$ in the $H$ plane (Fig. 696).

*Solution*. Since $a\alpha = p_x a_x$, the radius of revolution of the point $(a, a')$ is equal to $P_x a'$. Hence, drop a perpendicular from the horizontal projection $(a)$ of the point to the horizontal trace $(P_h)$ of the plane and lay off on it $\alpha A_0 = P_x a'$. We give one alternative solution.

**Example 213**

Given: the vertical trace $(P_v)$ of the plane $P$ and the point $A$ in this plane. Required: to find the coincident position of the point $A$ in the $V$ plane without determining the horizontal trace of the plane (Fig. 697).

*Solution*. Through the point $(a, a')$ pass a plane $R$ perpendicular to the axis of revolution $P_v$ and find the centre of revolution $(\beta, \beta')$ at the intersection of the plane $R$ and the trace $P_v$. Determine the true length of the radius of revolution $(a\ \beta, a'\beta')$; then from $\beta'$ as centre describe an arc of radius $r$ to intersect the trace $R_v$ at the desired point $A_0$. We give only one alternative solution.

*Conclusions*. In bringing a plane to coincidence with the $V$ plane:

(1) the radius of revolution is the hypotenuse of a right-angled triangle one of whose legs is the distance from the vertical projection of the point to the vertical trace of the plane, the other leg being the $y$-coordinate of the point;

(2) the position of any point in the plane is found on the vertical trace $(R_v)$ of the plane of revolution at a distance equal to the radius of revolution;

(3) in a particular case when the plane is a vertical projecting one, $a'\beta' = 0$ and $r = y$.

**Example 214**

Given: the horizontal projecting plane $P$ and the point $A$ in it. Required: to find the coincident position of the given point $A$ on the $V$ plane (Fig. 698).

*Solution*. Since $a'\beta' = P_x a_x$, the radius of revolution of the point $(a, a')$ equals $P_x a$. Hence, drop a perpendicular from the vertical projection $(a')$ of the point to the vertical trace $(P_v)$ of the plane and lay off on it $\beta' A_0 = P_x a$. We give only one alternative solution.

**Example 215**

Given: a point $A$ and a line-segment $MN$. Required: by revolving the point $A$ about $MN$, bring it to coincidence with a plane $T$ parallel to the horizontal plane of projection and passing through the line $MN$ (Fig. 699).

*Solution*. Through the point $(a, a')$ pass a plane $R$ perpendicular to the axis of revolution $(mn, m'n')$ and find the centre of revolution $(\alpha, \alpha')$ at the intersection of the line $(mn, m'n')$ and the plane $R$. Determine the true length of the radius of revolution $(a\alpha, a'\alpha')$; then from point $\alpha$ as centre describe an arc of radius $r$ to intersect the trace $R_h$ at the desired point $A_0$. The drawing shows only one alternative solution.

*Conclusion*. The radius of revolution is the hypotenuse of a right-angled triangle, one of whose legs is the distance from the horizontal projection of the point to the horizontal projection of a horizontal line, the other leg being the distance between the vertical projections of the point and the horizontal line.

**Example 216**

Given: a point $A$ and a line-segment $MN$. Required: by revolving the point $A$ about $MN$ bring it to coincidence with a plane $T$ parallel to the vertical plane of projection and passing through the line $MN$ (Fig. 700).

*Solution*. Through the point $(a, a')$ pass a plane $R$ perpendicular to the axis of revolution $(mn, m'n')$ and find the centre of revolution $(\beta, \beta')$ at the intersection of the plane $R$ and the line $(mn, m'n')$. Determine the true length of the radius of revolution $(a\beta, a'\beta')$; then from point $\beta'$ as centre describe an arc of radius $r$ to intersect the trace $R_v$ at the desired point $A_0$. The drawing shows only one alternative solution.

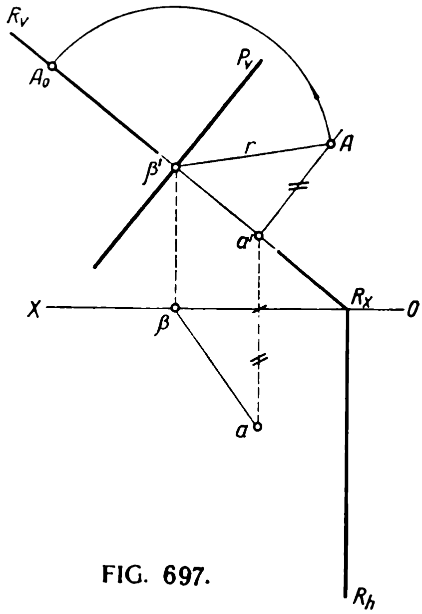



FIG. 697.

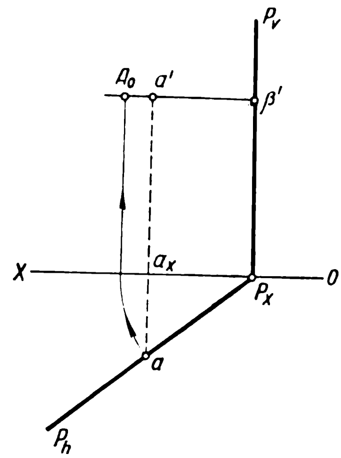



FIG. 698.

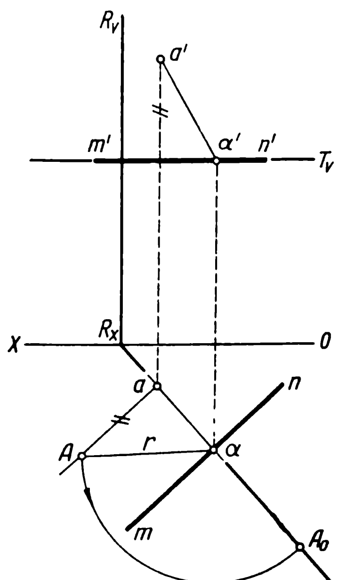



FIG. 699.

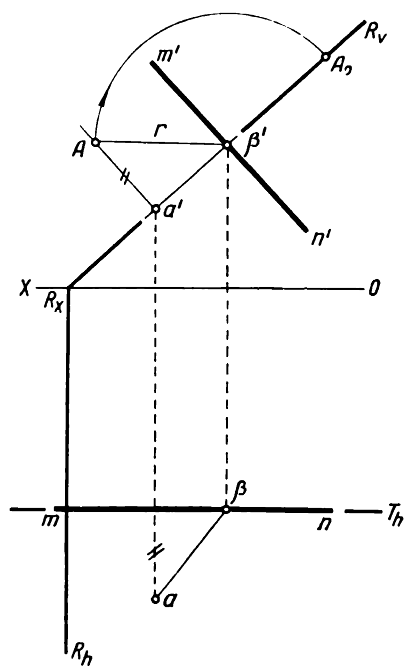



FIG. 700.

*Conclusion*. The radius of revolution is the hypotenuse of a right-angled triangle one of whose legs is the distance between the vertical projections of a point and a frontal line, the other leg being the distance between the horizontal projections of the point and the frontal line.

**Example 217**

Bring the plane $P$ to coincidence with the $H$ plane (Fig. 701).

*Solution*. Revolve the plane $P$ about its horizontal trace ($P_h$). To find the coincident vertical trace ($P_{v_1}$) of the plane, take an arbitrary point ($v$, $v'$) on the trace $P_v$ and find its coincident position $V_0$ in the $H$ plane (the construction is given in the accompanying drawing). Draw the desired coincident vertical trace ($P_{v_1}$) of the plane through the points $P_x$ and $V_0$; $\varphi$ is the true angle between the plane traces.

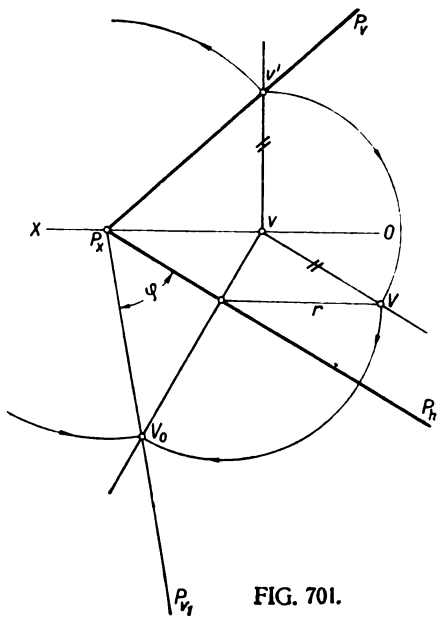


FIG. 701.

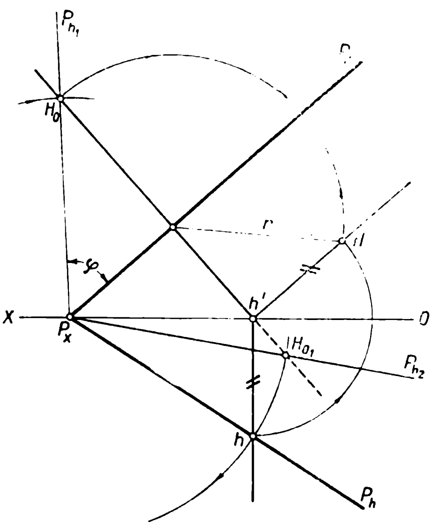


FIG. 702.

As is obvious from the drawing, $P_xV_0 = P_xv'$, which simplifies the solution of the problem. Drop a perpendicular from the point $v$ to the trace $P_h$, and then from the point $P_x$ as centre describe an arc of radius $P_xv'$ to intersect the perpendicular at the point $V_0$. Draw the coincident trace $P_{v_1}$ through the points $P_x$ and $V_0$.

**Example 218**

Bring the plane $P$ to coincidence with the $V$ plane (Fig. 702).

*Solution*. Revolve the plane $P$ about its vertical trace ($P_v$). To find the coincident horizontal trace ($P_{h_1}$) of the plane, take an arbitrary point ($h$, $h'$) on the trace $P_h$ and find its coincident position $H_0$ on the $V$ plane (the construction is given in the accompanying drawing). Draw the desired coincident horizontal trace ($P_{h_1}$) of the plane through the points $P_x$ and $H_0$; $\varphi$ is the true angle between the plane traces.

As is obvious from the drawing, $P_xH_0 = P_xh$, which simplifies the solution of the problem. Drop a perpendicular from the point $h'$ to the trace $P_v$, and then from point $P_x$ as centre describe an arc of radius $P_xh$ to intersect the perpendicular at points $H_0$ and $H_{0_1}$. Draw the coincident traces $P_{h_1}$ and $P_{h_2}$ through the respective pairs of points: $P_x$, $H_0$ and $P_x$, $H_{0_1}$.

**Example 219**

Given: a plane $P$ and a point $A$ belonging to it. Find the coincident position of the given point in the $H$ plane without determining the radius of revolution (Figs. 703, 704).

*Solution.*

*First method.* Using a horizontal line, find point $V_0$, which is the coincident position of the point $(v, v')$. Draw through it a coincident horizontal line parallel to the horizontal trace $(P_h)$ of the plane. Drop a perpendicular from the point $a$ to the trace $P_h$ to intersect the coincident horizontal line at the desired point $A_0$.

*Second method.* Using a frontal line, find the coincident position of the vertical trace $(P_{v_1})$ of the plane and draw a coincident vertical line through the point $h$ and parallel to the coincident vertical trace of the plane. Drop a perpendicular from point $a$ to trace $P_h$ to intersect the coincident frontal line at the desired point $A_0$.

*Conclusion.* In bringing a plane to coincidence with the $H$ plane, the position of any point contained in the plane is found at the intersection of a coincident principal line and a perpendicular dropped from the horizontal projection of the point to the horizontal trace of the plane.

**Example 220**

Given: a plane $P$ and a point $A$ belonging to it. Find the coincident position of the given point in the $V$ plane without determining the radius of revolution (Figs. 705, 706).

*Solution.*

*First method.* Find the coincident horizontal trace $(P_{h_1})$ of the plane and draw a coincident horizontal line through the point $v'$ and parallel to the trace $P_{h_1}$. Drop a perpendicular from point $a'$ to the trace $P_v$ to intersect the coincident horizontal line at the desired point $A_0$.

*Second method* (making use of a frontal line). Find point $H_0$, which is the coincident position of the point $(h, h')$, and draw a coincident frontal line through $H_0$ and parallel to the vertical trace $(P_v)$ of the plane. Drop a perpendicular from the point $a'$ to the trace $P_v$ to intersect the coincident frontal line at the desired point $A_0$.

*Conclusion.* In bringing a plane to coincidence with the $V$ plane, the position of any point contained in the plane is found at the intersection of a coincident principal line and a perpendicular dropped from the vertical projection of the point to the vertical trace of the plane.

*Note.* On the basis of the above examples and drawings it is easy to solve the converse problem, i.e. to find the projections of a point given its coincident position in the horizontal (vertical) plane of projection (see examples given below).

**Example 221**

Given: the plane $P$ and the coincident position of its point $A_0$ in the $H$ plane. Find the projections of this point (Fig. 707).

*Solution.* Find the coincident vertical trace $(P_{v_1})$ of the plane and draw a coincident horizontal line through the point $A_0$ and parallel to the horizontal trace $(P_h)$ of the plane to intersect the trace $P_{v_1}$ at the point $V_0$. From the point $V_0$ determine its projections $(v, v')$ and draw through them the projections of the horizontal line (how?). Then drop a perpendicular from the point $A_0$ to the trace $P_h$ to intersect the horizontal projection of the horizontal line at point $a$, which is the horizontal projection of the point; now determine its vertical projection $(a')$.

The problem may also be solved with the aid of a frontal line.

**Example 222**

Given: a plane $P$ and the coincident position of its point $A_0$ in the $H$ plane. Find the projections of this point (Fig. 708).

*Solution.* The point $A_0$ lies in the rear half-plane of the $H$ plane. Determine the coincident vertical trace $(P_{v_1})$ of the given plane and draw a coincident frontal line

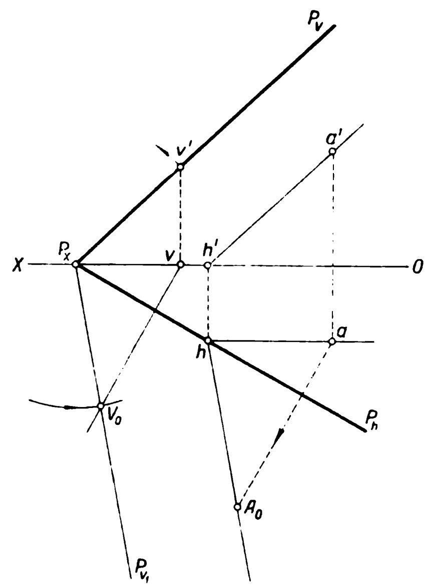



FIG. 703.

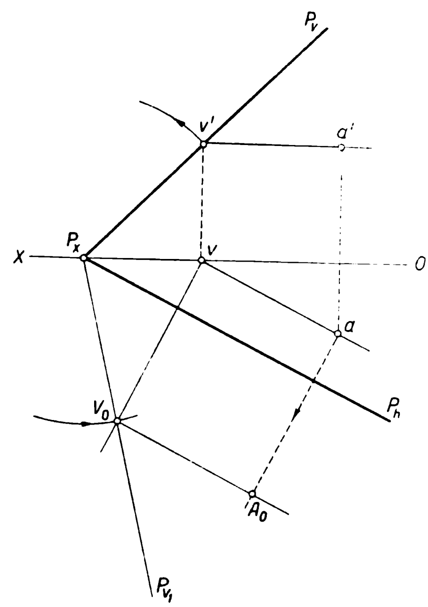



FIG. 704.

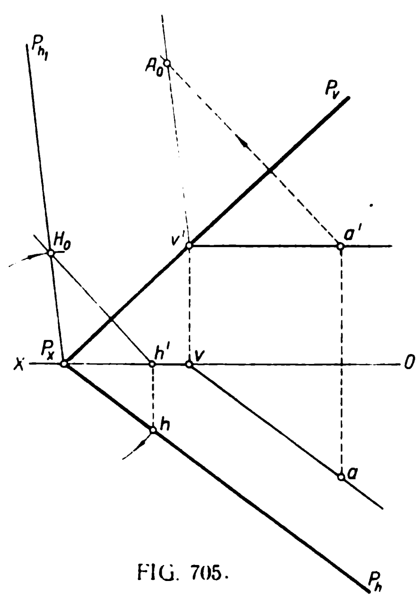



FIG. 705.

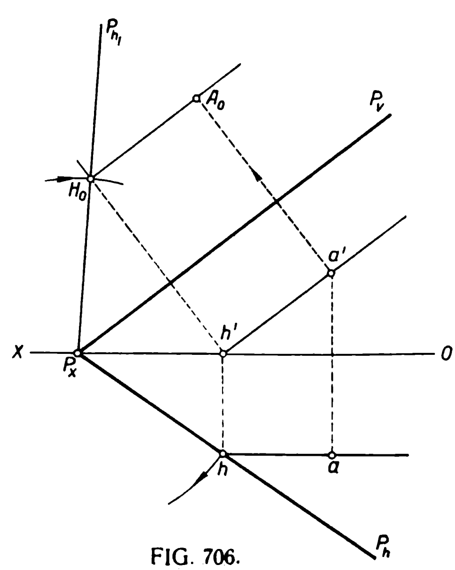



FIG. 706.

through the point $A_0$ and parallel to the vertical trace ($P_{v_1}$) to intersect the trace $P_h$ at the point $H_0$. Determine the projections ($h$, $h'$) of the latter point and draw through them the projections of the frontal line (how?). Drop a perpendicular from the point $A_0$ to the trace $P_h$ to intersect the horizontal projection of the frontal line at point $a$, which is the horizontal projection of the point; then determine its vertical projection ($a'$).

The problem may also be solved with the aid of a horizontal line.

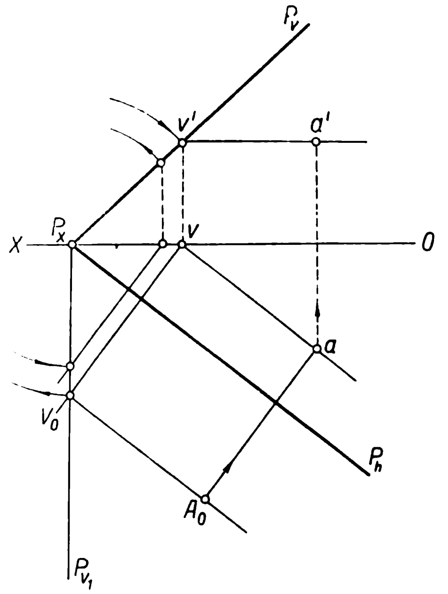



FIG. 707.

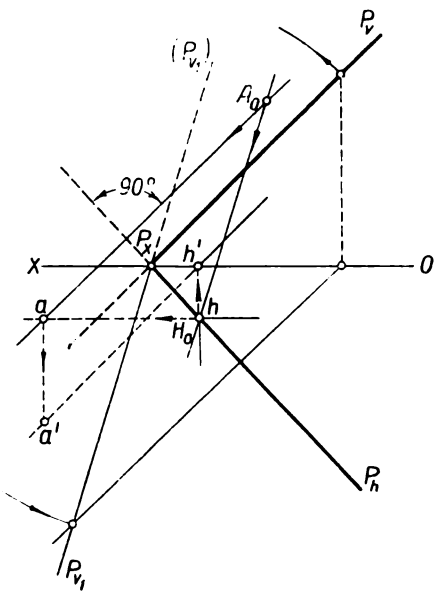



FIG. 708.

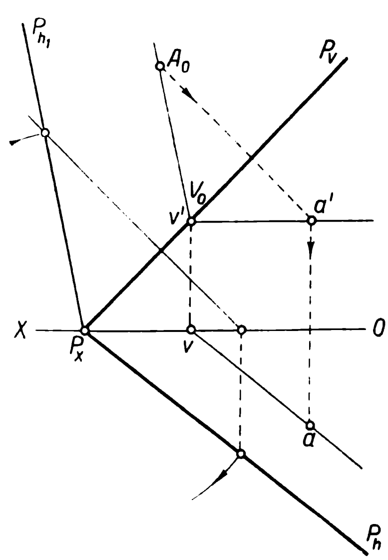



FIG. 709.

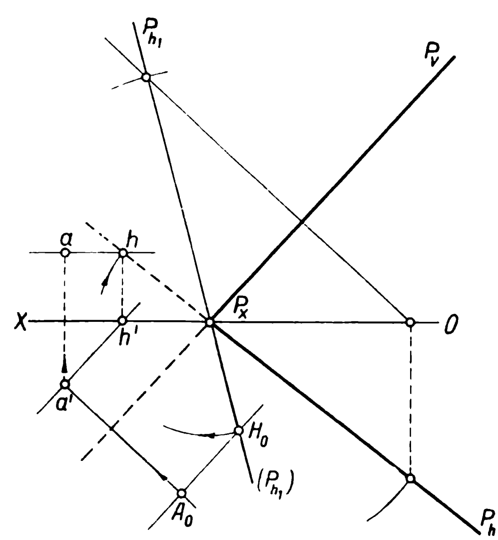



FIG. 710.

**Example 223**

Given: a plane $P$ and the coincident position of its point $A_0$ in the $V$ plane. Required: to find the projections of this point (Fig. 709).

*Solution.* Find the coincident horizontal trace ($P_{h_1}$) of the plane and draw a coincident horizontal line through the point $A_0$ and parallel to the coincident horizontal trace ($P_{h_1}$) of the plane to intersect the trace $P_v$ at point $V_0$. Then determine the projections ($v$, $v'$) of the latter point and draw through them the projections of the horizontal line (how?). Drop a perpendicular from the point $A_0$ to the trace $P_v$ to intersect the vertical projection of the horizontal line at point $a'$, which is the vertical projection of the point; finally, determine its horizontal projection ($a$).



15—1063

**Example 224**

Given: a plane $P$ and the coincident position of its point $A_0$ in the $V$ plane. Required: to find the projections of this point (Fig. 710).

*Solution.* The point $A_0$ lies in the lower half-plane of the $V$ plane. Determine the coincident horizontal trace ($P_{h_1}$) of the given plane and draw a coincident frontal line through the point $A_0$ and parallel to the trace $P_v$ to intersect the trace $P_{h_1}$ at point $H_0$. Now find the projections ($h$, $h'$) of the latter point and draw through them the projections of the frontal line (how?). Drop a perpendicular from the point $A_0$ to the trace $P_v$. Its intersection with the vertical projection of the frontal line yields the vertical projection ($a'$) of the point. Now find the horizontal projection ($a$) of the point.

The problem may also be solved with the aid of a horizontal line.

**Example 225**

Construct the projections of the right triangle $ABC$ lying in the plane $P$, given the vertical projection of the hypotenuse $AC$ and the angle $C = 60°$ (Fig. 711).

*Solution.* First find the coincident position $A_0$ and $C_0$ of the points ($a$, $a'$) and ($c$, $c'$) in the $H$ plane. Construct the triangle $A_0B_0C_0$ true size and then, knowing the point $B_0$, find its projections ($b$, $b'$). Join the point ($b$, $b'$) to the points ($a$, $a'$) and ($c$, $c'$) to obtain the desired projections ($abc$) and ($a'b'c'$) of the triangle.

The same problem is solved in Fig. 712, where the plane $P$ is brought to coincidence with the $V$ plane. The construction is obvious from the drawing.

**Example 226**

Construct the projections of the equilateral triangle $ABC$ contained in the plane $P$, given the horizontal projection of its side $AB$ (Fig. 713).

*Solution.* Find the coincident position $A_0B_0$ of the side of the triangle in the $V$ plane. Construct the triangle $A_0B_0C_0$ true size and then, knowing the point $C_0$, find its projections ($c$, $c'$). Joining the point ($c$, $c'$) to the endpoints of the side ($ab$, $a'b'$), we obtain the desired projections of the triangle.

The same problem is solved in Fig. 714, where the plane $P$ is brought to coincidence with the $H$ plane. The construction is obvious from the drawing.

**Example 227**

Determine the true size of the triangle $ABC$ contained in the plane $P$ parallel to the coordinate axis, given the horizontal projection (Fig. 715).

*Solution.* Find the projections ($a'b'c'$) and ($a''b''c''$) of the given triangle. Bring the plane $P$ to coincidence with the $W$ plane; the position of the point $A_0$ is found by dropping a perpendicular from the profile projection of the point to the trace $P_w$ and marking off on it a line-segment $a''A_0$ equal to the $x$-coordinate of the point ($a$, $a'$). Points $B_0$ and $C_0$ are found likewise. Join these points to obtain the triangle $A_0B_0C_0$, which is the true size of the given triangle.

This problem can be solved by bringing the plane $P$ to coincidence with the horizontal (Fig. 716) or vertical (Fig. 717) plane. The construction is obvious from the drawings.

The radii of revolution of the vertices of the triangle are determined with the aid of the profile plane of projection (how?).

*Conclusion.* The radius of revolution of any point in a plane parallel to the coordinate axis, when brought to coincidence

(1) with the $W$ plane, is equal to the $x$-coordinate of the point;

(2) with the $H$ plane, is equal to $P_ya''$, $P_yb''$, $P_yc''$, etc.

(3) with the $V$ plane, is equal to $P_za''$, $P_zb''$, $P_zc''$, etc.

**Example 228**

Construct the projections of the square $ABCD$ lying in the plane $P$, given the vertical projection of its diagonal $AC$ (Fig. 718).

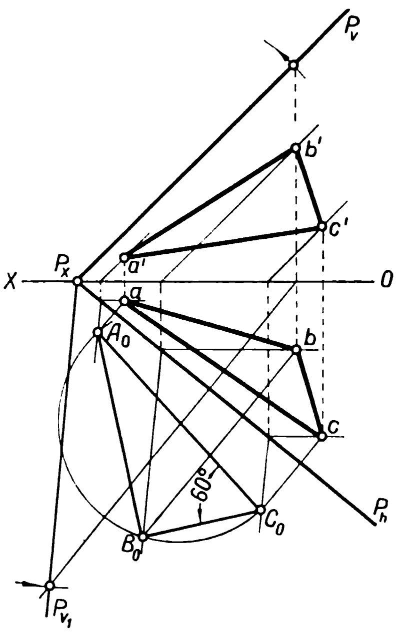



FIG. 711.

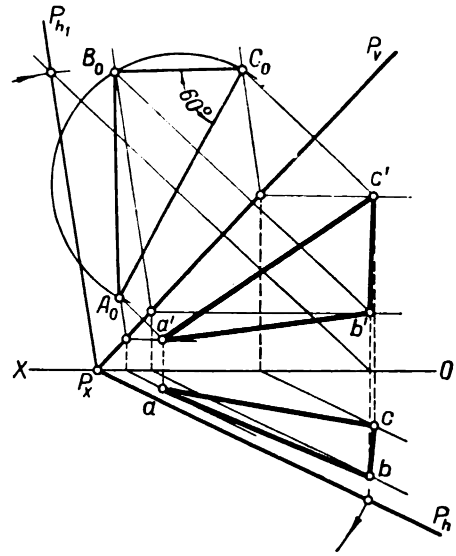



FIG. 712.

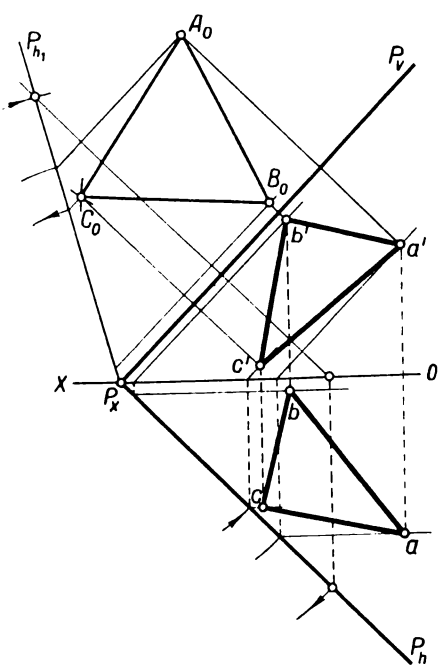



FIG. 713.

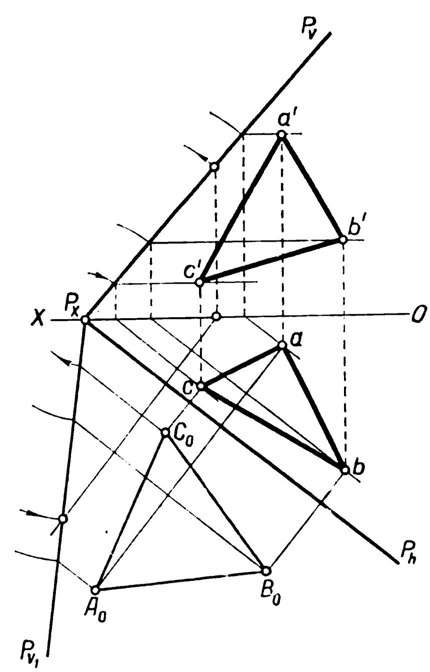



FIG. 714.

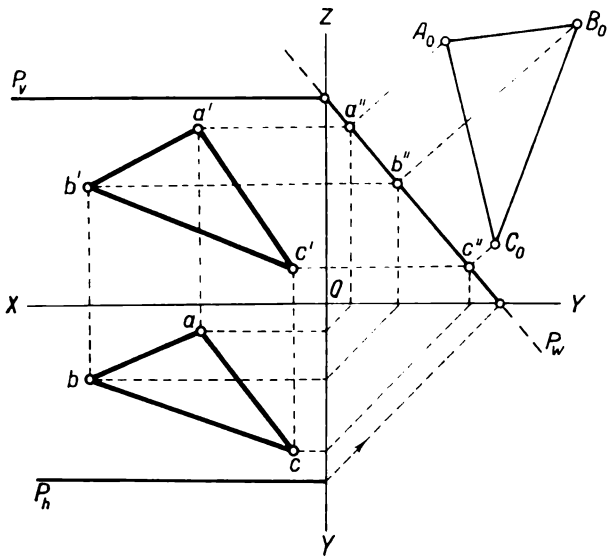



FIG. 715.

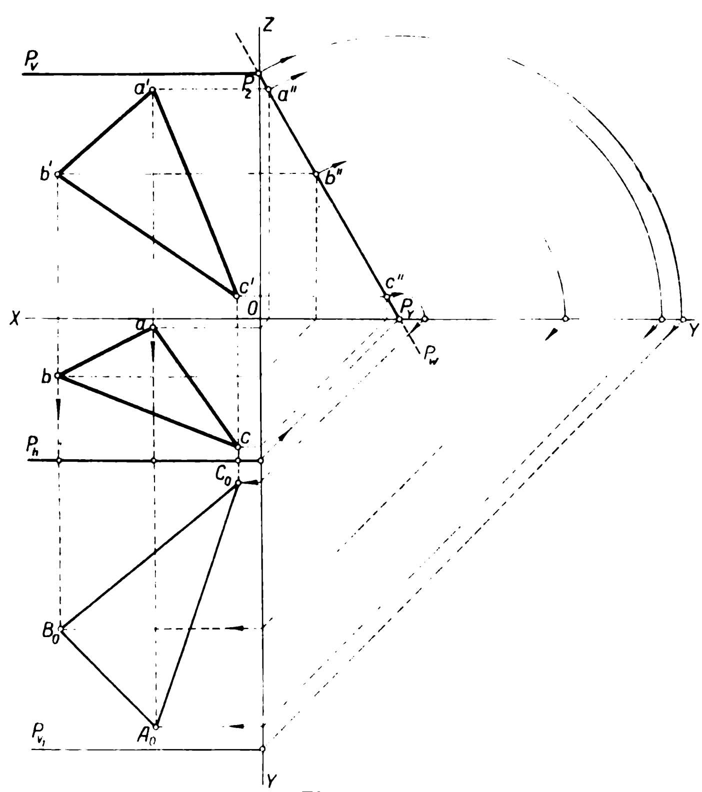



FIG. 716.

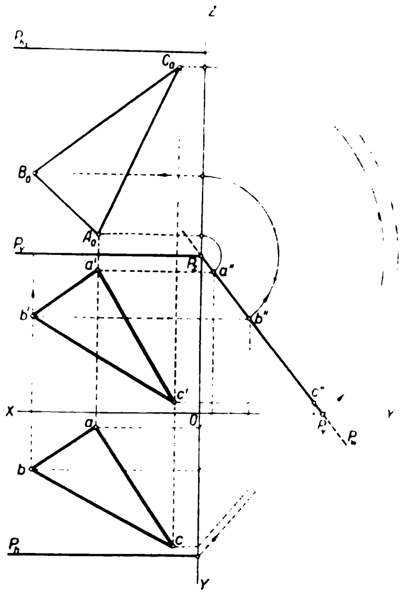



FIG. 717.

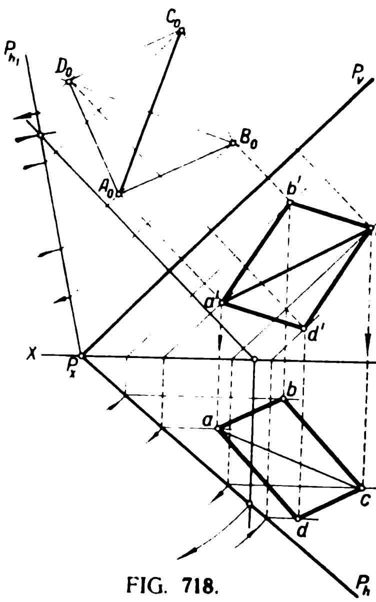



FIG. 718.

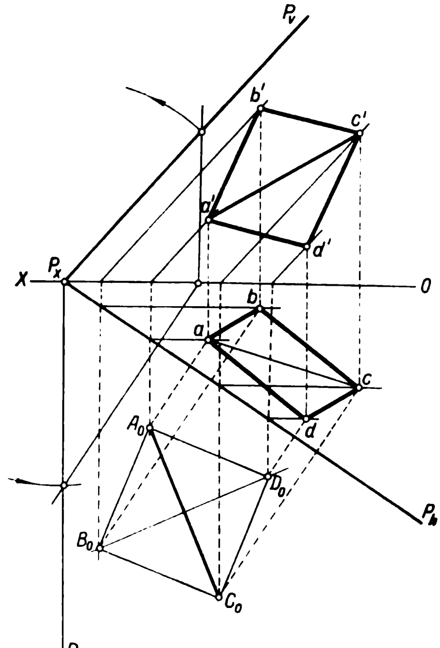



FIG. 719.

*Solution.* Bring the plane $P$ to coincidence with the $V$ plane and find the position of points $A_0$ and $C_0$. Construct the square $A_0B_0C_0D_0$ on the diagonal $A_0C_0$ and then find the projections ($abcd$) and ($a'b'c'd'$) of the square in the reversed order. The same problem is solved in Fig. 719, where the plane $P$ is brought to coincidence with the $H$ plane.

**Example 229**

Construct the projections of a circle lying in the plane $P$, given its centre $C$ and radius equal to 20 mm (Fig. 720).

*Solution.* A circle is projected on the planes of projection as an ellipse, its major axis always being equal to the diameter of the circle. Since the circle lies in an oblique

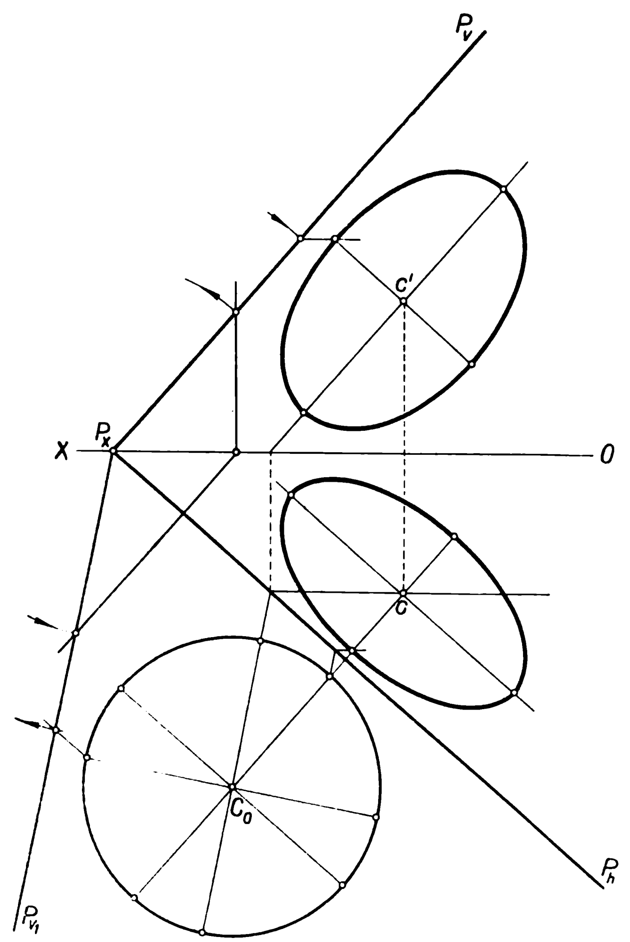



FIG. 720.

plane it is impossible to single out the diameter which is projected true length on both planes. Hence, the principal axes of the ellipses should be found on the $H$ and $V$ planes *independently*. The diameter lying on the horizontal line in the plane $P$ is projected true length onto the horizontal plane of projection to represent the major axis of the ellipse; the diameter perpendicular to it is foreshortened and represents the minor axis of the ellipse perpendicular to the major axis (according to the theorem on projecting a right angle). Likewise, the diameter lying on the frontal line in the given plane is projected true length onto the vertical plane of projection to represent the major axis of the ellipse; the diameter perpendicular to it is foreshortened and represents the minor axis of the ellipse, which is perpendicular to the major axis.

Hence the following sequence of construction:

(1) bring the given plane to coincidence, say, with the $H$ plane and find the position of the point $C_0$, the centre of the circle;

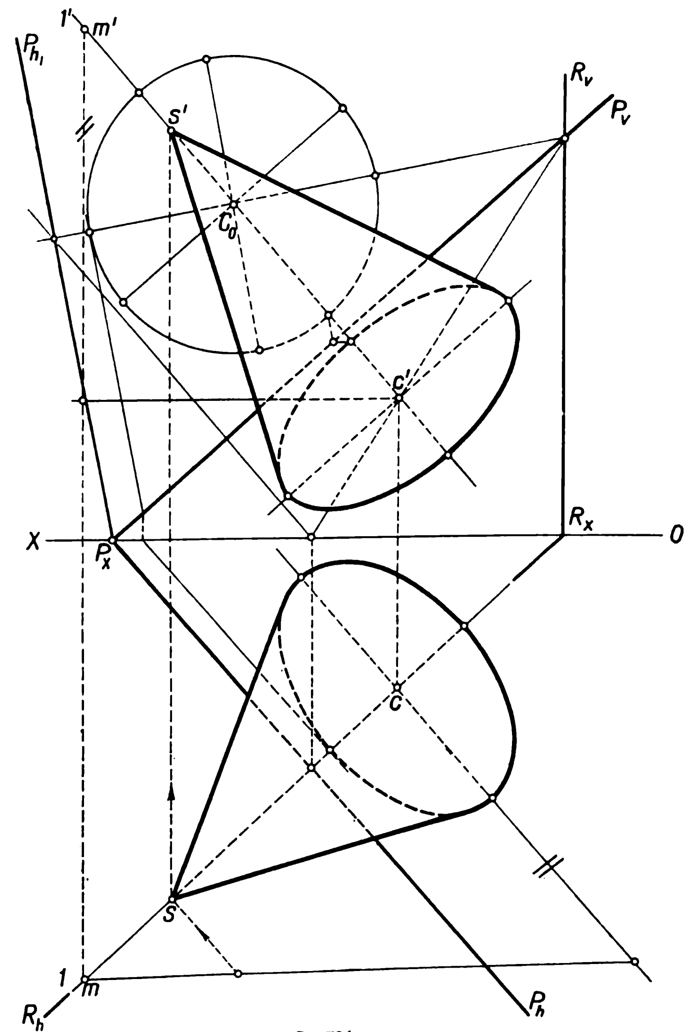



FIG. 721.

(2) from the point $C_0$ as centre describe a circle of the given radius and draw two pairs of mutually perpendicular diameters: (1) a diameter parallel to the trace $P_h$ and another diameter perpendicular to the former one; (2) a diameter parallel to the trace $P_{v_1}$ and another diameter perpendicular to the former one. Then find the horizontal projections of the first pair of diameters and the vertical projections of the second pair of diameters.

The further construction is obvious from the drawing.

*Conclusions*:

1. The principal axes of an ellipse in the $H$ plane are the horizontal projections of two mutually perpendicular diameters, one of which is located parallel to the $H$ plane.

2. The principal axes of an ellipse in the $V$ plane are the vertical projections of two mutually perpendicular diameters, one of which is located parallel to the $V$ plane.

**Example 230**

Construct the projections of a right circular cone with the base lying in the plane $P$, given that the radius of its base is equal to 20 mm, altitude $h = 55$ mm and its axis coincides with the line $(1, 1')$ (Fig. 721).

*Solution.* Find the point $(c, c')$ of intersection of the line $(1, 1')$ and the plane $P$, i.e. the centre of the base of the cone. Bring the plane $P$ to coincidence with the $V$ plane, find point $C_0$, and from the latter as centre describe a circle of the given radius (20 mm). Knowing the coincident position of the base, determine its projections (for more details see Example 229). To find point $(s, s')$—the vertex of the cone—lay off on the line $(1, 1')$ from the point $(c, c')$ a segment 55 mm long. Draw the projections of the extreme elements through the points $s$ and $s'$ and tangent to each ellipse.

**Example 231**

Construct the projections of a cube with the base $ABCD$ and with one of its side edges coinciding with the line $(1, 1')$, given the vertical projection of the diagonal $AC$ (Fig. 722).

*Solution.* The base $ABCD$ of the cube lies in the plane $P$, which is perpendicular to the line $(1, 1')$ passing through point $(a, a')$. Knowing the vertical projection $(a')$ of the vertex $(a, a')$, find its horizontal projection $(a)$ on the horizontal projection of the line $(1, 1')$. Pass a plane $P$ through the point $(a, a')$ and perpendicular to the line $(1, 1')$, and find the horizontal projection $(ac)$ of the diagonal. Bring this plane to coincidence with the $H$ plane, find the position of the diagonal $A_0C_0$ and construct on it the square $A_0B_0C_0D_0$, the base of the cube. Knowing the coincident position of the base of the cube, find its projections $(abcd)$ and $(a'b'c'd')$. Erect at points $(b, b')$, $(c, c')$ and $(d, d')$ perpendiculars to the plane $P$, lay off on them segments equal to the side of the base of the cube and join the endpoints of the perpendiculars obtained. The drawing shows visible and invisible lines.

**Example 232**

Drop a perpendicular from the point $C$ to the line $AB$. Solve the problem by revolving about a horizontal and a frontal line (Figs. 723, 724).

*Alternative solutions.*

1. *Revolving about a horizontal line*

Through the point $(c, c')$ draw a horizontal line intersecting the line $(ab, a'b')$ at the point $(k, k')$, and, revolving the given line about a horizontal line, bring it to coincidence with the plane $R$ passing through this horizontal line and parallel to the $H$ plane. Since the point $(k, k')$ of the line $(ab, a'b')$ already lies in the plane $R$, it is necessary to find the position of its other point $(a, a')$ in this plane. To do this, drop a perpendicular from point $a$ to the horizontal projection $(ck)$ of the horizontal line, and from point $\alpha$ as centre describe an arc of radius $\alpha A$, equal to the radius of revolution of the point $(a, a')$, to intersect the perpendicular at point $A_0$. Draw a straight line through the points $k$ and $A_0$ and drop a perpendicular on it from point $c$. The point of their intersection yields the point $D_0$, which is the foot of the perpendicular. Drop from the point $D_0$ a perpendicular to the horizontal projection $(ck)$ of the horizontal line to intersect the line $ab$ at point $d$, the horizontal projection of the foot of the perpendicular. Now find its vertical projection $(d')$ on the line $a'b'$. Join the points $(c, c')$ and $(d, d')$ to obtain the desired line $(cd, c'd')$.

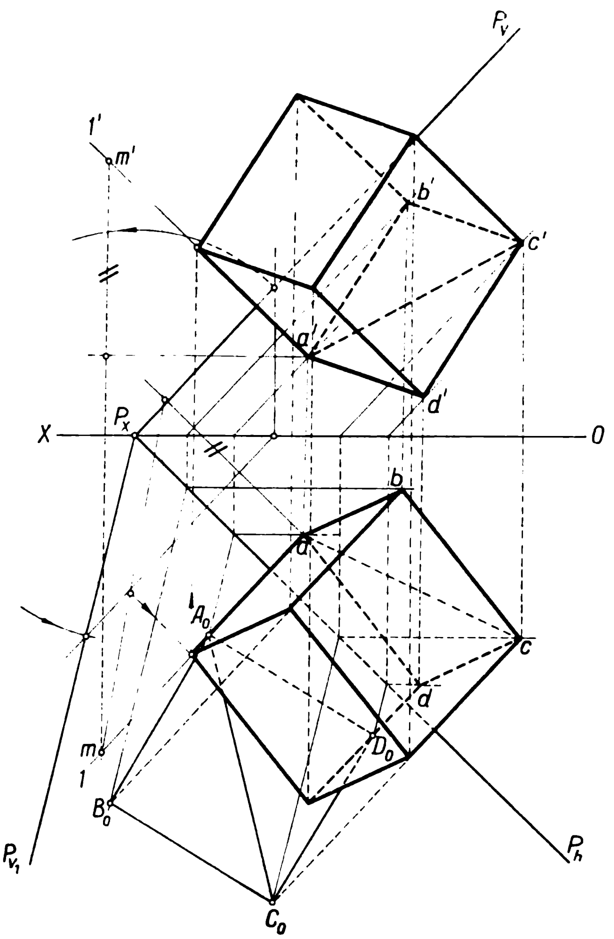


FIG. 722.

2. *Revolving about a frontal line*

Through the point ($c$, $c'$) draw a frontal line intersecting the line ($ab$, $a'b'$) at point ($k$, $k'$) and, revolving the given line ($ab$, $a'b'$) about the frontal line, bring it to coincidence with a plane $R$ passing through the frontal line and parallel to the $V$ plane. Since the point ($k$, $k'$) of the line ($ab$, $a'b'$) already lies in the plane $R$, it is necessary to determine the position of another point ($a$, $a'$) in it. To do this, drop a perpendicular from the point $a'$ to the vertical projection ($c'k'$) of the frontal line, and from $\beta'$ as centre and with a radius $\beta'A$ equal to the radius of revolution of the point ($a$, $a'$) describe an arc to intersect the perpendicular at point $A_0$. Join $k'$ to $A_0$ to obtain a straight line $k'A_0$. Then drop on it a perpendicular from the point $c'$. Their intersection will yield point $D_0$, the foot of the perpendicular. Now drop a perpendicular from the point $D_0$ onto the vertical projection ($c'k'$) of the frontal line to intersect the line $a'b'$ at point $d'$, which is the vertical projection of the foot of the perpendicular. Knowing the point $d'$, find the horizontal projection ($d$) on the line $ab$. Join the points ($c$, $c'$) and ($d$, $d'$) to obtain the desired line ($cd$, $c'd'$).

*Conclusion*. The method of revolution about a horizontal or a frontal line is useful in solving problems where the given elements lie in one and the same plane.

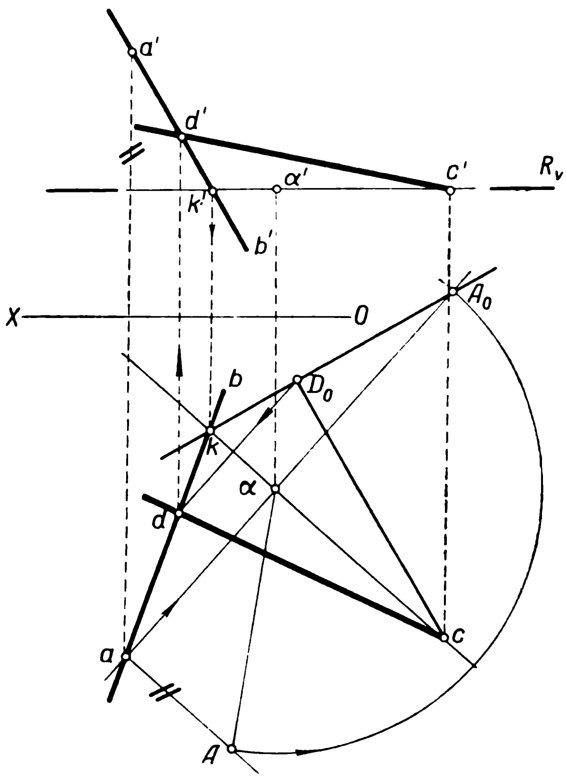


FIG. 723.

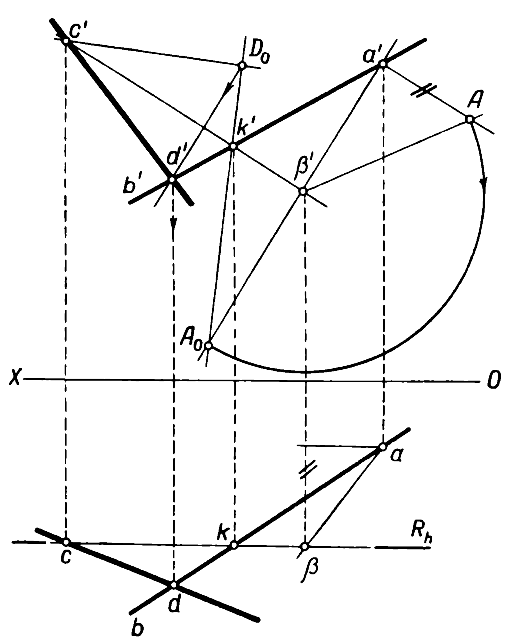


FIG. 724.

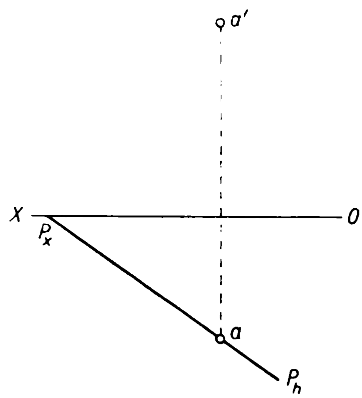



FIG. 725.

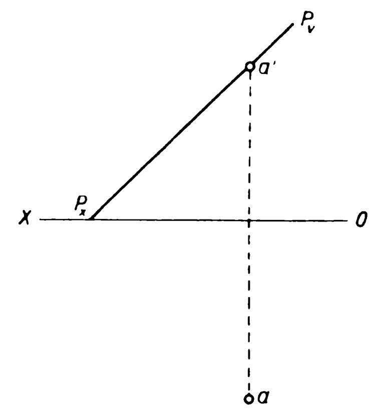



FIG. 726.

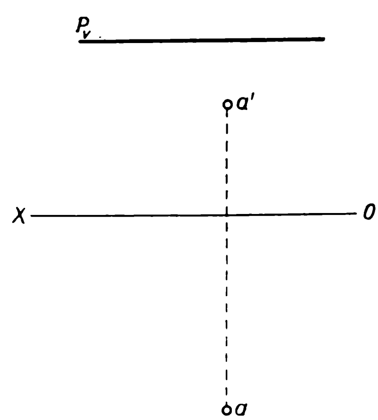



FIG. 727.

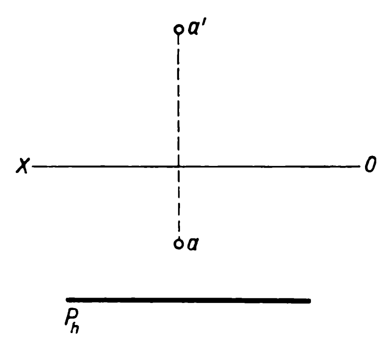



FIG. 728.

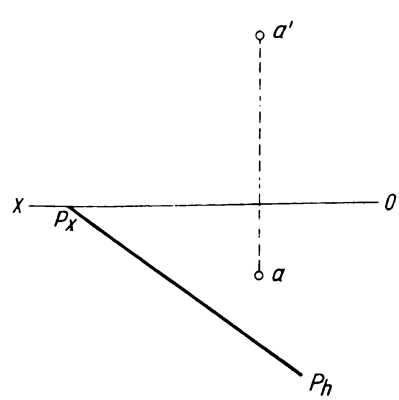



FIG. 729.

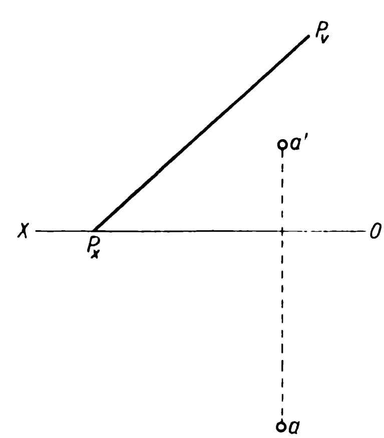



FIG. 730.

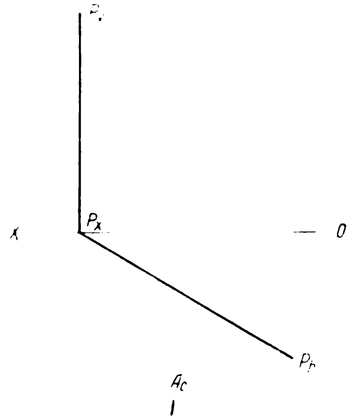


FIG. 731.

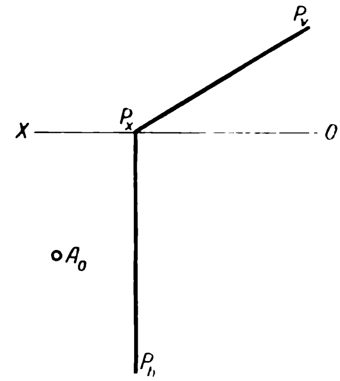



FIG. 732.

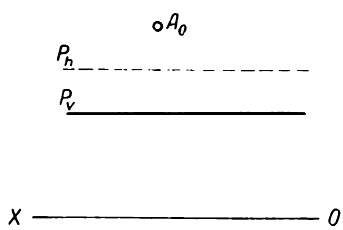



FIG. 733.

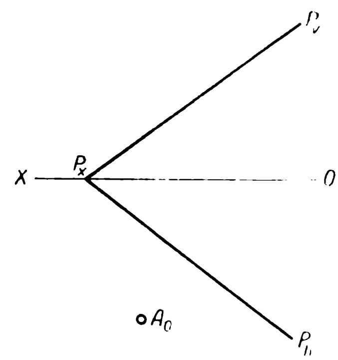



FIG. 734.

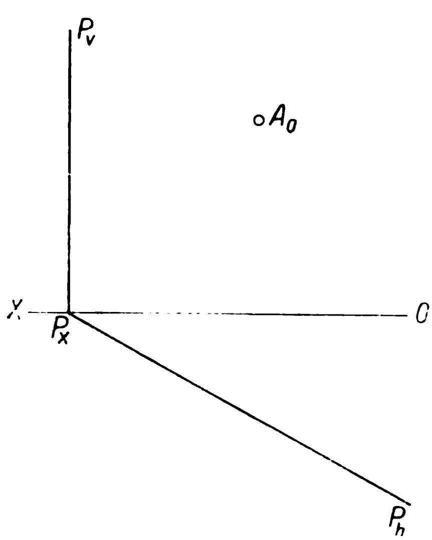



FIG. 735.

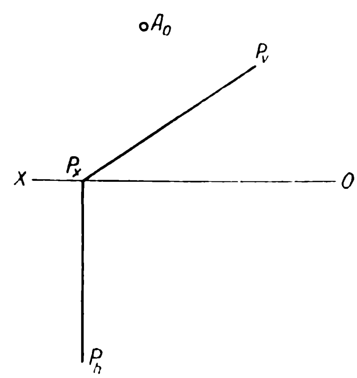



FIG. 736.

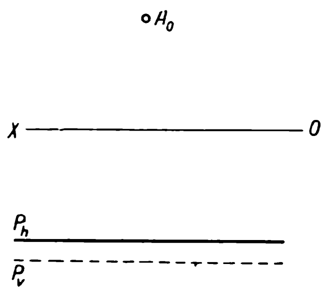


FIG. 737.

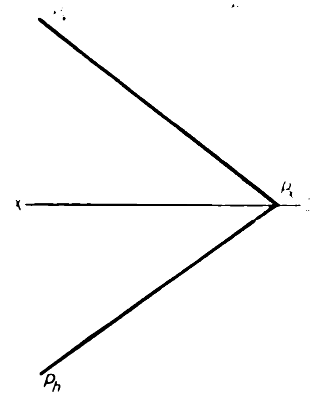


FIG. 738.

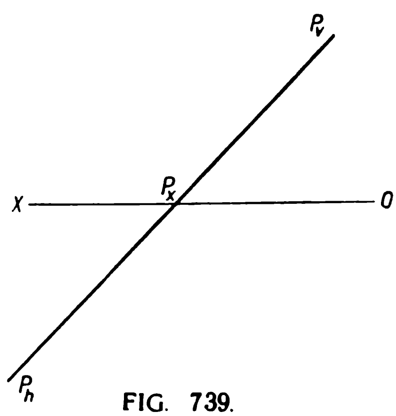


FIG. 739.

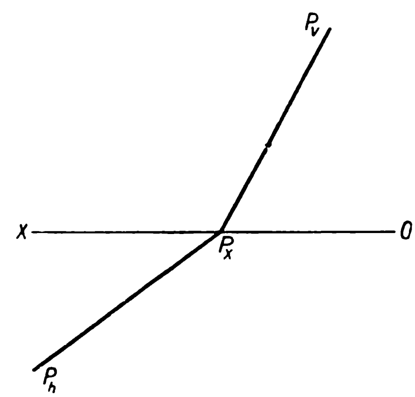


FIG. 740.

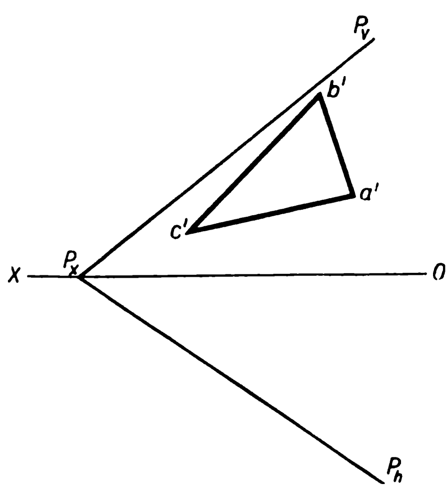


FIG. 741.

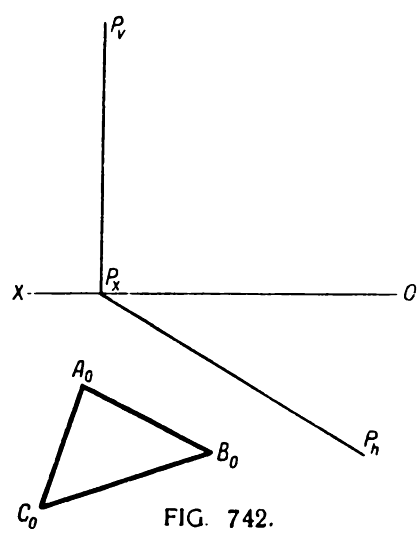


FIG. 742.

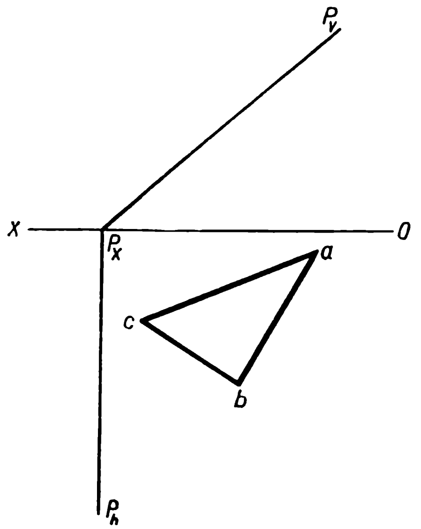



FIG. 743.

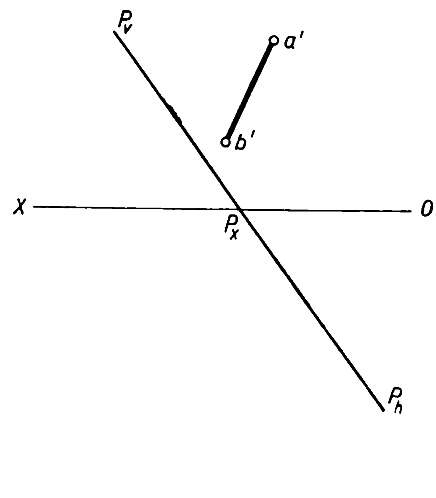



FIG. 744.

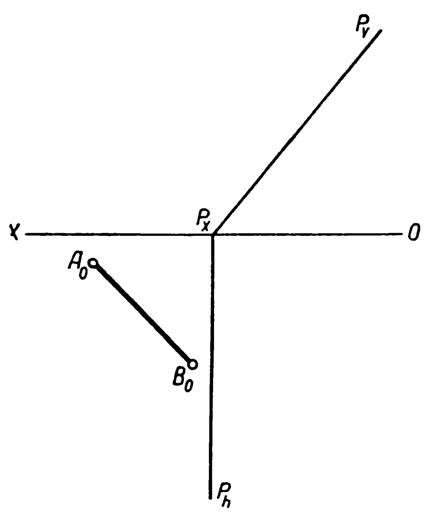



FIG. 745.

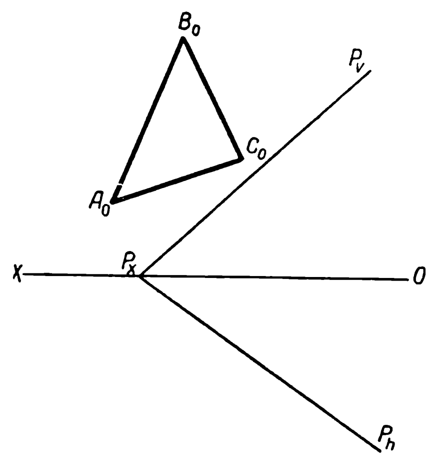



FIG. 746.

PROBLEMS

384. Bring the plane $P$ to coincidence with the $H$ plane and find the coincident position of the point $A$ lying in this plane (Figs. 725 to 730).

385. Bring the plane $P$ to coincidence with the $V$ plane and find the coincident position of the point $A$ lying in this plane (Figs. 725 to 730).

386. Construct the projections of the point $A$ lying in the plane $P$, given the coincident position of the point in the $H$ plane (Figs. 731 to 734).

387. Construct the projections of the point $A$ lying in the plane $P$, given the coincident position of the point in the $V$ plane (Figs. 735 to 738).

388. Construct on the plane $P$ the locus of points equidistant from the traces of the plane (Figs. 739, 740).

389. Construct the locus of points in space equidistant from the vertices of the triangle $ABC$ (Fig. 741).

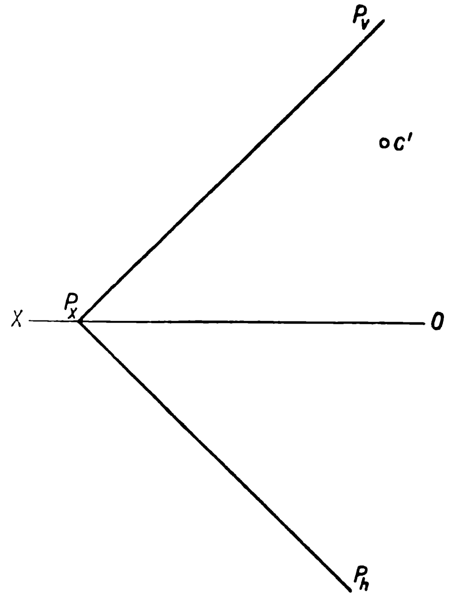



FIG. 747.

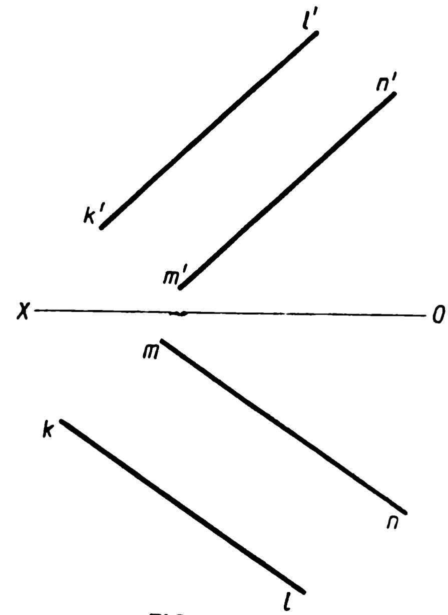



FIG. 748.

390. Construct the projections of the triangle $ABC$ lying in the plane $P$, given the coincident position of the triangle in the $H$ plane (Fig. 742).

391. Determine the centre of a circle described about the triangle $ABC$ contained in the plane $P$ (Fig. 743).

392. Construct the projections of the square $ABCD$ lying in the plane $P$, given the side $AB$ (Fig. 744).

393. Construct the projections of the equilateral triangle $ABC$ lying in the plane $P$, given the coincident position of the side $A_0B_0$ in the $H$ plane (Fig. 745).

394. Construct the projections of the triangle $ABC$ lying in the plane $P$, given the coincident position of the triangle in the $V$ plane (Fig. 746).

395. Construct on the plane $P$ the locus of points 20 mm distant from its point $C$ (Fig. 747).

396. Construct in the plane $P$ a circle (with point $C$ as centre) tangent to the vertical trace of the plane (Fig. 747).

397. Construct in the plane $P$ a circle of radius 20 mm tangent to the traces of the plane (Fig. 739).

398. Construct a circle inscribed in the triangle $ABC$ lying in the plane $P$ (Fig. 741).

399. Construct a circle circumscribed about the triangle $ABC$ lying in the plane $P$ (Fig. 741).

400. Construct a circle tangent to the traces of the plane $P$ and passing through the point $A$ lying in the plane. One alternative solution is sufficient (Fig. 730).

401. Determine the true size of the triangle $ABC$ (Fig. 615).

(Problems 401 to 412 should be solved by the method of revolution about a horizontal or a frontal line.)

402. Draw the bisector of the angle $A$ of the triangle $ABC$ (Fig. 615).

403. Find the centre of a circle inscribed in the triangle $ABC$ (Fig. 615).

404. Find the centre of a circle circumscribed about the triangle $ABC$ (Fig. 615).

405. Drop a perpendicular from the point $C$ to the line $AB$ (Figs. 636, 637).

406. Find on the line $AB$ a point 25 mm distant from the point $C$ (Figs. 636, 637).

407. Through point $C$ draw a straight line intersecting the line $AB$ at a given angle $\varphi$ equal to 30°, 45°, 60° (Fig. 637).

408. Construct an equilateral triangle $ABC$ with the base $BC$ on the line $MN$ (Fig. 651).

409. Construct an isosceles triangle $ABC$ with the base $BC$ on the line $MN$, given the angle $B = C = \varphi$ (Fig. 651).

410. Construct a square $ABCD$ with the side $BC$ on the line $MN$ (Fig 652).

411. Intersect the parallel lines $KL$ and $MN$ with two other straight lines so as to obtain a square $ABCD$ (Fig. 748).

412. Intersect the parallel lines $KL$ and $MN$ with two other straight lines so as to obtain a rhombus $ABCD$ with the acute angle $B = 60°$ (Fig. 748).

CHAPTER XIX

# REPLACING PLANES OF PROJECTION

When replacing a plane of projection with a new one, perpendicularity of the projection planes should *always* be preserved in the new system. In this case both projections of a point, in the new system too, will be situated on a common perpendicular to the new coordinate axis.

When replacing the horizontal plane of projection with a new one $\left(\frac{V}{H} \to \frac{V}{H_1}\right)$:

(1) a point may pass over from the first to the fourth quadrant, from the second to the third, and back;

(2) the position of the vertical projection of a point remains unchanged;

(3) the distance from the horizontal projection of a point to the coordinate axis in the new system also remains unchanged ($a_1a_{x1} = aa_x$).

When replacing the vertical plane of projection with a new one $\left(\frac{V}{H} \to \frac{V_1}{H}\right)$:

(1) a point may pass over from the first to the second quadrant, from the third to the fourth quadrant, and back;

(2) the position of the horizontal projection of a point remains unchanged;

(3) the distance from the vertical projection of a point to the coordinate axis in the new system also remains unchanged ($a_1'a_{x_1} = a'a_x$).

**Rule.** To determine the vertical (horizontal) projection of a point in a new vertical (horizontal) plane of projection, drop a perpendicular from the horizontal (vertical) projection of the point onto the new coordinate axis and, from the point of intersection, lay off on it (in the *appropriate direction*—see examples below) a line-segment equal to the distance from the old vertical (horizontal) projection of the point to the old coordinate axis.

EXAMPLES

**Example 233**

Given: the point $A$ in the first quadrant. Required: to find its projections in the system: $\frac{V_1}{H}$ (Fig. 749).

*Solution.* First determine the quadrant in which the point will be found in the new system. The horizontal projection ($a$) of the point in the old system is found *below* the coordinate axis, i.e. in the front half-plane of the horizontal plane of projection, whereas in the new system it appears *above* the coordinate axis ($O_1X_1$), i.e. in the rear half-plane of the $H$ plane. Hence, in the new system the required point is situated in the *second quadrant.*

Proceed as follows: through the point $a$ draw a perpendicular to the axis $O_1X_1$ and lay off on the perpendicular, from the point $a_{x1}$ *upwards*, a segment equal to $a'a_x$. Its endpoint will be the new vertical projection ($a_1'$) of the given point.

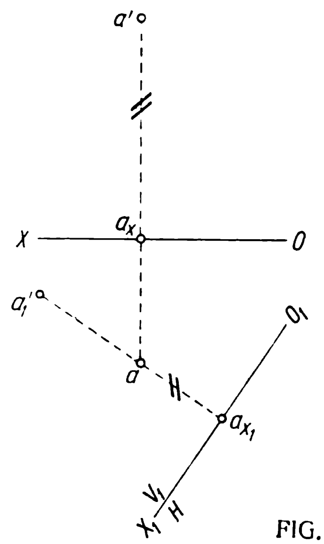


FIG. 749.

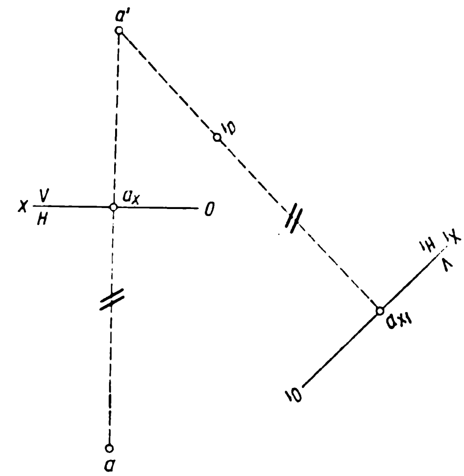


FIG. 750.

**Example 234**

Given: the point $A$ in the first quadrant. Required: to find its projections in the system $\frac{V}{H_1}$ (Fig. 750).

*Solution.* Determine the quadrant in which the point will be found in the new system. The vertical projection ($a'$) of the point in the old system is found *above* the coordinate axis, i.e. in the upper half-plane of the vertical plane of projection, whereas in the new system it appears *below* the coordinate axis ($O_1X_1$), i.e. in the lower half-plane of the $V$ plane. Hence, in the new system the required point is situated in the *fourth* quadrant.

Proceed as follows: through the point $a'$ draw a perpendicular to the axis $O_1X_1$ and lay off on the perpendicular, from the point $a_{x_1}$ *downwards*, a segment equal to $aa_x$. Its endpoint will be the new horizontal projection ($a_1$) of the given point.

**Example 235**

Given: a line-segment $AB$. Find its projections, given that in a new system it must be located parallel to the vertical plane of projection (Fig. 751).

*Solution.* For a line-segment to be parallel to the $V$ plane, its horizontal projection must be parallel to the coordinate axis. Hence, draw the coordinate axis $O_1X_1$ parallel to the horizontal projection ($ab$) of the segment and, as indicated above, find the vertical projections ($a_1'$) and ($b_1'$) of the points to obtain the vertical projection ($a_1'b_1'$) of the segment in the new system.

**Example 236**

Given: a line-segment $AB$. Find its projections, given that in a new system it must be located parallel to the horizontal plane of projection (Fig. 752).

*Solution.* For a line-segment to be parallel to the $H$ plane, its vertical projection must be parallel to the coordinate axis. Hence, draw the coordinate axis $O_1X_1$ parallel to the vertical projection ($a'b'$) of the segment, and, as indicated above, find the horizontal projections ($a_1$) and ($b_1$) of the points to obtain the horizontal projection ($a_1b_1$) of the segment in the new system.

**Example 237**

Given: a line-segment $AB$. Find its projections, given that in a new system it must be situated perpendicular to the horizontal plane of projection (Fig. 753).

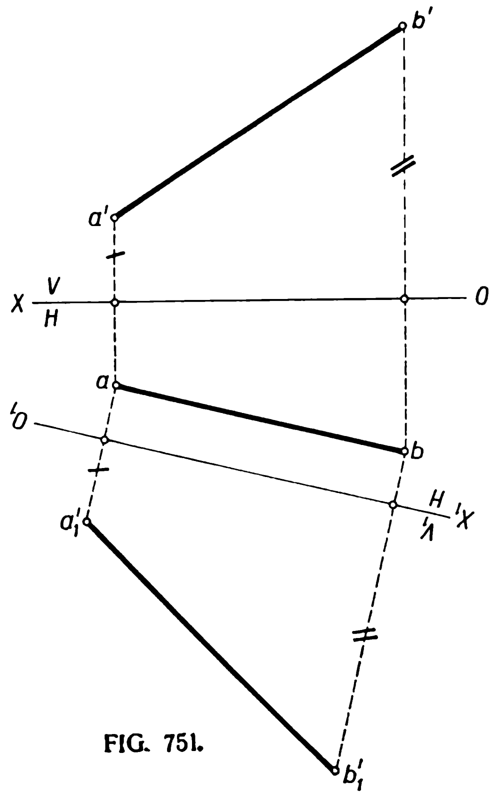



FIG. 751.

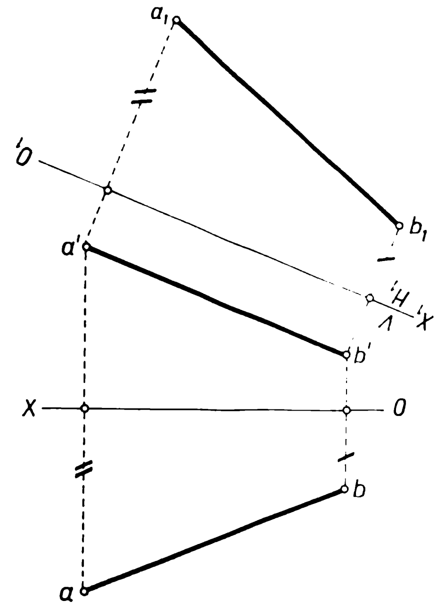



FIG. 752.

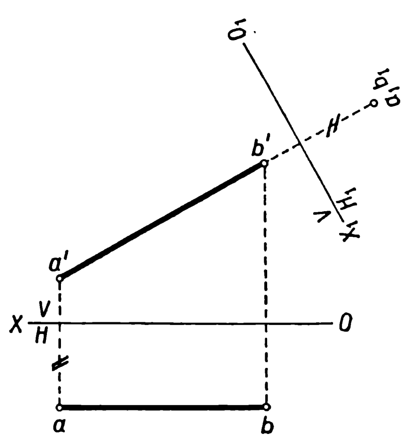



FIG. 753.

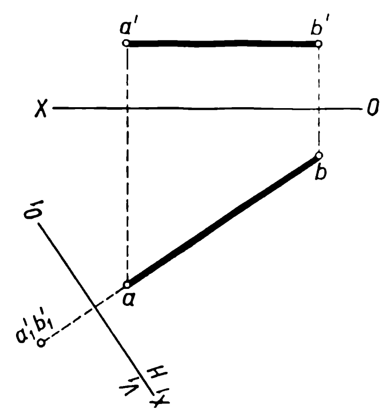



FIG. 754.

*Solution.* For a line-segment to be perpendicular to the horizontal plane of projection, its vertical projection must be perpendicular to the coordinate axis. Hence, draw the coordinate axis $O_1X_1$ perpendicular to the vertical projection $(a'b')$ of the segment, and, as indicated above, find its horizontal projection $(a_1b_1)$, which is a point.

**Example 238**

Given: a line-segment $AB$. Find its projections, given that in a new system it must be situated perpendicular to the vertical plane of projection (Fig. 754).

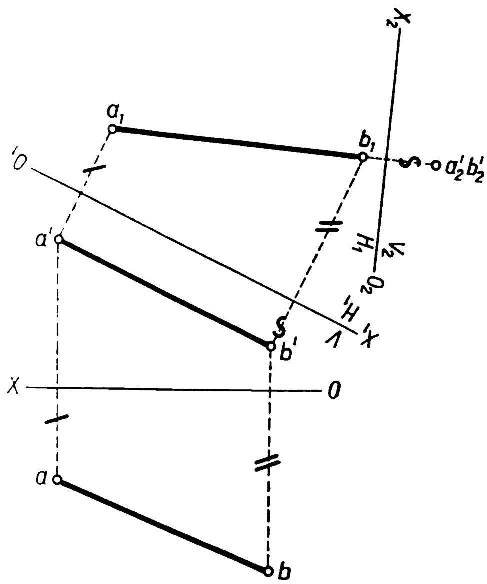



FIG. 755.

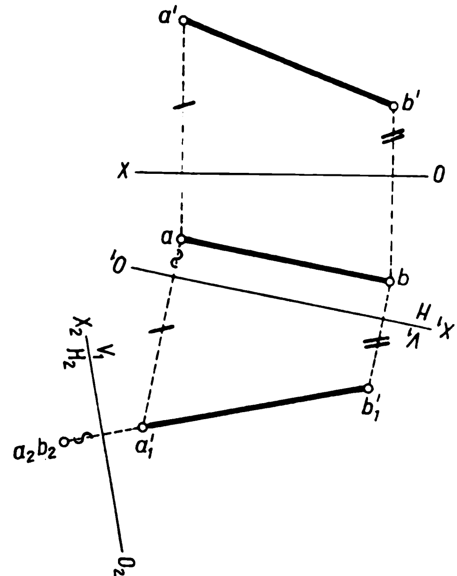



FIG. 756.

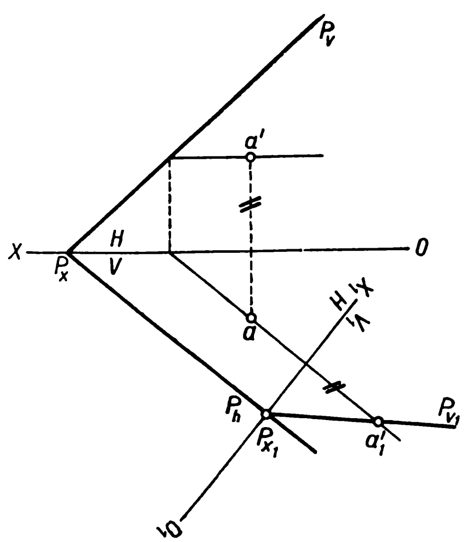



FIG. 757.

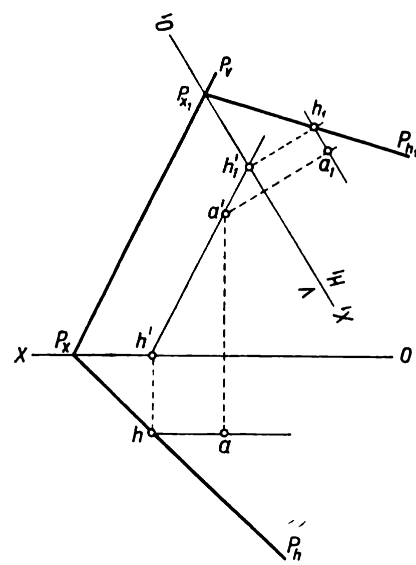



FIG. 758.

*Solution.* For a line-segment to be perpendicular to the vertical plane of projection, its horizontal projection must be perpendicular to the coordinate axis. Hence, draw the coordinate axis $O_1X_1$ perpendicular to the horizontal projection ($ab$) of the segment, and, as indicated above, find its vertical projection ($a_1'b_2'$), which is a point.

**Example 239**

Given: a line-segment $AB$. Find its projections, given that in a new system it must be perpendicular to the vertical plane of projection (Fig. 755).

*Solution.* To solve the problem, *two* consecutive replacements of the projection planes are required: $\left(\frac{V}{H_1};\frac{V_2}{H_1}\right)$. Construct the projections of the line-segment ($ab$, $a'b'$) in the system $\frac{V}{H_1}$, in which it is parallel to the horizontal plane of projection; then construct the projections of the line-segment ($a_1b_1$, $a'b'$) in the system $\frac{V_2}{H_1}$, in which it is perpendicular to the $V$ plane. The construction is obvious from the drawing.

**Example 240**

Given: a line-segment $AB$. Find its projections, given that in a new system it must be perpendicular to the horizontal plane of projection (Fig. 756).

*Solution.* To solve the problem, *two* consecutive replacements of the projection planes are required: $\left(\frac{V_1}{H}; \frac{V_1}{H_2}\right)$. Construct the projections of the line-segment ($ab$, $a'b'$) in the system $\frac{V_1}{H}$, in which it is parallel to the $V$ plane. Then construct the projections of the segment ($ab$, $a_1'b_1'$) in the system $\frac{V_1}{H_2}$, in which it is perpendicular to the $H$ plane. The construction is obvious from the drawing.

**Example 241**

Construct the traces of the plane $P$ in the system $\frac{V_1}{H}$ (Fig. 757).

*Solution.* The horizontal trace ($P_h$) of the plane remains unchanged. The intersection of the horizontal trace ($P_h$) of the plane and the new axis of projection $O_1X_1$ yields the new vanishing point ($P_{x_1}$) of the traces. By hypothesis, $P_h$ is perpendicular to $O_1X_1$, which means that in the new system the given plane must be a *vertical projecting one.* To determine the direction of the new vertical trace ($P_{v_1}$), take an arbitrary point ($a$, $a'$) in the plane and find the new vertical projection ($a_1'$). Through the points $P_{x_1}$ and $a_1'$ draw the new vertical trace ($P_{v_1}$) of the plane.

**Example 242**

Construct the traces of the plane $P$ in the system $\frac{V}{H_1}$(Fig. 758).

*Solution.* The vertical trace ($P_v$) of the plane remains unchanged. The intersection of the vertical trace ($P_v$) of the plane and the new coordinate axis $O_1X_1$ yields the new vanishing point ($P_{x_1}$) of the traces. To determine the direction of the new horizontal trace ($P_{h_1}$) of the plane, take an arbitrary point ($a$, $a'$) in the plane by means of a frontal line, find it in the new system and draw a frontal line through the point. Then determine the horizontal trace of this frontal line and draw the new horizontal trace ($P_{h_1}$) of the plane through points $P_{x_1}$ and $h_1$.

*Note.* Sometimes the point $P_{x_1}$ is found outside the limits of the drawing; in such cases use two frontal lines, find points $h_1$ and $h_2$, and draw through them the trace $P_{h_1}$.

**Example 243**

Given: a plane $P$. Construct its traces, given that in a new system it must be perpendicular to the $H$ plane (Fig. 759).

*Solution.* As is known, the vertical trace of a horizontal projecting plane must be perpendicular to the coordinate axis. Draw a new coordinate axis $O_1X_1$ perpendicular to the trace $P_v$. The point $P_{x_1}$ of their intersection is the new vanishing point. To determine the direction of the new horizontal trace ($P_{h_1}$) of the plane, assume an arbitrary point ($h$, $h'$) on the trace $P_h$ and find its new horizontal projection ($h_1$). Through points $P_{x_1}$ and $h_1$ draw the new horizontal trace ($P_{h_1}$) of the plane.

**Example 244**

Drop a perpendicular from the point $C$ to the line $AB$ (Fig. 760).

*Solution.* To drop a perpendicular from a point to a line directly on an orthographic drawing is possible only if the given line is parallel to a projection plane (why?). Hence: replace, for instance, the vertical plane of projection with a new one $V_1$,

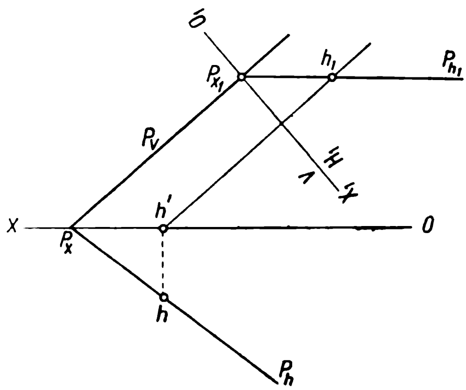



FIG. 759.

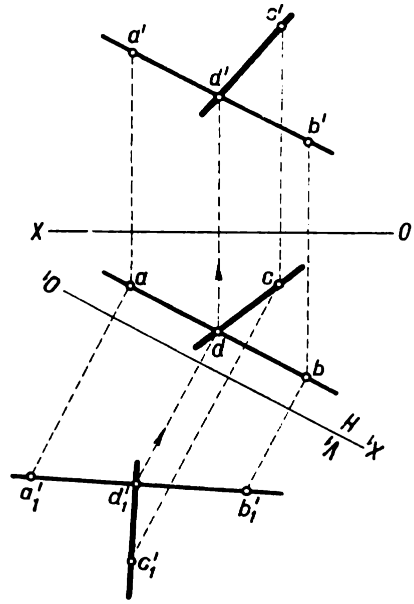



FIG. 760.

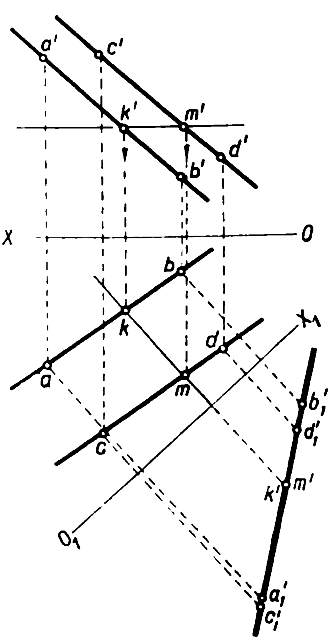



FIG. 761.

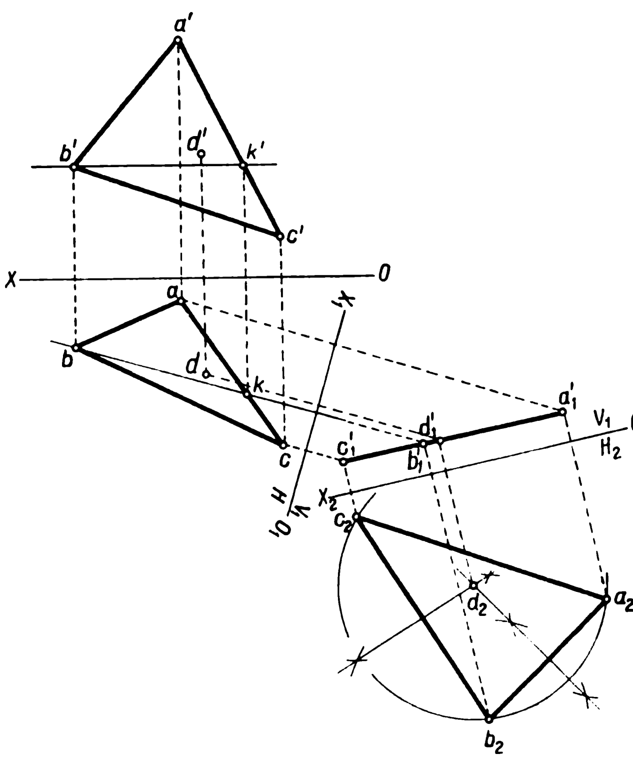



FIG. 762.

which is parallel to the line $AB$. Draw the new coordinate axis ($O_1X_1$) parallel (why?) to the line $ab$ and find the vertical projections of the line ($a_1'b_1'$) and point ($c_1'$). Drop a perpendicular from the point $c_1'$ to line $a_1'b_1'$. Their intersection will yield the vertical projection ($d_1'$) of the point, which is the foot of the perpendicular. Find its projections ($d$, $d'$) in the initial system and draw the projections of the desired perpendicular: the horizontal one through the points $c$ and $d$, and the vertical one through the points $c'$ and $d'$.

**Example 245**

Replace the projection planes with a new system in which the vertical projections of the parallel lines $AB$ and $CD$ will merge into one line (Fig. 761).

*Solution.* The parallel lines $AB$ and $CD$ specify a plane, and their vertical projections will merge into one line if this plane is perpendicular to the $V$ plane in the new system of projection planes. Intersect the lines ($ab$, $a'b'$) and ($cd$, $c'd'$) with an arbitrary horizontal line ($km$, $k'm'$), draw the new coordinate axis ($O_1X_1$) perpendicular (why?) to the like $km$ and find their new vertical projections ($a_1'b_1'$) and ($c_1'd_1'$), which, when *accurately* drawn, will merge into one line.

**Example 246**

Find the centre of a circle circumscribed about the triangle $ABC$ (Fig. 762).

*Solution.* The centre of a circle circumscribed about a triangle is found at the intersection of perpendiculars erected at the mid-points of its sides. To draw these perpendiculars the true size of the triangle must be determined. To do this, one of the projection planes, say the $H$ plane, must be parallel to the plane of the triangle. Make two consecutive replacements of projection planes: first replace the vertical one with $V_1$, which is perpendicular to the plane of the triangle, and then replace the horizontal plane with $H_2$, which is parallel to the plane of the triangle. Draw a horizontal line ($bk$, $b'k'$) of the triangle and, taking a new coordinate axis $O_1X_1$ perpendicular (why?) to the horizontal projection ($bk$) of the horizontal line, determine the vertical projection ($a_1'b_1'c_1'$) of the triangle, which is a straight line. Then draw the coordinate axis $O_2X_2$ parallel (why?) to the vertical projection ($a_1'b_1'c_1'$) of the triangle and find its horizontal projection. The triangle $a_2b_2c_2$ represents the true size of the given triangle $ABC$. Determine the centre ($d_2$) of the circle circumscribed about the triangle $a_2b_2c_2$, and find the projections ($d$, $d'$) of this point in the initial system of projection planes. The construction is obvious from the drawing.

**Example 247**

Intersect the skew lines $AB$ and $CD$ with the line $MN$ perpendicular to them (Fig. 763).

*Solution.* Replacing the projection planes with new ones, obtain a new system in which one of the planes, say the $V$ plane, is perpendicular to the line $AB$ (or $CD$). Replace the horizontal plane of projection with a new one ($H_1$), which is parallel to the line $AB$; draw a new coordinate axis ($O_1X_1$) parallel to $a'b'$ and find the horizontal projections ($a_1b_1$) and ($c_1d_1$) of the given lines. Then replace the vertical plane of projection with a new one $V_2$, which is perpendicular to $A_1B_1$; draw a new coordinate axis $O_2X_2$ perpendicular to $a_1b_1$ and find the vertical projections ($a_2'b_2'$) and ($c_2'd_2'$) of the given lines. Intersect the obtained lines ($a_1b_1$, $a_2'b_2'$) and ($c_1d_1$, $c_2'd_2'$) with a perpendicular line ($m_1n_1$, $m_2'n_2'$), and then determine the projections ($mn$, $m'n'$) of the desired line in the initial system of projection planes. The construction is obvious from the drawing.

*Note.* If one of the lines is parallel to a projection plane, one replacement of projection planes is sufficient.

**Example 248**

Find the line of intersection of a pyramid and the plane $P$ (Fig. 764).

*Solution.* To construct the desired line we must find the points of intersection of lateral edges of the pyramid with the plane. Since the plane $P$ is an oblique one, it is

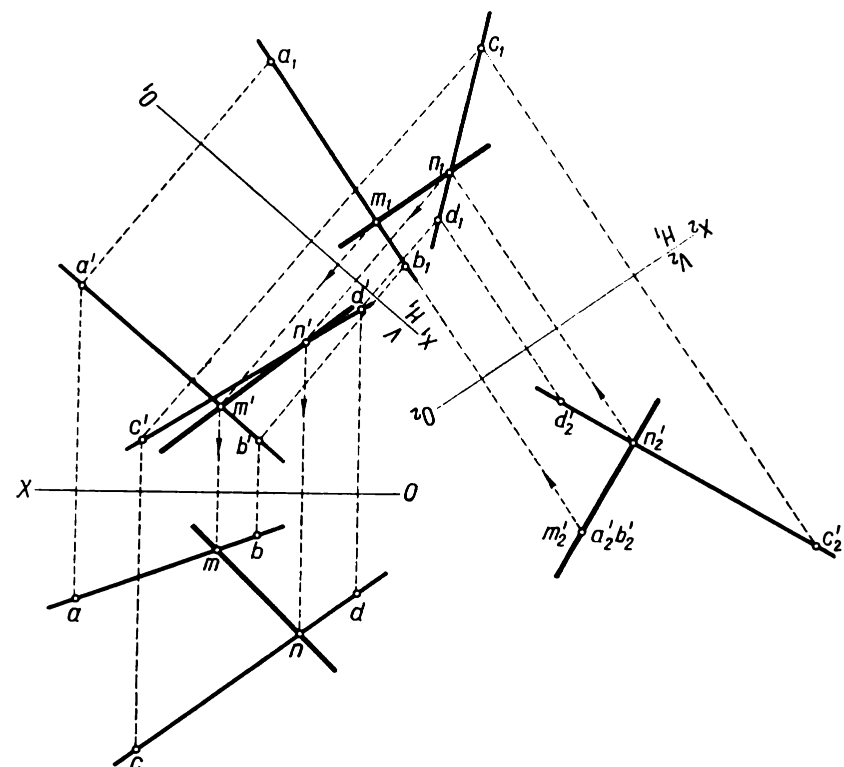



FIG. 763.

*advisable* to replace the projection planes so that in a new system the given plane becomes a vertical projecting one. Replace the $V$ plane with $V_1$; draw the new coordinate axis ($O_1X_1$) perpendicular to the trace $P_h$, find the vertical trace $P_{v_1}$ and the vertical projection of the pyramid. Then determine the vertical projection ($a'_1b'_1c'_1d'_1$) of the line of intersection, and obtain the projections ($abcd$) and ($a'b'c'd'$) of the desired line in the initial system of projection planes. The construction is clear from the drawing.

**Example 249**

Replace the projection planes with new ones so that the planes $P$ and $Q$ become vertical projecting planes (Fig. 765).

*Solution.* For the planes $P$ and $Q$ to become vertical projecting ones, the $V$ plane in a new system of projection planes must be perpendicular to the line of intersection of these planes. Find the line ($hv$, $h'v'$) of intersection of the planes $P$ and $Q$. Replace the $H$ plane with $H_1$, which is parallel to the obtained line of intersection of the planes; draw the new coordinate axis $O_1X_1$ parallel to the line $h'v'$ and determine the horizontal traces ($P_{h_1}$) and ($Q_{h_1}$) of the given planes, which are parallel to $h_1v_1$. Then replace the $V$ plane with $V_2$; draw the coordinate axis $O_2X_2$ perpendicular to $h_1v_1$ and find the vertical traces ($P_{v_2}$) and ($Q_{v_2}$) of the given planes, which traces pass through the point $h'_2v'_2$.

*Note.* If the given planes intersect along a horizontal or a frontal line, one replacement of projection planes (which one?) is sufficient.

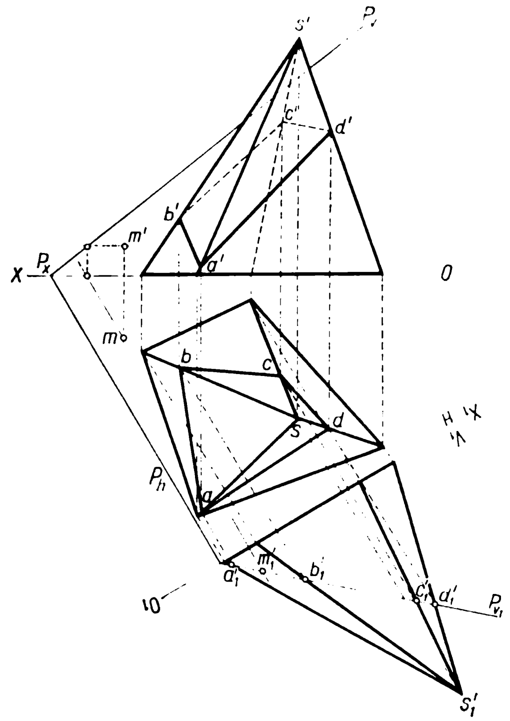



FIG. 764.

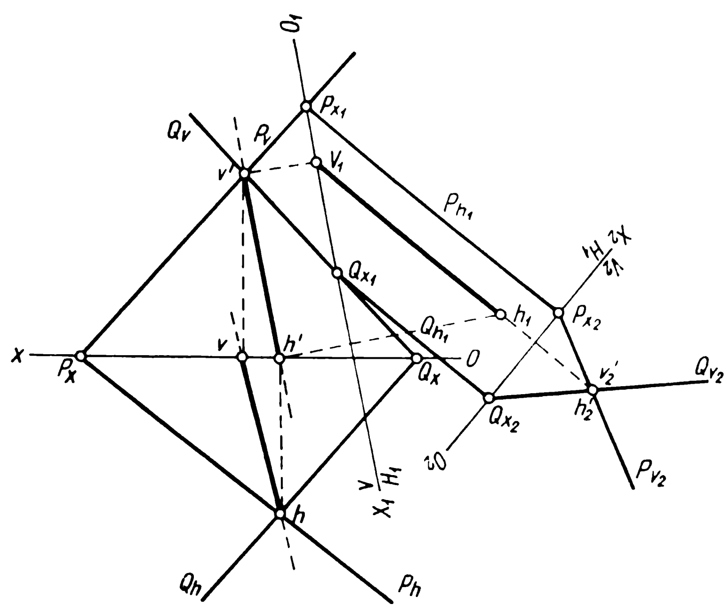



FIG. 765.

An analysis of the problems solved by the method of replacing projection planes [the same goes for the method of revolution (displacement)] shows that the main procedure consists in determining "the most favourable position" of the given elements. This leads to a simple solution of the problem.

## Check-Up Questions

1. What determines the distance from a point to a horizontal projecting line in an orthographical drawing?
2. What determines the distance from a point to a vertical projecting line in an orthographical drawing?
3. What position of the elements is the most favourable in determining the distance from a point to a straight line?
4. What determines the distance between two horizontal projecting lines in an orthographical drawing?
5. What determines the distance between two vertical projecting lines in an orthographical drawing?
6. What position of the elements is the most favourable in determining the distance between two parallel lines?
7. What determines the distance between two skew lines in an orthographical drawing if one of the lines is a horizontal projecting line?
8. What determines the distance between two skew lines in an orthographical drawing if one of the lines is a vertical projecting line?
9. What position of the elements is the most favourable in determining the distance between two skew lines?
10. What determines the distance from a point to a horizontal projecting plane in an orthographical drawing?
11. What determines the distance from a point to a vertical projecting plane in an orthographical drawing?
12. What position of the elements is the most favourable in determining the distance from a point to a plane?
13. What determines the distance between two parallel horizontal projecting planes in an orthographical drawing?
14. What determines the distance between two parallel vertical projecting planes in an orthographical drawing?
15. What position of the elements is the most favourable in determining the distance between two parallel planes?
16. What determines the angle of inclination of a horizontal projecting plane to the $V$ plane in an orthographical drawing?
17. What determines the angle of inclination of a vertical projecting plane to the $H$ plane in an orthographical drawing?
18. What position of a plane is the most favourable in determining its angles of inclination to the projection planes?
19. What determines the angle between two horizontal projecting planes in an orthographical drawing?
20. What determines the angle between two vertical projecting planes in an orthographical drawing?
21. What position of planes is the most favourable in determining the angle formed by them?

## PROBLEMS

413. Construct the projections of the points $A$ and $B$ in a given new system (Figs. 766 to 773).

414. Construct the projections of the straight line $AB$ in a new system, given that the line must be parallel to the $H$ plane (Figs. 774, 775).

415. Construct the projections of the straight line $AB$ in a new system, given that the line must be parallel to the $V$ plane (Figs. 774, 775).

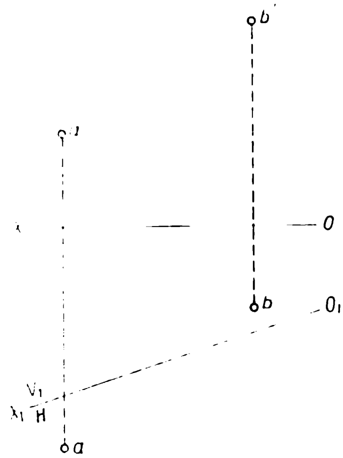



FIG. 766.

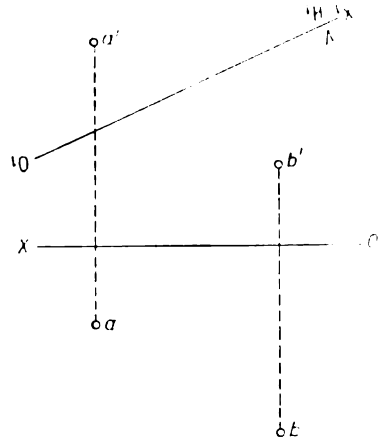



FIG. 767.

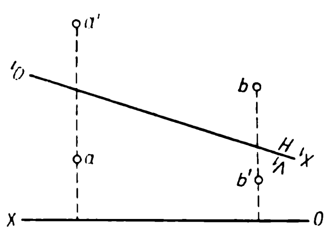



FIG. 768.

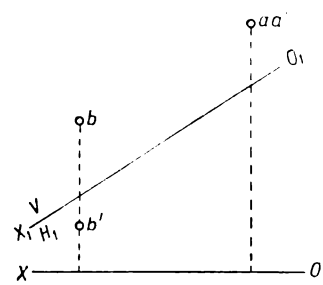



FIG. 769.

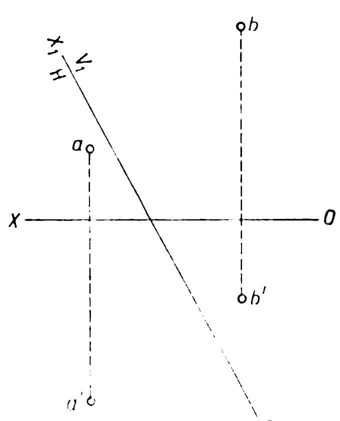



FIG. 770.

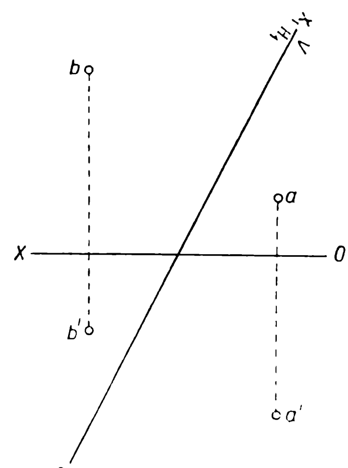



FIG. 771.

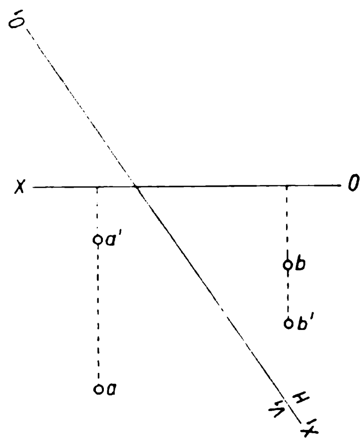



FIG. 772.

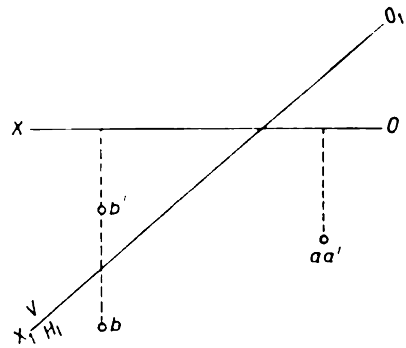



FIG. 773.

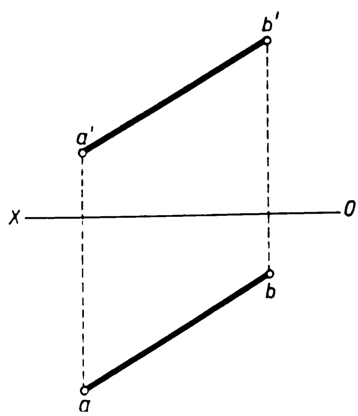



FIG. 774.

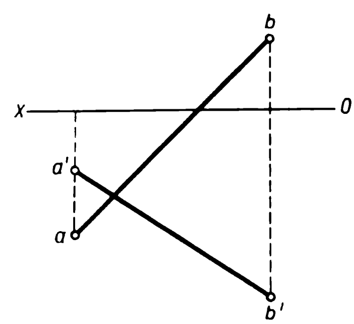



FIG. 775.

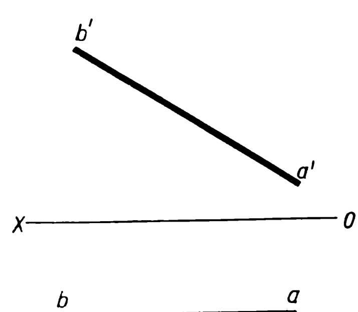



FIG. 776.

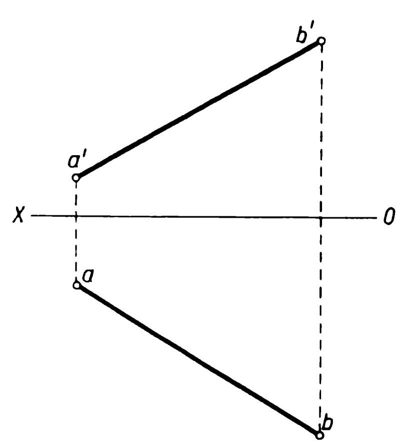



FIG. 777.

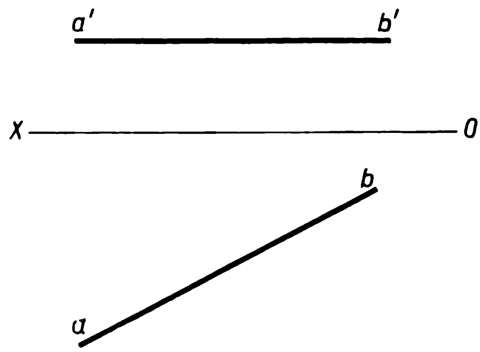



FIG. 778.

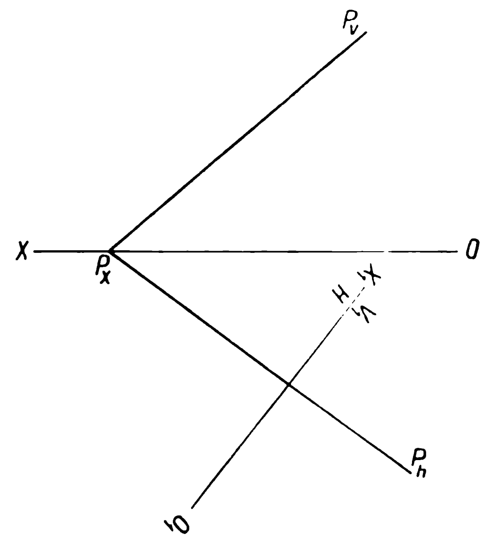



FIG. 779.

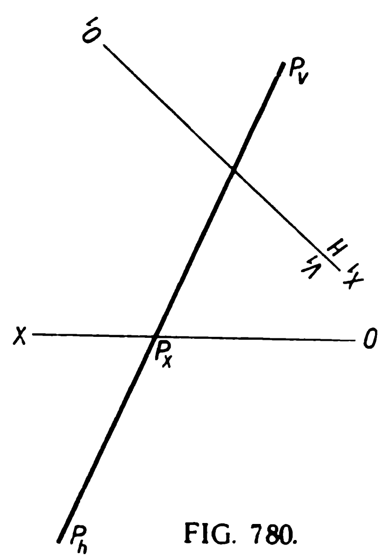



FIG. 780.

416. Construct the projections of the straight line $AB$ in a new system, given that the line must be perpendicular to the $H$ plane (Figs. 776, 777).

417. Construct the projections of the straight line $AB$ in a new system, given that the line must be perpendicular to the $V$ plane (Figs. 777, 778).

418. Construct the traces of the plane $P$ in a given new system (Figs. 779, 780).

419. Construct the traces of the plane $P$ in a new system so that it becomes a horizontal projecting plane (Figs. 739, 740).

420. Construct the traces of the plane $P$ in a new system so that it becomes a vertical projecting plane (Figs. 739, 740).

421. Construct the projections of the parallel lines $AB$ and $CD$ in a new system, given that their horizontal projections merge into one line (Fig. 639).

422. Construct the projections of the parallel lines $AB$ and $CD$ in a new system, given that their vertical projections merge into one line (Fig. 639).

423. Construct the projections of the lines $AB$ and $CD$ in a new system, given that their horizontal projections are parallel (Fig. 578).

424. Construct the projections of the lines $AB$ and $CD$ in a new system, given that their vertical projections are parallel (Fig. 578).

425. Construct the projections of the triangle $ABC$ in a new system, given that its horizontal projection merges into a line (Fig. 600).

426. Construct the projections of the triangle $ABC$ in a new system, given that its vertical projection merges into a line (Fig. 600).

427. Find the point of intersection of the line $AB$ and the plane $P$ (Fig. 466, 467).

428. Construct the projections of the triangle $ABC$ in a new system, given that its horizontal projection represents its true size (Fig. 600).

429. Construct the projections of the triangle $ABC$ in a new system, given that its vertical projection represents its true size (Fig. 600).

430. Find the centre of gravity of the perimeter of the triangle $ABC$ (Fig. 600).

431. Find the centre of a circle inscribed in the triangle $ABC$ (Fig. 600).

432. Find the centre of a circle circumscribed about the triangle $ABC$ (Fig. 600).

433. Construct the projections of the lines $AB$ and $CD$ in a new system, given that the line $AB$ is perpendicular to the $V$ plane (Fig. 578).

434. Construct the projections of the lines $AB$ and $CD$ in a new system, given that the line $AB$ is perpendicular to the $V$ plane (Fig. 144).

435. Intersect the lines $AB$ and $CD$ with the line $MN$, which is perpendicular to the line $AB$, so that the line-segment between the given lines is 20 mm long (Fig. 578).

436. Construct the traces of the planes $P$ and $Q$ in a new system, given that the planes must be horizontal projecting (Figs. 393, 394).

437. Construct the traces of the planes $P$ and $Q$ in a new system, given that the planes must be vertical projecting (Figs. 400, 408).

## CHAPTER XX

# DETERMINING DISTANCES

The *distance between two points*, as measured by the length of a line-segment joining these points, can be determined by any one of the following methods:

(1) *constructing a right-angled triangle* (see Example 33);

(2) *revolving* or *displacing*; the line-segment is brought to a position parallel to a projection plane (see Examples 196, 197);

(3) *coincidence*; the line-segment is placed in a plane (for the sake of simplicity in a horizontal- or vertical projecting one), which is brought to coincidence with a projection plane;

(4) *replacing the planes of projection*; one of the projection planes is replaced with a new one, which must be parallel to the given line-segment (see Examples 235 and 236).

### EXAMPLES

**Example 250**

Given: points $A$ and $B$. Required: to determine the distance between them (Fig. 781).

*Solution.* Let us use the coincidence method. Through the points $(a, a')$ and $(b, b')$ draw a horizontal projecting plane $R$ and bring this plane to coincidence with the $H$ plane. Find the positions $A_0$ and $B_0$ of the points $(a, a')$ and $(b, b')$. Join these points to obtain the desired distance $(A_0B_0)$.

The *distance from a point to a line* can be determined by any one of the following methods:

(1) *direct method*: through the point pass a plane perpendicular to the line-segment; find the point of intersection of the given line and this plane; then determine the length of the line-segment joining the obtained point to the given one (see Example 170);

(2) *revolving* or *displacing*: bring the given system to a position in which the given line is perpendicular to a projection plane, or the plane specified by the line and the point is parallel to a plane of projection;

(3) *coincidence*: find one of the traces of the plane specified by the line and the point and, by revolving about this trace, determine the coincident position of the point and the line;

(4) *revolving about a horizontal or a frontal line*: through a horizontal (frontal) line of the plane specified by the given elements pass a plane $R$ parallel to the horizontal (frontal) plane of projection, and find the coincident position of the point and the line in this plane by revolving about the horizontal (frontal) line;

(5) *replacing the projection planes*: replace the projection planes with new ones, one of them perpendicular to the given line or parallel to the plane specified by the line and the point.

**Example 251**

Determine the distance from the point $C$ to the line $AB$ (Figs. 782 to 784).

*Alternative solutions*:

1. By *revolving about a horizontal line* (Fig. 782). Through the point $(c, c')$ draw a horizontal line $(ck, c'k')$ to intersect the line $(ab, a'b')$ at point $(k, k')$ and put it into a plane $R$ parallel to the horizontal plane of projection. The points $(c, c')$ and $(k, k')$ already lie in the plane $R$; to determine the coincident position of a line in the plane $R$, it will suffice to find the position of one more arbitrary point of this line, say $(b, b')$. Drop a perpendicular from the point $b$ to the line $ck$, and from point $\alpha$ as centre describe an arc of radius equal to $\alpha B$ to intersect the perpendicular at point $B_0$. Join the points $k$ and $B_0$ to obtain the coincident position $(kB_0)$ of the given line. The line-segment $l$ is the desired distance.

2. By the *method of displacement* (Figs. 783, 784). Displace the given elements parallel to the $H$ plane, and bring the line to a position parallel to the $V$ plane. Then displace them parallel to the $V$ plane and bring the line to a position perpendicular to the $H$ plane. The desired distance $l$ is the distance between the points $c_2$ and $a_2b_2$. The construction is obvious from the drawing.

Using the same method, the problem can be solved in a different way: construct a triangle $(abc, a'b'c')$ from the point $(c, c')$ and the line $(ab, a'b')$ and, by two consecutive displacements, bring it to a position parallel, say, to the horizontal plane of projection. The altitude $c_2d_2$ of the triangle $a_2b_2c_2$ is the desired distance. The construction is obvious from the drawing.

The *distance between two parallel lines* can be determined by either one of the methods recommended for determining the distance from a point to a line.

**Example 252**

Find the distance between the parallel lines $AB$ and $CD$ (Figs. 785, 786).

*Alternative solutions*:

1. By the *coincidence method* (Fig. 785). Determine the horizontal traces $(h, h')$ and $(h_1, h_1')$ of the lines $(ab, a'b')$ and $(cd, c'd')$, and through these points pass the horizontal trace $(P_h)$ of the plane specified by the given lines. By revolving about the trace $P_h$ find the coincident positions of the lines $(ab, a'b')$ and $(cd, c'd')$ in the $H$ plane. Since one point $(h, h')$ of the line $(ab, a'b')$ lies in the $H$ plane, find the coincident position $B_0$ of the point $(b, b')$; the points $h$ and $B_0$ determine the coincident position $hB_0$ of the line $(ab, a'b')$. The coincident position of the line $(cd, c'd')$ is found by drawing a line through the point $h_1$ and parallel to the line $hB_0$ (why?); the distance $(l)$ between them is the desired distance.

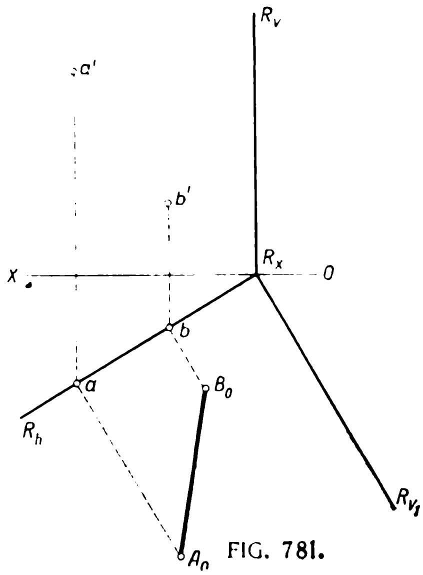


FIG. 781.

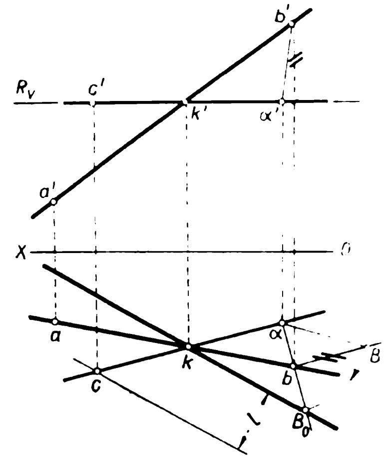


FIG. 782.

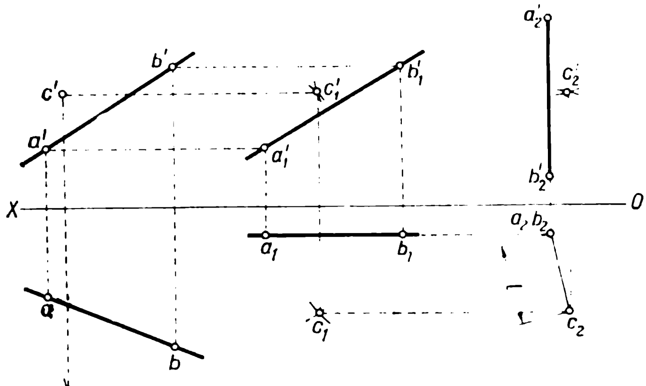


FIG. 783.

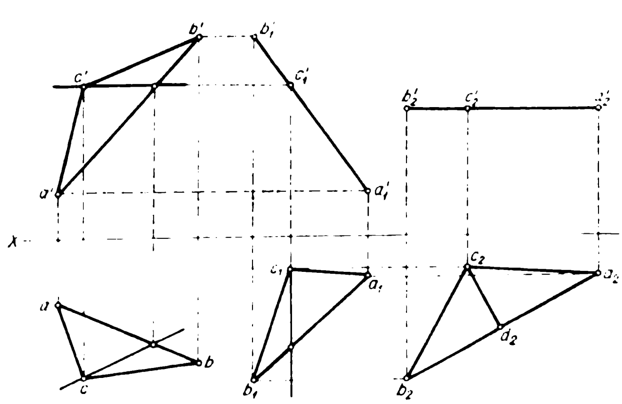


FIG. 784.

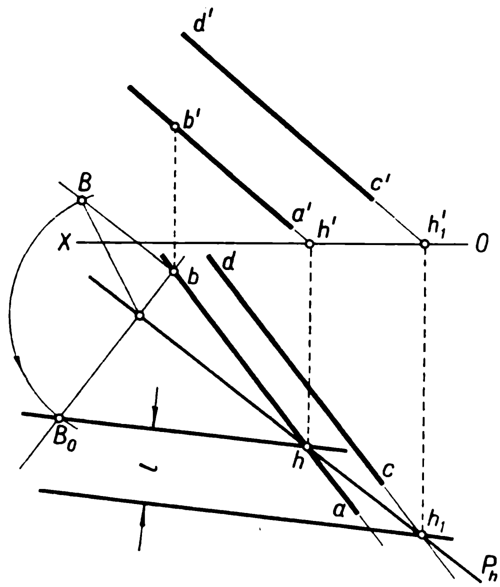



FIG. 785.

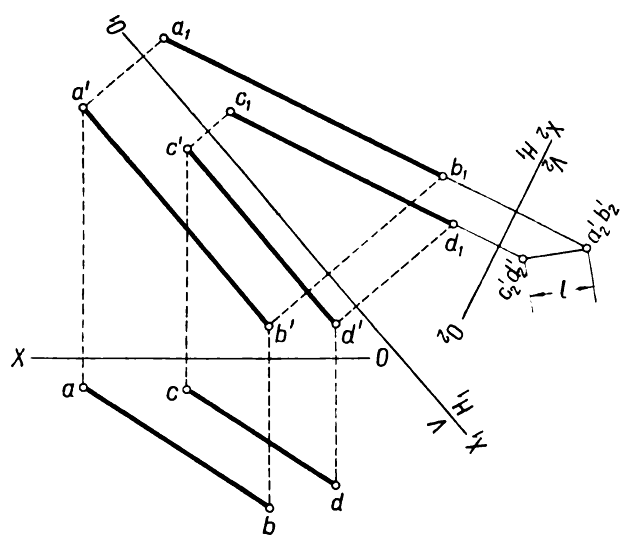



FIG. 786.

2. By *replacing the projection planes* (Figs. 786, 787). Replace the $H$ plane by a new one ($H_1$), which is parallel to the given lines; then replace the $V$ plane by $V_2$, which is perpendicular to the same lines. The distance $l$ between the vertical projections ($a_2'b_2'$) and ($c_2'd_2'$) of the lines is the desired distance. The construction is obvious from the drawing.

Using the same method, the problem can be solved in a different way: replace the $H$ plane with a new one ($H_1$), which is perpendicular to the plane specified by the given lines $AB$ and $CD$. Then replace the $V$ plane with $V_2$, which is parallel to the given plane. The distance $l$ between the parallel lines $a_2'b_2'$ and $c_2'd_2'$ is the desired distance. The construction is obvious from the drawing.

The *distance between skew lines* can be determined by any one of the following methods:

(1) *direct method*: through one of the given lines pass a plane $R$ parallel to the other line and determine the distance from an arbitrary point on the second line to the plane;

(2) *revolving* or *displacing*: bring the given lines to a position in which one of them is perpendicular to a projection plane;

(3) *replacing the projection planes*: replace the planes of projection so that one of them becomes perpendicular to either one of the given lines.

**Example 253**

Determine the distance between the skew lines $AB$ and $CD$ (Fig. 788).

*Solution.* By the method of replacing the projection planes.

Replace the $V$ plane with $V_1$, which is parallel to $AB$. Then replace the $H$ plane with $H_2$, which is perpendicular to the same line. The distance $l$ between the horizontal projections of the lines is the desired distance. The construction is obvious from the drawing.

The distance *from a point to a plane* can be determined by any one of the following methods:

(1) *direct method*: drop a perpendicular from the point to the plane, find the foot of the perpendicular and determine the distance between the points—the given and the obtained one.

(2) *revolving* or *displacing*: bring the given system to a position, in which the given plane is perpendicular to a projection plane;

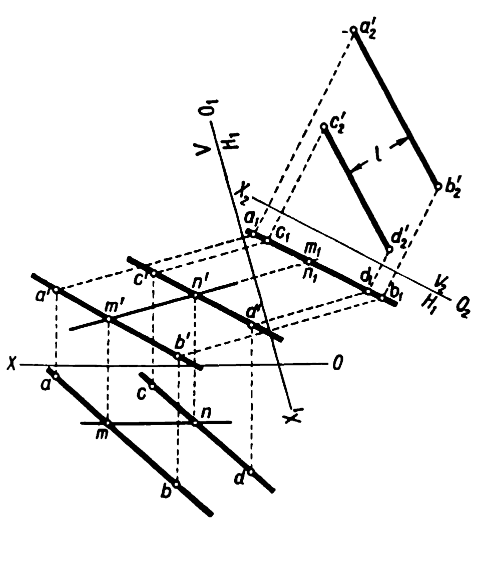


FIG. 787.

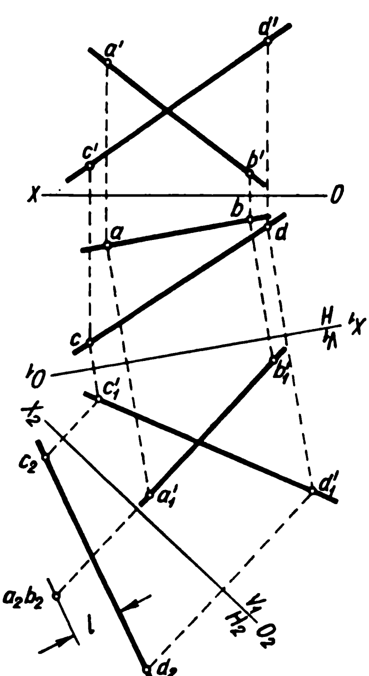


FIG. 788.

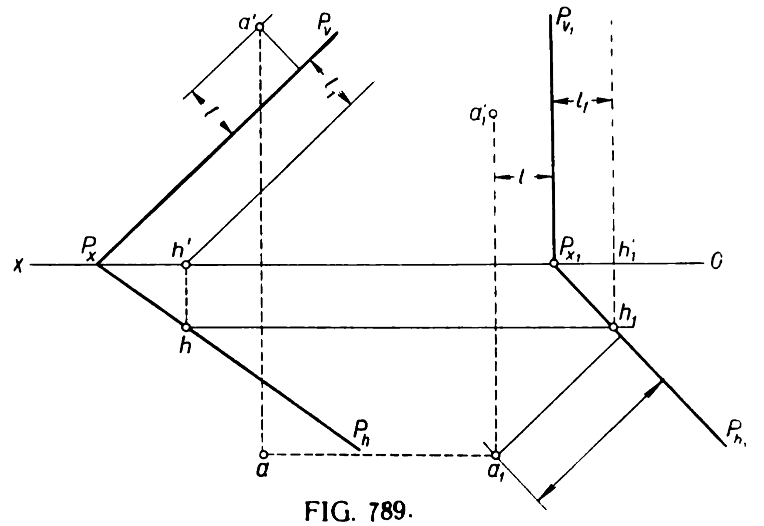



FIG. 789.

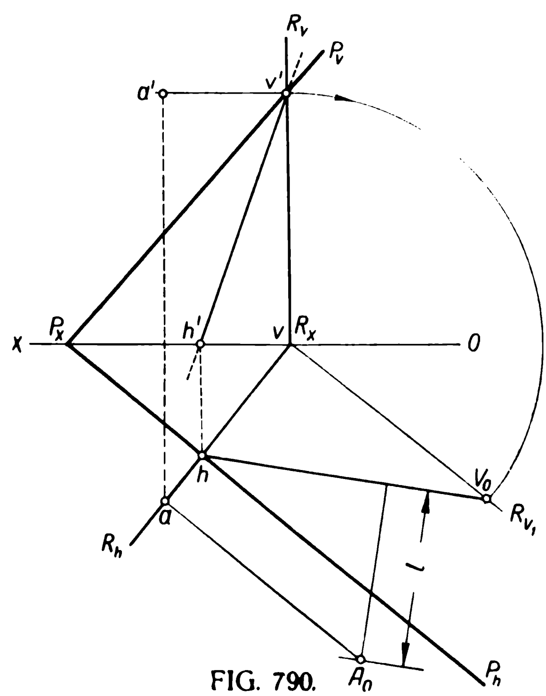



FIG. 790.

(3) the *coincidence method*: pass a horizontal (or vertical) projecting plane through the given point and perpendicular to the given plane and, bringing an auxiliary plane to coincidence with a projection plane, find the position of the given point and the line of intersection of the planes; the distance $l$ between them is the desired distance;

(4) *replacing the projection planes*: replace one of the planes of projection with a new one, which must be perpendicular to the given plane.

**Example 254**

Determine the distance from the point $A$ to the plane $P$ (Figs. 789, 790).

*Alternative solutions*:

1. By the *method of displacement* (Fig. 789). Draw in the plane $P$ an arbitrary frontal line and, by displacing it, together with the whole system, parallel to the $V$ plane, bring it to a position perpendicular to the $H$ plane. Assume an arbitrary point $P_{x_1}$ on the coordinate axis and through this point draw the vertical trace ($P_{v_1}$) perpendicular to the coordinate axis and, parallel to the trace, at a distance $l_1$, the vertical projection of the frontal line. The new position $a'_1$ of the point ($a$, $a'$) is found at a distance $l$ from the trace $P_{v_1}$ and at an arbitrary distance from the coordinate axis. Knowing $a'_1$, find $a_1$ (see the drawing). Draw the horizontal trace ($P_{h_1}$) through the points $P_{x_1}$ and $h$; a perpendicular dropped from the point $a_1$ onto the trace $P_{h_1}$ determines the desired distance.

2. By the *coincidence method* (Fig. 790). Through the point ($a$, $a'$) pass a horizontal projecting plane $R$ perpendicular to the plane $P$. Find the line ($hv$, $h'v'$) of intersection of the planes $P$ and $R$ and, bringing the plane $R$ to coincidence with the horizontal (vertical) plane of projection, find the position $A_0$ of the point ($a$, $a'$) and $hV_0$ of the line ($hv$, $h'v'$). Drop a perpendicular from the point $A_0$ to the line $hV_0$. The line-segment $l$ is the desired distance.

*Note*. We may pass a vertical projecting plane $R$ through point $A$, and so on.

**Example 255**

Determine the distance from the point $K$ to the plane of the triangle $ABC$ (Fig. 791).

*Solution*. By the method of replacing the projection planes.

Replace the $V$ plane with $V_1$. Through the vertex ($a$, $a'$) draw a horizontal line ($am$, $a'm'$) lying in the plane of the triangle and draw the new coordinate axis $O_1X_1$ perpendicular to the horizontal projection ($am$) of the horizontal line. Find the vertical projections of the point ($kk'$) and of the triangle ($abc$, $a'b'c'$)—($k'_1$) and ($a'_1b'_1c'_1$), respectively. Drop a perpendicular from the point $k'_1$ to the vertical projection $a'_1b'_1c'_1$ of the triangle. The line-segment $l$ is the desired distance.

The distance *between two parallel planes* can be determined by any one of the indicated methods for determining the distance from a point to a plane.

**Example 256**

Determine the distance between the parallel planes $P$ and $Q$ (Fig. 792).

*Solution*. Displacing the given planes, bring them to a position perpendicular to the vertical (or horizontal) plane of projection. In the plane $P$ draw a horizontal line and, displacing the whole system parallel to the $H$ plane, bring the line to a position perpendicular to the $V$ plane. The distance $l$ between the vertical traces ($P_{v_1}$) and ($Q_{v_1}$) of the planes is the desired distance. The construction is obvious from the drawing.

*Note*. In the same way we can determine the distance *between two skew lines* as the distance between the planes of parallelism.

**Example 257**

Given: a point $A$ and the horizontal trace ($P_h$) of a plane $P$, which is 18 mm distant from the point $A$. Required: to find the vertical trace of this plane (Figs. 793, 794).

*Alternative solutions*:

1. By the *coincidence method* (Fig. 793). Through the point ($a$, $a'$) pass a horizontal projecting plane $R$ perpendicular to the plane $P$ and, bringing this plane to coincidence with the horizontal (or vertical) plane of projection, find point $A_0$ and trace $R_{v_1}$. Since the distance from a point to a plane is measured by the distance from the point to the line of intersection of the planes $P$ and $R$, describe, from the point $A_0$ as centre, a circle of radius equal to 18 mm and draw through the point ($h$) of intersection of the traces $P_h$ and $R_h$ a straight line tangent to the circle to intersect the trace $R_{v_1}$ at point $V_0$ (the drawing gives one alternative solution). Then, having $V_0$, find point $v'$ on the trace $R_v$ and draw through the points $P_x$ and $v'$ the desired vertical trace ($P_v$) of the plane.

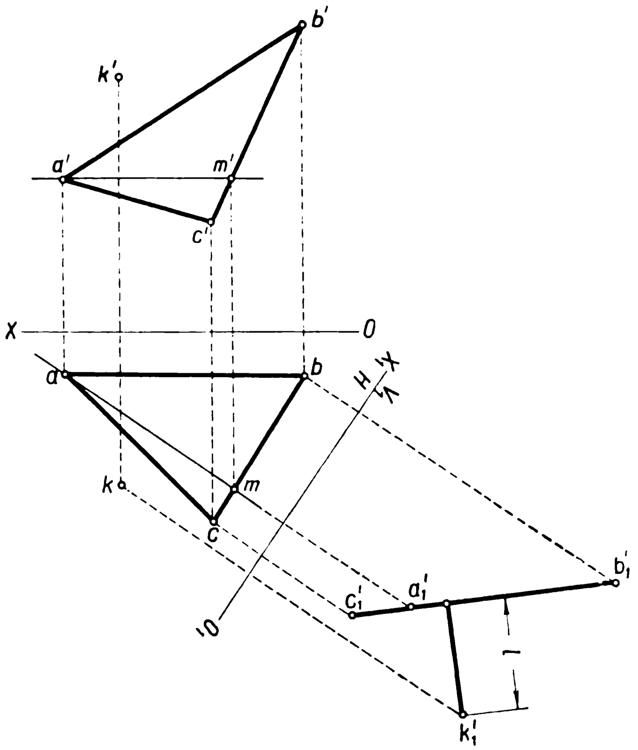



FIG. 791.

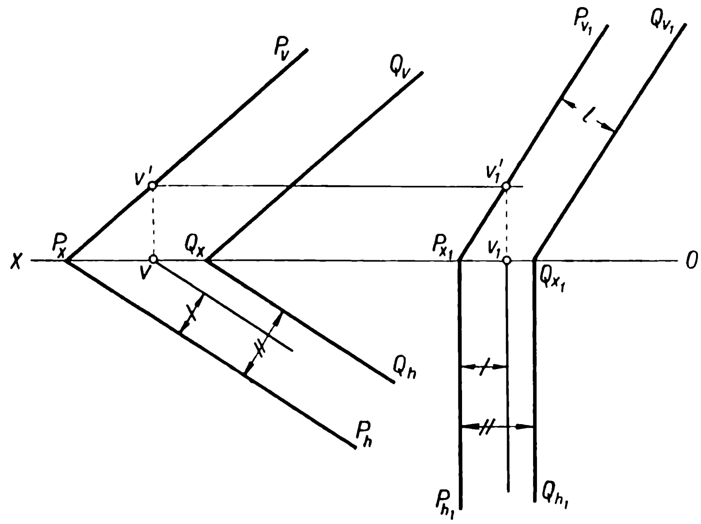



FIG. 792.

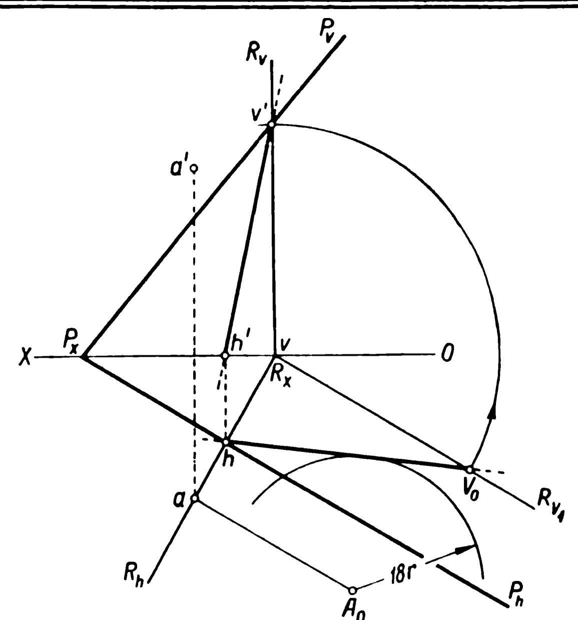



FIG. 793.

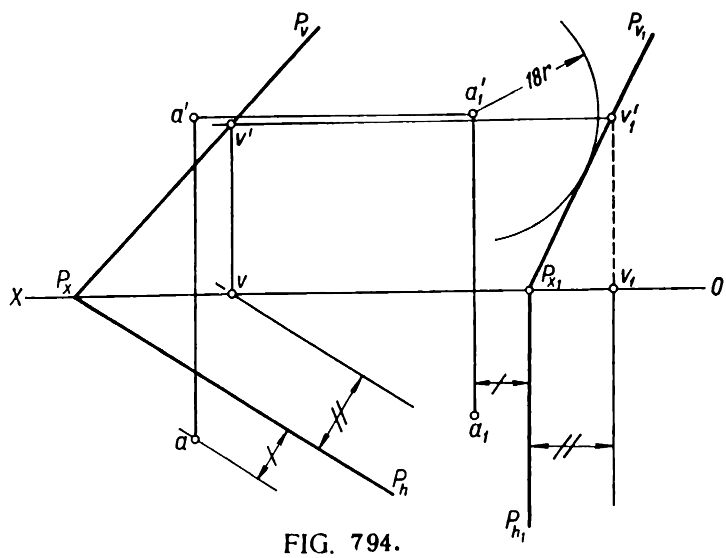



FIG. 794.

2. By the *displacement method* (Fig. 794). Displacing the given system parallel to the $H$ plane, bring it to a position in which the plane $P$ is a vertical projecting one. Assume an arbitrary position of the horizontal projection ($a_1$) of the point and the horizontal trace ($P_{h_1}$) of the plane and find the vertical projection ($a'_1$) of the point. Since the distance from a point to a vertical projecting plane is orthographically measured by the distance from the vertical projection of the point to the vertical trace of the plane, describe from point $a_1$ as centre a circle of radius 18 mm and draw the trace $P_{v_1}$ through point $P_{x_1}$ and tangent to the circle (we give one alternative solution).

Draw an arbitrary horizontal line in the plane $P_1$, find it in the initial system, and draw through the points $P_x$ and $v'$ the desired vertical trace ($P_v$) of the plane.

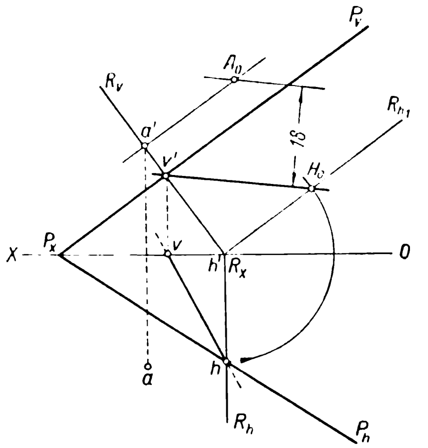


FIG. 795.

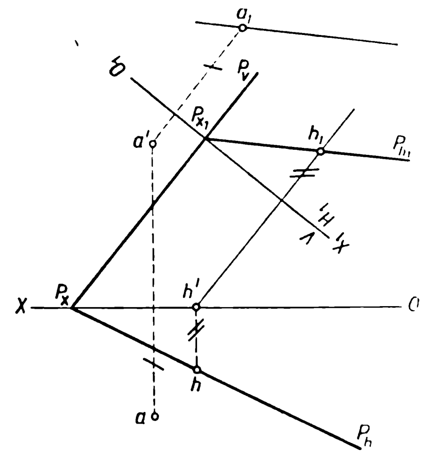


FIG. 796.

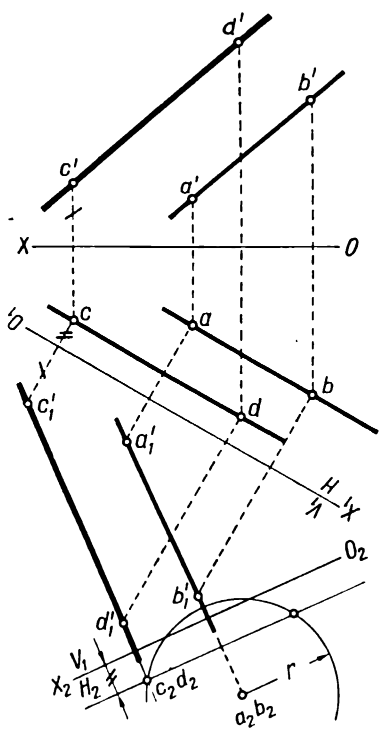


FIG. 797.

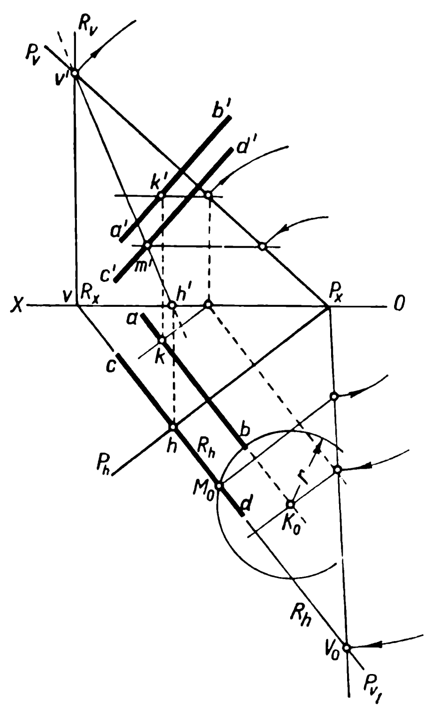


FIG. 798.

Example 258

Given: a plane $P$ and the vertical projection of a point $A$, which is 18 mm distant from the plane $P$. Required: to find the horizontal projection of the point (Figs. 795, 796).

*Alternative solutions*:

1. By the *coincidence method* (Fig. 795). Through the point $(a, a')$ draw a vertical projecting plane $R$ perpendicular to the plane $P$. Bring the plane $R$ to coincidence with the $V$ plane and find the line $v'H_0$ of intersection of the planes $P$ and $R$. Since the coincident position of the point $A_0$ must be at a distance of 18 mm from the line $v'H_0$ and on the perpendicular erected at the point $a'$ to the trace $R_v$, draw an auxiliary line at a distance of 18 mm from the line $v'H_0$ and parallel to it to intersect the perpendicular at point $A_0$ (we give one alternative solution). Knowing the point $A_0$, find the horizontal projection $(a)$ of the point $A$.

2. By *replacing the projection planes* (Fig. 796). Replace the horizontal plane of projection with a new one $(H_1)$, which is perpendicular to the plane $P$. Draw an auxiliary line, parallel to the trace $P_{h_1}$ and at a distance of 18 mm from it, to obtain the point $a_1$ at the intersection of the auxiliary line and a perpendicular dropped from the vertical projection $(a')$ of the point to $O_1X_1$. (We give one solution.) From the point $a_1$ determine the desired horizontal projection $(a)$.

Example 259

Given: a line $AB$ and the horizontal projection $(cd)$ of a line $CD$ parallel to $AB$. Required: to determine the vertical projection of the line $CD$, given that the distance between the lines is equal to $l$ mm (Figs. 797 to 799).

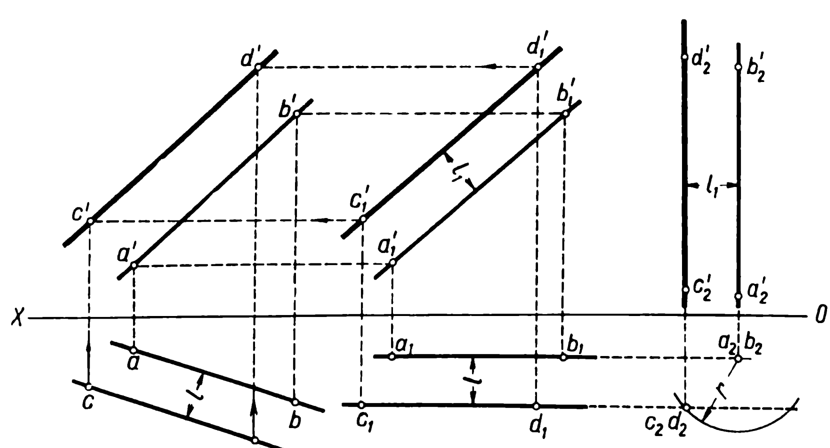



FIG. 799.

*Alternative solutions*:

1. By *replacing the projection planes* (Fig. 797). Replacing the projection planes with new ones, bring them to a position in which the $H$ plane is perpendicular to the given lines. In this case the distance between the lines is measured by the distance between their horizontal projections $(a_2b_2)$ and $(c_2d_2)$ represented as points. Hence, replace the $V$ plane with $V_1$, which is parallel to the given lines, and find first only the vertical projection $(a_1'b_1')$ of the line $AB$. Then replace the $H$ plane with $H_2$, which is perpendicular to the given lines, and determine again only the

horizontal projection ($a_2b_2$) of the line $AB$. From this point as centre describe a circle of radius $l$ mm. Then draw a line parallel to the axis $O_2X_2$ and at a distance $cc_{x_1}$ (see the accompanying drawing) to intersect the circle at point $c_2$ (or $d_2$), which is the horizontal projection ($c_2d_2$) of the line $CD$ (the drawing gives one alternative solution). Find the vertical projection ($c_1'd_1'$) of the line by reverse construction and obtain its vertical projection ($c'd'$) in the initial system.

2. By the *coincidence method* (Fig. 798). Assume an arbitrary point ($k$, $k'$) on the line ($ab$, $a'b'$) and through this line pass a plane $P$ perpendicular to it. To find the foot ($m$, $m'$) of the line ($cd$, $c'd'$) on the plane $P$, put this line into a horizontal projecting plane $R$ and find the line ($hv$, $h'v'$) of intersection of these planes. Bring the plane $P$ to coincidence with the horizontal (or vertical) plane of projection to find point $K_0$ and from this point as centre describe a circle of radius $l$ mm to intersect the line $hV_0$ at point $M_0$, which is the foot of the line ($cd$, $c'd'$) in the plane $P$. Find the vertical projection ($m'$) of this point and, through it, draw the vertical projection ($c'd'$) of the line. Here we give one alternative solution.

3. By *displacement* (Fig. 799). Displace the given system parallel to the horizontal plane of projection to bring it to a position in which the line $AB$ is parallel to the $V$ plane. Then displace the system parallel to the $V$ plane to bring it to a position in which the same line $AB$ is perpendicular to the $H$ plane. The horizontal projections of the straight lines in this position are represented as points ($a_2b_2$) and ($c_2d_2$), and the distance between them must be equal to $l$ mm. Hence, to find the point ($c_2d_2$) describe from the point ($a_2b_2$) a circle of radius $l$ mm; at its intersection with the line $c_1d_1$ parallel to the coordinate axis we obtain point ($c_2d_2$), which is the horizontal projection of the line $CD$. (We give one alternative solution.) Determine the vertical projection ($c_2'd_2'$) of the line and obtain its vertical projection ($c'd'$) in the initial system by reverse construction.

## PROBLEMS

438. Determine the distance between the point $A$ and $B$ (Figs. 800 to 803).

439. Find the lacking projection of the point $B$, given that the distance between the points $A$ and $B$ is 25 mm (Figs. 804, 805).

440. Determine the distance from the point $C$ to the line $AB$ (Figs. 636, 637).

441. Find the lacking projection of the point $A$, given that the distance from the point $A$ to the line $BC$ is 25 mm (Figs. 806, 807).

442. Determine the distance between the parallel lines $AB$ and $CD$ (Figs. 599, 639).

443. Find the lacking projection of the line $CD$, if the distance between the parallel lines $AB$ and $CD$ is 25 mm (Figs. 808, 809).

444. Construct the projections of the line $MN$ parallel to the lines $AB$ and $CD$ and 20 mm distant from $AB$ and 30 mm distant from $CD$ (Figs. 599, 639).

445. Construct the projections of the line $MN$ parallel to the line $AB$ and 20 mm distant from this line and 30 mm distant from the point $C$ (Figs. 636, 637).

446. Construct the projections of the line $MN$ parallel to the lines $AB$, $CD$, $EF$ and equidistant from them (Fig. 810).

447. Construct the projections of the line $MN$ parallel to the line $AB$ and equidistant from this line and from the points $C$ and $D$ (Fig. 811).

448. Determine the distance between the skew lines $AB$ and $CD$ (Figs. 560, 578).

449. Find on the line $AB$ a point 25 mm distant from the line $CD$ (Figs. 560, 578).

450. Determine the distance from the point $K$ to the given plane (Figs. 641 to 645).

451. Describe, from the point $C$ as centre, a sphere tangent to the given plane (Figs. 582 to 587).

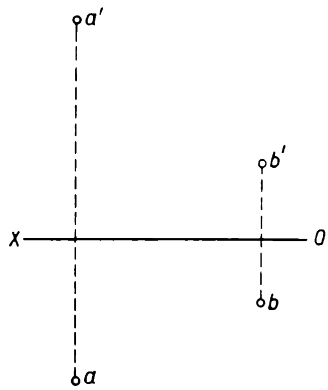



FIG. 800.

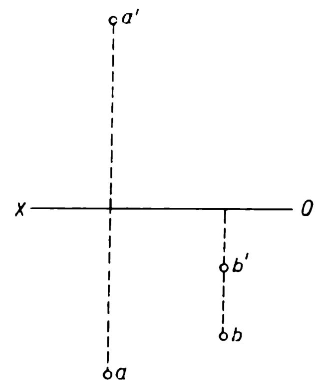



FIG. 801.

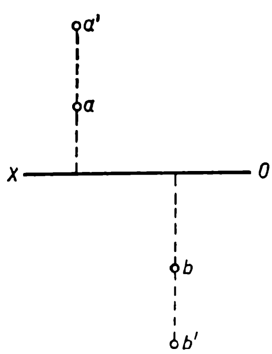



FIG. 802.

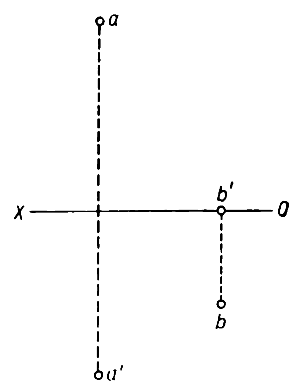



FIG. 803.

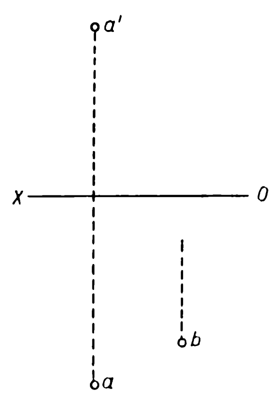



FIG. 804.

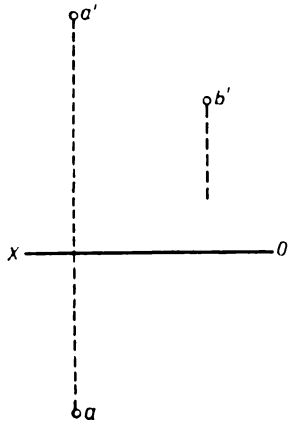



FIG. 805.

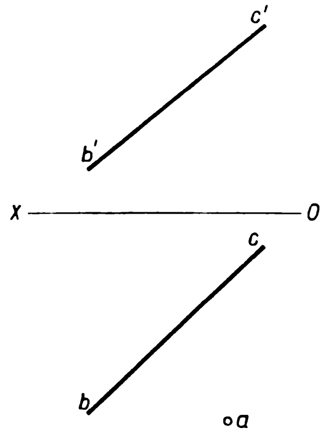



FIG. 806.

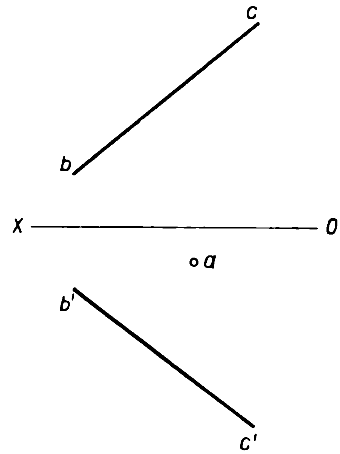



FIG. 807.

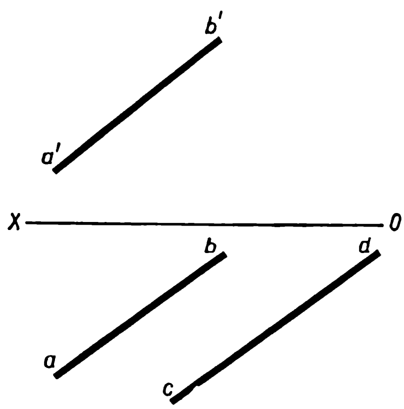



FIG. 808.

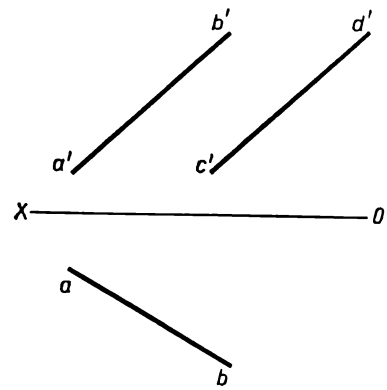



FIG. 809.

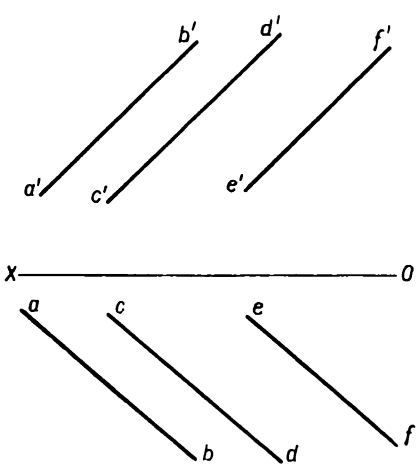



FIG. 810.

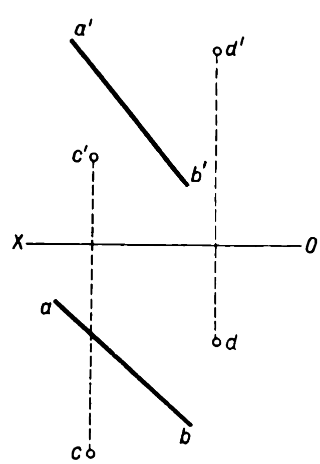



FIG. 811.

452. Determine the lacking projection of the point $K$ 25 mm distant from the given plane (Figs. 557, 588 to 590).

453. Find the lacking trace of the plane $P$, given that the distance from the point $A$ to this plane is 20 mm (Figs. 812, 813).

454. Find the lacking trace of the plane $P$, given that it is tangent to the sphere (Fig. 814).

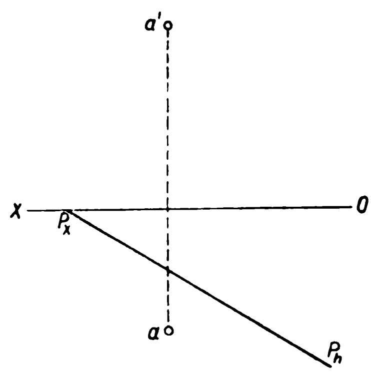



FIG. 812.

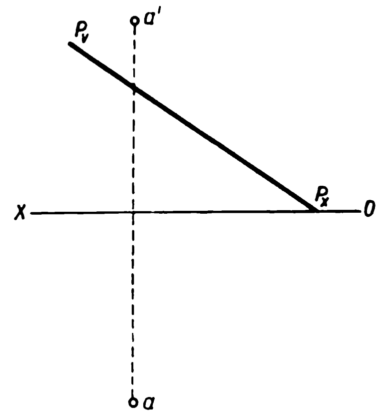



FIG. 813.

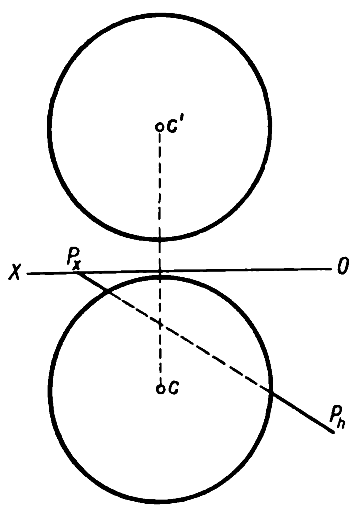



FIG. 814.

455. Determine the distance between the parallel planes $P$ and $Q$ (Fig. 577).

456. Construct the locus of points in space 25 mm distant from the given plane (Figs. 596 to 600).

457. Find on the line $MN$ a point 25 mm distant from the given plane (Figs. 606 to 608).

CHAPTER XXI

# DETERMINING ANGLES

The *angle between two intersecting lines* can be found by any one of the following methods:

(1) *constructing a triangle containing the desired angle*: intersect the sides of the angle with an arbitrary line and determine the true size of the triangle thus obtained; then determine the true size of the angle;

(2) *revolving* (or *displacing*): bring the plane of the angle to a position parallel to a projection plane;

(3) *coincidence*: find one of the traces (horizontal or vertical) of the plane of the angle and, by revolving about the trace, bring the angle to coincidence with the corresponding plane of projection;

(4) *revolving about a horizontal or frontal line*: bring the angle to coincidence with a plane $R$ parallel to the horizontal (vertical) plane of projection and passing through a horizontal (frontal) line of the plane of the angle;

(5) *changing the projection planes*: replace the planes of projection so that one of them becomes parallel to the plane of the angle.

Of the above-listed methods the fourth is the most effective.

The angle formed by two lines *not contained in one and the same plane* is measured by the angle between two intersecting lines parallel to the given ones.

## EXAMPLES

**Example 260**

Determine the true size of the angle $ABC$ (Fig. 815).

*Solution.* By revolving about a horizontal line.

Draw a horizontal line ($mn$, $m'n'$) in the plane of the given angle and pass through it a plane $T$ parallel to the $H$ plane. Bring the sides ($mb$, $m'b'$) and ($nb$, $n'b'$) of the angle to coincidence with the plane $T$. Since the points ($m$, $m'$) and ($n$, $n'$) already lie in the plane $T$, it remains to bring the vertex ($b$, $b'$) to coincidence with this plane; the angle $mB_0n$ is the desired angle. The construction is obvious from the drawing.

**Example 261**

Draw the bisector of the angle $C$ in the triangle $ABC$ (Fig. 816).

*Solution.* By the displacement method.

To draw the bisector of an angle, the true size of the angle should be found. For this purpose bring the plane of the given triangle to a position parallel to a projection plane, say, to the $H$ plane.

Through the point ($a$, $a'$) draw a horizontal line in the plane of the triangle and displace the line parallel to the horizontal plane of projection to bring it to a position perpendicular to the $V$ plane. As is known, when a triangle is displaced parallel to the $H$ plane, its horizontal projection remains unchanged. Thus, bring the horizontal projection of the triangle to the position $a_1b_1c_1$ so that the horizontal projection of the horizontal line becomes perpendicular to the coordinate axis. From $a_1b_1c_1$ find the vertical projection ($a_1'b_1'c_1'$) of the triangle, which is a straight line. Then displace the triangle ($a_1b_1c_1$, $a_1'b_1'c_1'$) parallel to the $V$ plane and bring the vertical projection ($a_2'b_2'c_2'$) of the triangle to a position parallel to the coordinate axis; from $a_2'b_2'c_2'$ find $a_2b_2c_2$. Draw the bisector of the angle $a_2c_2b_2$ and find its projections ($cd$, $c'd'$) by reverse construction in the initial system (the construction is clear from the drawing).

The *angle between a line and a plane* is an *acute* angle formed by this line and its projection on the given plane.

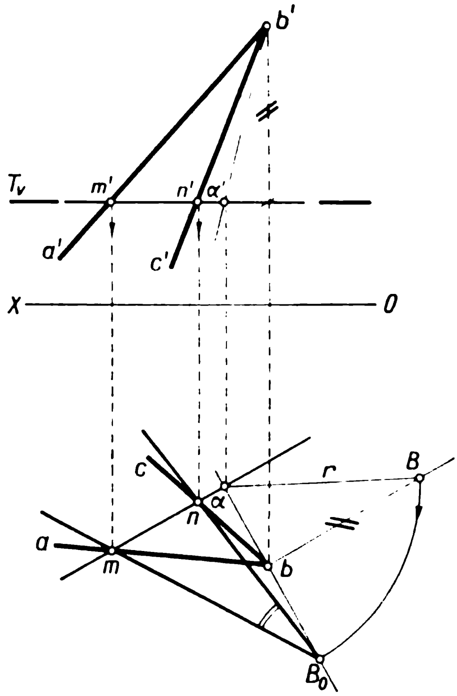



FIG. 815.

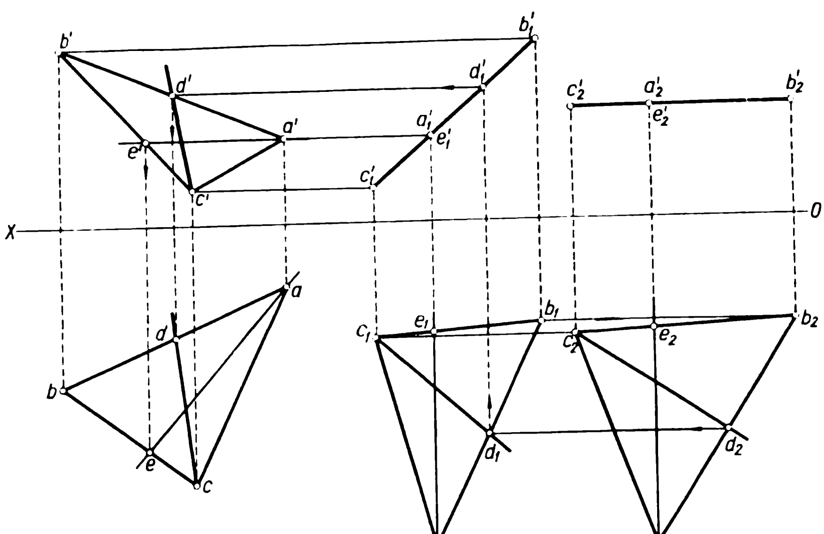



FIG. 816.

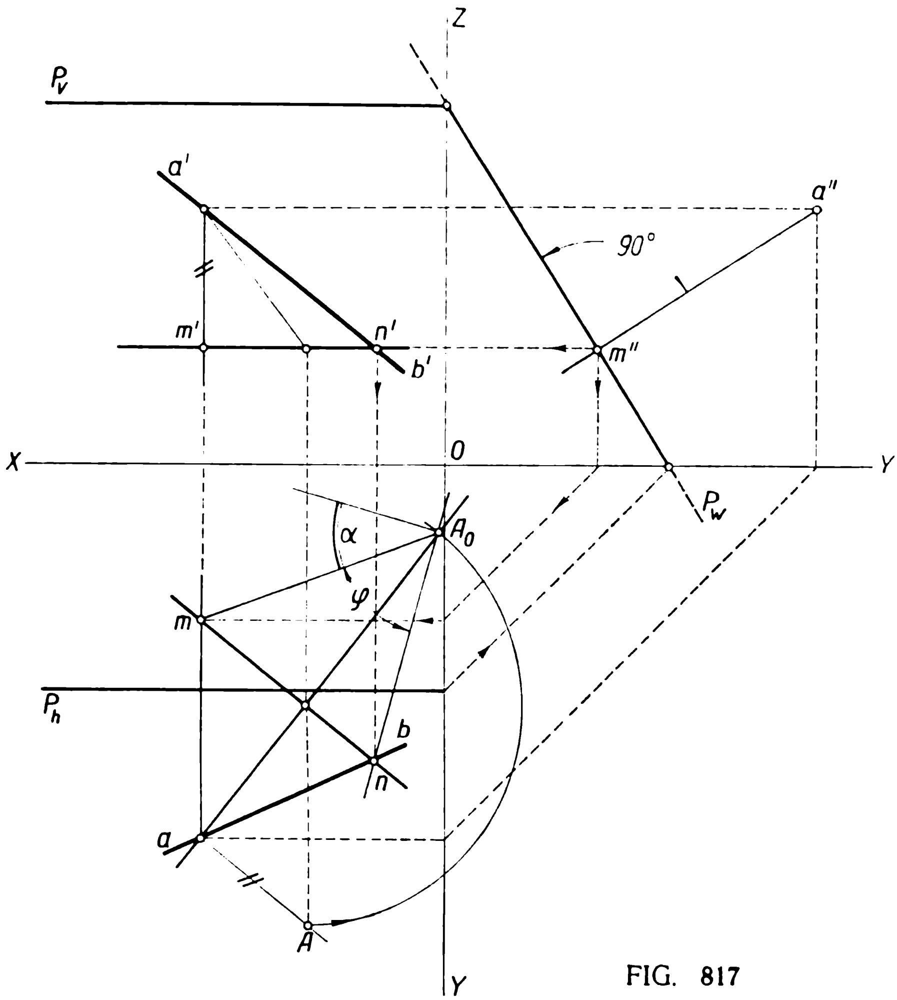



FIG. 817

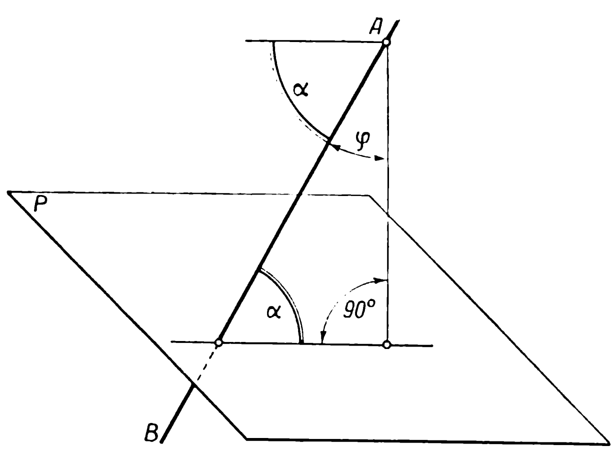



FIG.817a.

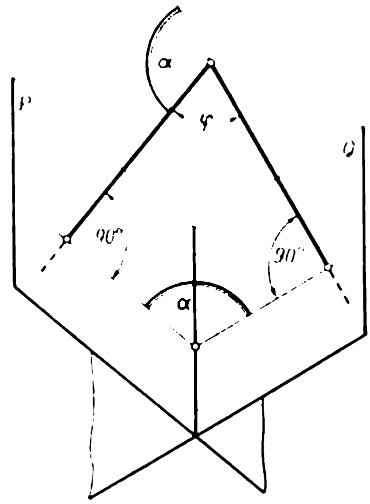



FIG. 818.

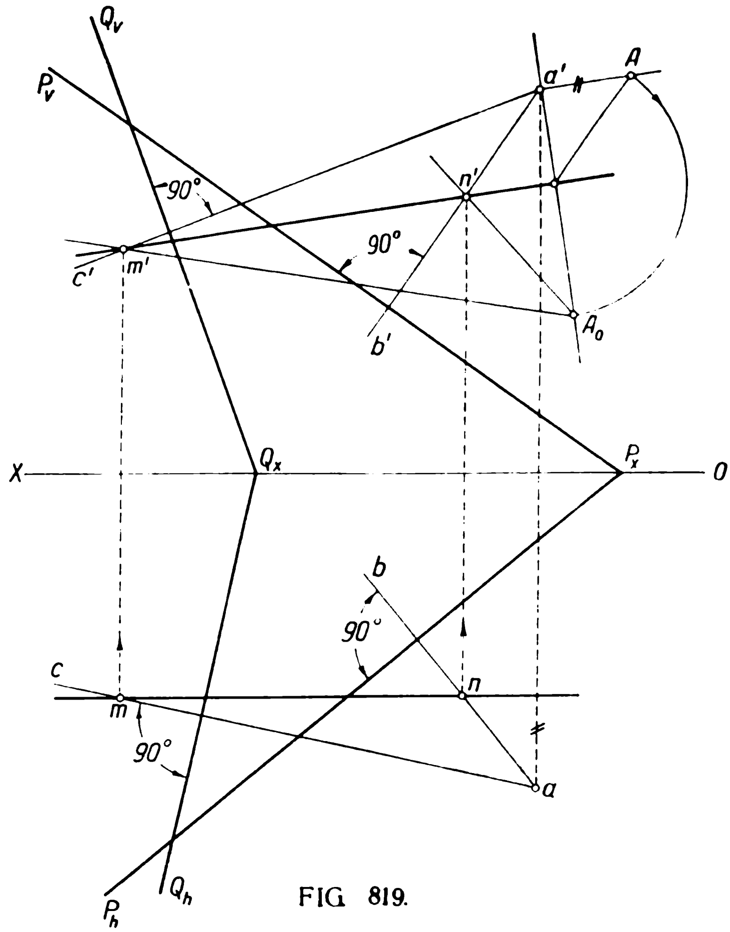



FIG 819.

Direct determination of this angle requires a number of auxiliary constructions (which ones?) and is too tedious. The task can be considerably simplified by determining not the desired angle $\alpha$ but the complementary angle $90° - \alpha$, i.e. the acute angle $\varphi$ formed by the given line and a perpendicular dropped to the plane from an arbitrary point on the line. The angle $\varphi$ can be determined by any one of the above methods, but, as has already been stated, the simplest way is revolution about a horizontal or frontal line.

If the obtained angle $\varphi$ is *obtuse*, the desired angle $\alpha$ is determined by subtracting 90° from $\varphi$ (Fig. 817a) (why?).

**Example 262**

Determine the angle between the line $AB$ and the plane $P$ (Fig. 817).

*Solution.* Drop a perpendicular from the point ($a$, $a'$) to the plane $P$. Since the latter is a profile projecting plane, construct the profile trace ($P_w$) of the plane and the profile projection ($a''$) of the point; draw a line $a''m''$ through the point $a''$ and perpendicular to the trace $P_w$, and find the horizontal ($am$) and vertical ($a'm'$) projections of the perpendicular. Then find the true size of the angle $mA_0n$ by revolving about the horizontal line ($mn$, $m'n'$) and obtain the desired angle $\alpha$, which is equal to $90° - mA_0n$. (The angle between a line and a plane specified other than by traces is determined in the same way.)

The angle between two intersecting planes is measured by one of its linear angles, usually by the *smaller* one ($\alpha°$). Determining a linear angle requires a series of auxiliary constructions (name them), therefore, direct determination is too tedious. The task can be considerably simplified by determining the angle $\varphi$ made by perpendiculars dropped from an arbitrary point to the given planes. The obtained angle $\varphi$ is the desired angle, provided it is acute. If the angle $\varphi$ is obtuse, then the desired angle is equal to $180° - \varphi$ (Fig. 818).

In determining the *angle between two intersecting planes with directed sides* (for instance, between faces of a polyhedron), perpendiculars to the given planes should

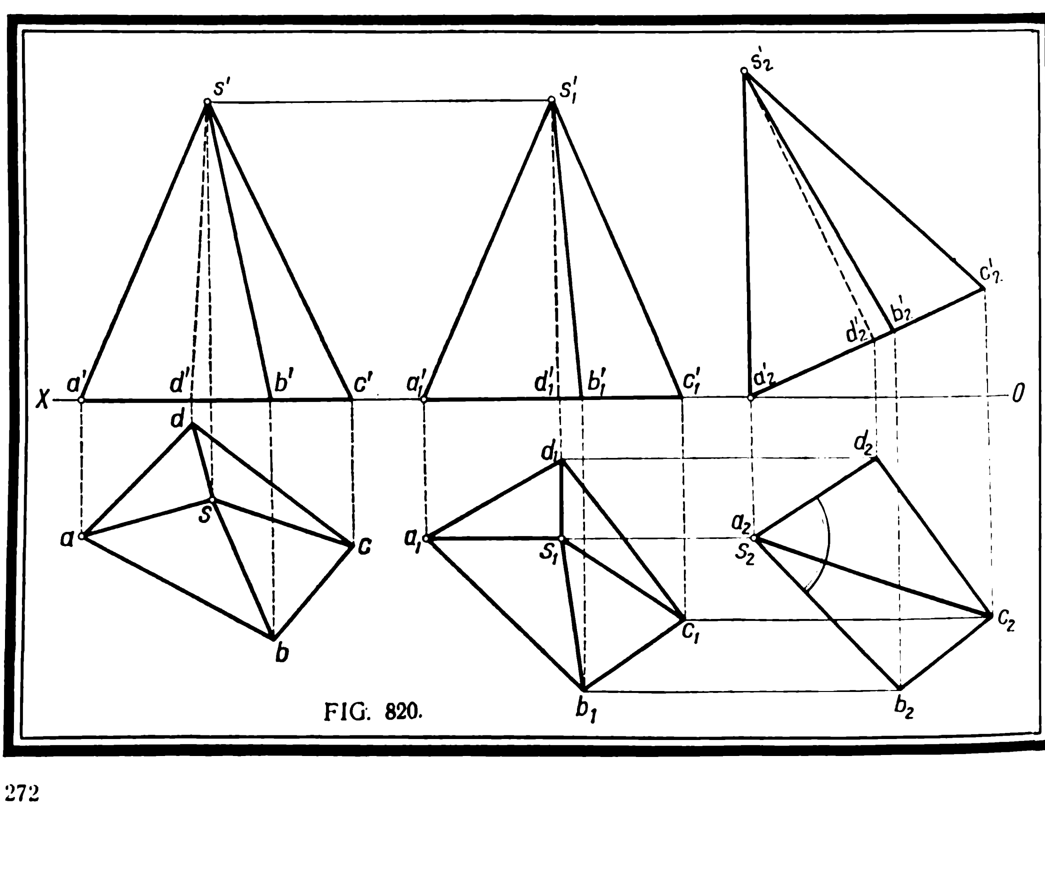



FIG. 820.

be dropped from an arbitrary point taken within the dihedral angle. The desired angle is always equal to $180° - \varphi$.

The angle between two planes can alternatively be determined by:

(1) *revolving* or *displacement*: bring the given planes (i.e. the line of their intersection) to a position perpendicular to a projection plane (see Example 210);

(2) *changing the projection planes*: replace the planes of projection so that one of them becomes perpendicular to the given planes, i.e. to the line of their intersection (see Example 249).

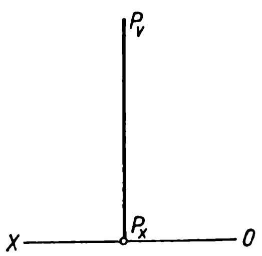


FIG. 821.

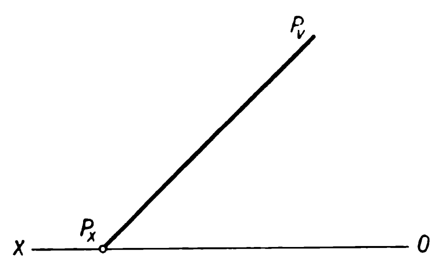


FIG. 822.

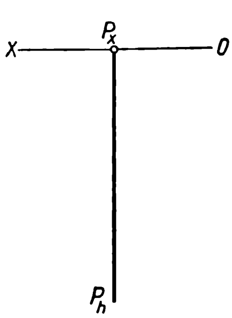


FIG. 823.

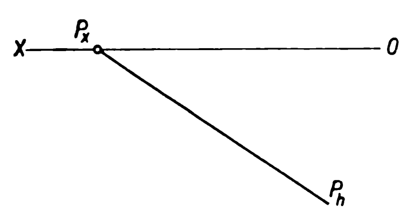


FIG. 824.

**Example 263**

Determine the angle between the planes $P$ and $Q$ (Fig. 819).

*Solution.* Drop perpendiculars ($ab$, $a'b'$) and ($ac$, $a'c'$) from an arbitrary point ($a$, $a'$) to the planes $P$ and $Q$ and, by revolving about a frontal line ($mn$, $m'n'$), determine the true size of the angle ($bac$, $b'a'c'$).

**Example 264**

Determine the angle at the edge $SA$ of the pyramid $SABCD$ (Fig. 820).

*Solution.* By the displacement method.

Displace the pyramid parallel to the $H$ plane to bring it to a position in which the edge $SA$ becomes parallel to the $V$ plane. Then displace the pyramid parallel to the $V$ plane to a position in which the edge $SA$ is perpendicular to the $H$ plane. The angle $b_2a_2d_2$ is the desired angle. The construction is obvious from the drawing.

## PROBLEMS

458. Determine the angle between the intersecting lines $AB$ and $AC$ (Figs. 164, 165).

459. Determine the angle between the skew lines $AB$ and $CD$ (Figs. 560, 578).

460. Determine the angle between the line $AB$ and the given plane (Figs. 451 to 458; 470 to 473).

461. Determine the angles of inclination of the given plane to the projection planes (Figs. 596 to 600).

462. Find the lacking trace of the plane $P$, given that the angle between this plane and the $V$ plane is 60° (Figs. 821, 822).

463. Find the lacking trace of the plane $P$, given that the angle between this plane and the $H$ plane is 45° (Figs. 823, 824).

464. Through the point $A$ (20,30) pass an arbitrary plane $P$ inclined at an angle of 45° to the horizontal (vertical) plane of projection.

465. Determine: the dihedral angle of the pyramid at the edge $SA$ ($SB$, $SC$, $SD$); the angle of inclination of the face $SAB$ ($SBC$, $SCD$, $SAD$) to the base (Fig. 820).

466. Determine the angle between the given planes (Figs. 393, 394, 398 to 401, 408, 409, 474, 601 to 603).

# PART FOUR

CHAPTER XXII

## A POLYHEDRON CUT BY A PLANE. THE DEVELOPMENT OF POLYHEDRONS

The *development of a polyhedron* is a plane figure obtained by successively bringing all the faces into coincidence with the plane of the drawing. The area of the figure thus obtained is equal to the surface of the developed polyhedron.

*Conclusion.* In the development of a polyhedron all its faces should be constructed true size.

EXAMPLES

**Example 265**

Determine the line of intersection of a prism with the plane $P$ (Fig. 825).

*Solution.* To construct the desired line of intersection it is necessary to find the points at which the edges of the prism intersect the given plane. Find the point $(a, a')$ of intersection of the edge $(1, 1')$ with the plane. The horizontal projection $(a)$ of this point coincides with the horizontal projection of the edge; knowing this, find the vertical projection $(a')$ of the point, given that the point $(a, a')$ lies in the plane $P$ as well. Likewise find the points $(b, b')$, $(c, c')$, and $(d, d')$ of intersection of the remaining edges with the plane $P$. Join the obtained points *consecutively* to determine the projections of the desired line of intersection: horizontal $(abcd)$ and vertical $(a'b'c'd')$. As is seen from the drawing, the horizontal projection $(abcd)$ of the intersection line coincides with the horizontal projection $(1, 2, 3, 4)$ of the prism.

**Example 266**

Determine the line of intersection of a prism with the plane $P$ (Fig. 826).

*Solution.* It is necessary to find the points of intersection of the edges of the prism with the plane $P$. Determine the point $(a, a')$ of intersection of the edge $(1, 1')$ with the plane; pass, through the edge, a plane $R$ parallel to the $V$ plane and intersecting the plane $P$ along a frontal line. The intersection of the vertical projections of the edge and of the frontal line yields the vertical projection $(a')$ of the point; from it find the horizontal projection $(a)$ of the point on the horizontal projection of the edge. Similarly, find the points $(b, b')$, $(c, c')$, and $(d, d')$ of intersection of other edges with the plane. Join the obtained points *successively* to obtain the projections $(abcd)$ and $(a'b'c'd')$ of the desired lines.

**Example 267**

Determine the line of intersection of a prism and the plane $P$ (Fig. 827).

*Solution.* Since the lateral edges and the plane $P$ are oblique, the usual method of solution, i.e. finding the points of intersection of all lateral edges with the plane, is rather *complicated* (why?). To simplify the constructions the following should be taken into consideration: the base of the prism is placed in the $H$ plane, and consequently the horizontal projection of each side of the base is the horizontal trace of the corresponding lateral face of the prism, and the point of their intersec-

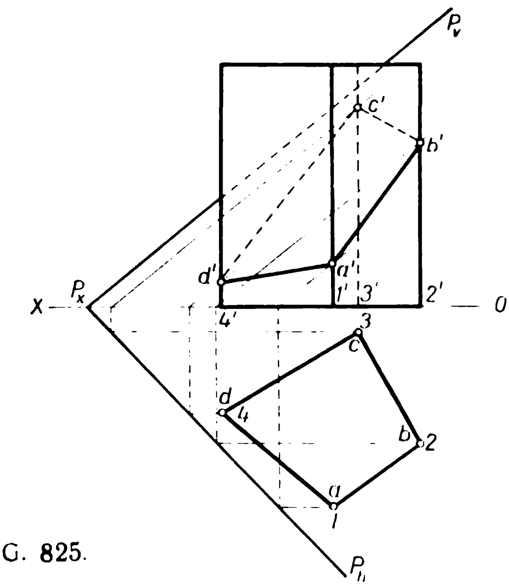



FIG. 825.

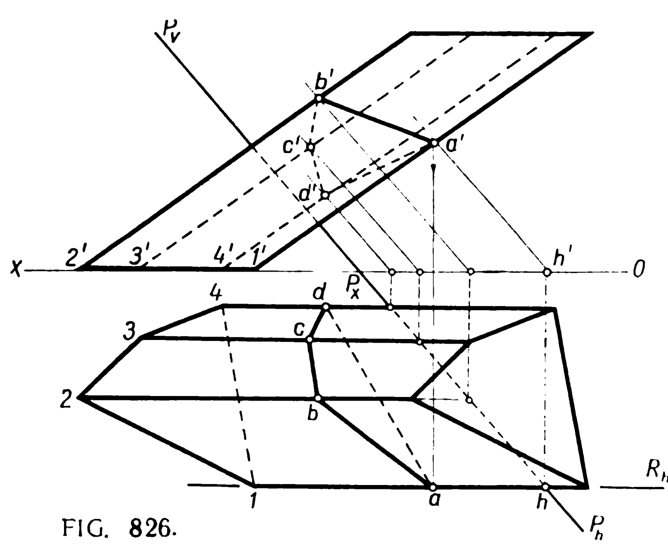



FIG. 826.

tion with the horizontal trace ($P_h$) of the plane is one of the points of the horizontal projection of the intersection line. Hence, find in the usual way, say, the point ($a$, $a'$) of intersection of the edge ($1$, $1'$) with the plane. Then extend the line $1$, $4$ to intersect the trace $P_h$ at point $h$. Join $a$ to $h$ with a line ($ah$) to intersect the horizontal projection of the edge ($4$, $4'$) and thus obtain the horizontal projection ($b$) of the point of intersection of the edge ($4$, $4'$) with the plane. Then find the vertical projection ($b'$) of the point. In the same fashion extend the line $4$, $3$ to intersect the trace $P_h$ at point $h_1$; join $b$ to $h_1$ with a straight line $bh_1$. The intersection of the line $bh_1$ and the horizontal projection of the edge ($3$, $3'$) yields the horizontal projection ($c$) of the point of intersection of the edge ($3$, $3'$) with the plane; then find the vertical projection ($c'$). Finally, extend the line $2$, $3$ to intersect the trace $P_h$ at point $h_2$; join $c$ and $h_2$ with a straight line. The intersection of the line $ch_2$ and the horizontal projection of the edge ($2$, $2'$) yields the horizontal projection ($d$)

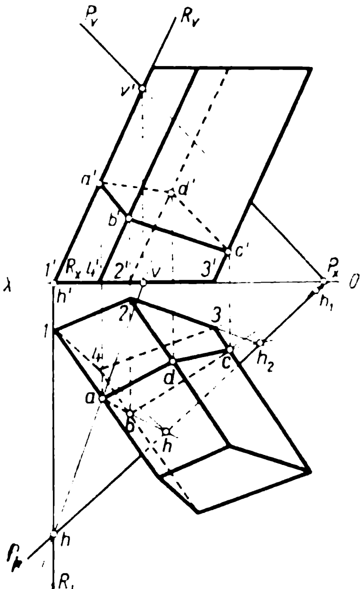



FIG. 827.

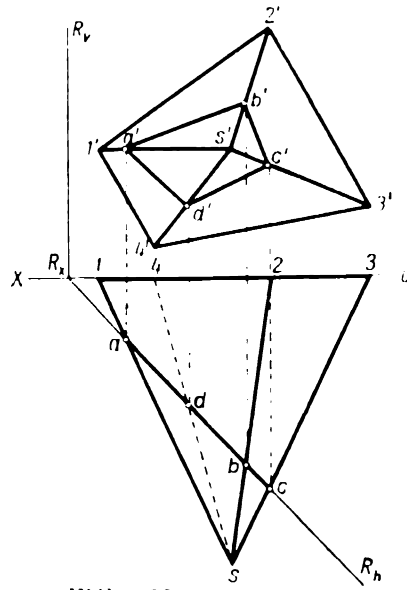



FIG. 828.

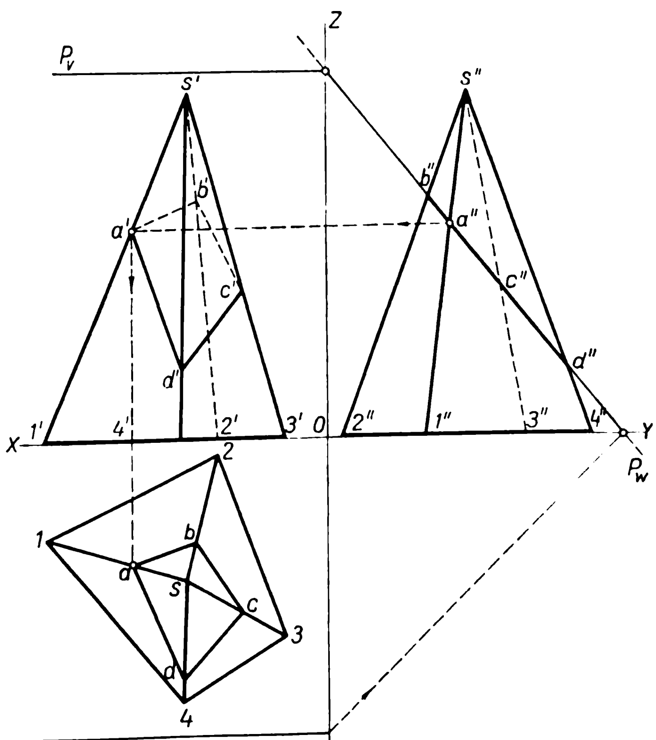



FIG 829.

of the point of intersection of the edge (*2*, *2′*) and the plane; now find the vertical projection (*d′*) of the point. Join the determined points *successively* to obtain the projections (*abcd*) and (*a′b′c′d′*) of the desired line of intersection.

**Example 268**

Determine the line of intersection of the pyramid with the plane $R$ (Fig. 828).

*Solution.* Find the points of intersection of the lateral edges of the pyramid with a horizontal projecting plane. Join the points thus obtained to determine the projections (*abcd*) and (*a′b′c′d′*) of the desired line. As is seen from the drawing, the horizontal projection (*abcd*) of the line of intersection merges with the trace $R_h$ (why?).

**Example 269**

Determine the line of intersection of a pyramid and the plane $P$ (Fig. 829).

*Solution.* Find the points of intersection of the lateral edges of the pyramid with a profile projecting plane. Construct the profile projection of the pyramid and the profile trace ($P_w$) of the plane. The intersection of the profile projections of the edges of the pyramid with the trace $P_w$ yields the profile projections of the points of intersection of the edges with the plane; now find the horizontal and verti-

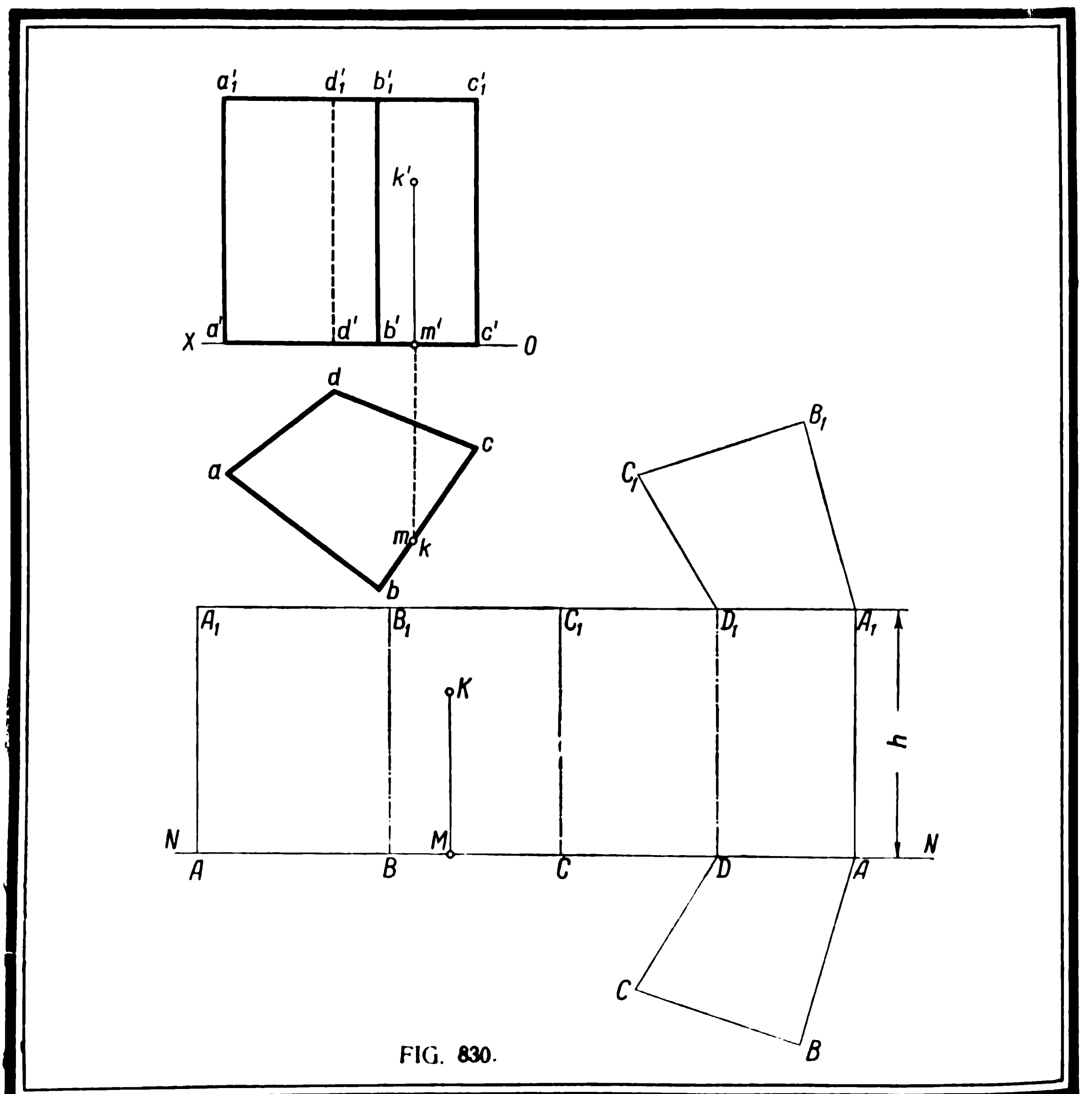



FIG. 830.

cal projections of these points. Join the determined points *successively* to obtain the horizontal ($abcd$) and vertical ($a'b'c'd'$) projections of the desired line of intersection. Without the use of the profile plane of projection the construction would be more *complicated* (why?).

**Example 270**

Construct a complete development of the quadrangular prism (Fig. 830).

*Solution*. The full surface of the given prism consists of four rectangles and two quadrilaterals. Draw an arbitrary line $NN$ and lay off on it (from point $A$) line-segments $AB$, $BC$, $CD$, $DA$ equal to the respective sides of the base, i.e. $AB = ab$, $BC = bc$, etc. Through the points $A$, $B$, $C$, $D$, $A$ draw perpendiculars to the line $NN$ and lay off on them *equal* (why?) segments of length $h$. Join the ends of the perpendiculars to obtain a straight line $A_1B_1C_1D_1A_1$ parallel to the line

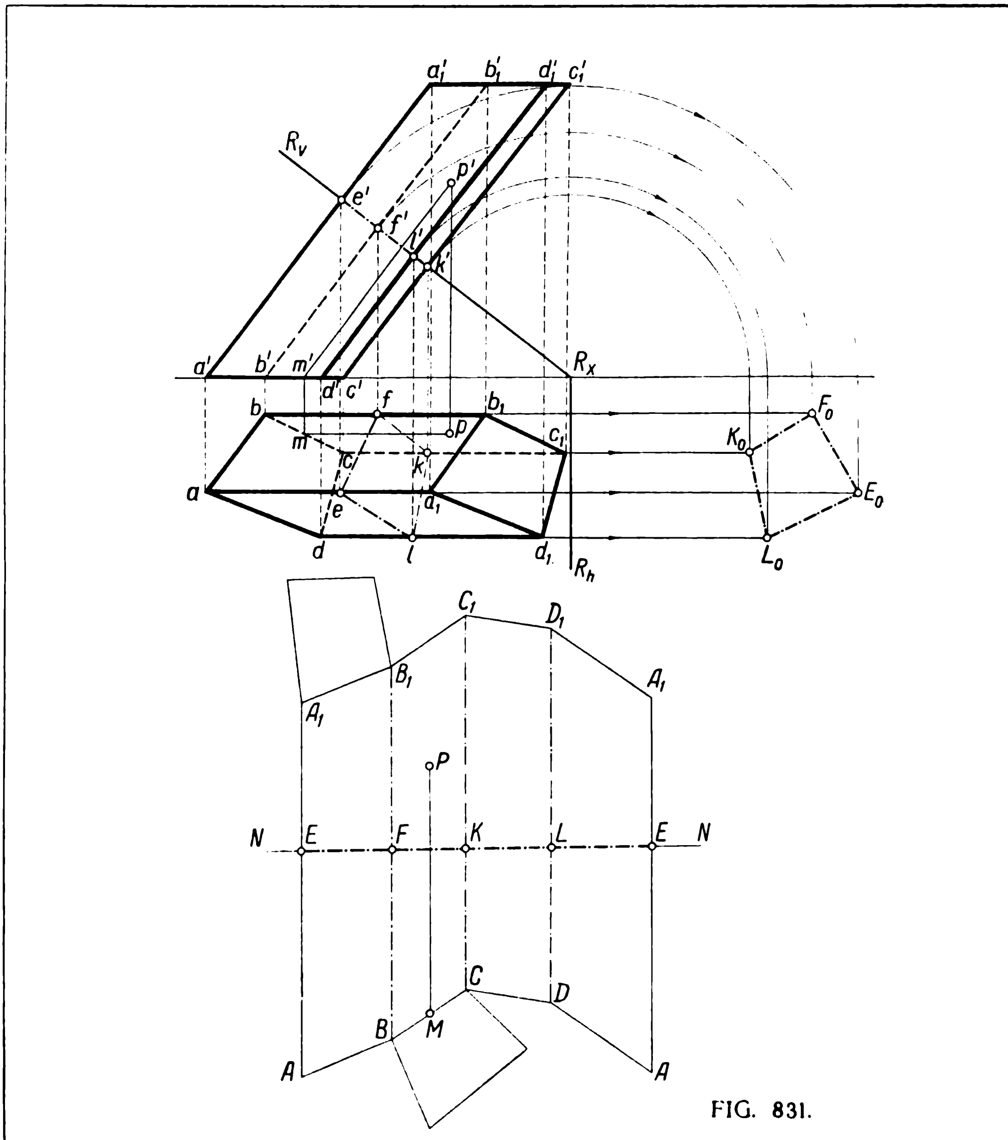



FIG. 831.

*ABCDA*. Then construct, say, at the side *AD*, the lower base of the prism, and, at the side $A_1D_1$, its upper base.

The figure thus obtained is a *complete development* of the prism.

Let us now see how the point ($k$, $k'$) given on the face $BB_1C_1C$ of the prism is transferred to its development. Mark off on the side *BC* a line-segment *BM* equal to *bm*, erect a perpendicular at the point *M* and lay off on it a segment *MK* equal to *m'k'*.

**Example 271**

Construct a complete development of the inclined quadrangular prism (Fig. 831).

*Solution.* Since the lateral faces of an inclined prism are parallelograms, to construct them true size it is necessary to determine for each parallelogram the angles between its sides or the length of one of its diagonals. This procedure can be avoided by introducing an *auxiliary* plane *R* perpendicular to the lateral edges, which cuts the inclined prism into two right truncated prisms with a common base lying in the plane of the perpendicular section. Find the projections of this section (so-called "normal" section) and its true size using the coincidence method; then begin developing the surface of the inclined prism which consists of the surfaces of two right prisms situated on both sides of the perpendicular section.

Draw an arbitrary line *NN* and lay off on it (from point *E*) line-segments *EF*, *FK*, *KL*, *LE* equal to the sides of the normal section of the inclined prism, i.e. $EF = E_0F_0$; $FK = F_0K_0$, etc. Through the points *E*, *F*, *K*, *L*, *E* draw perpendiculars to the line *NN* and mark off on them the lengths of the respective lateral

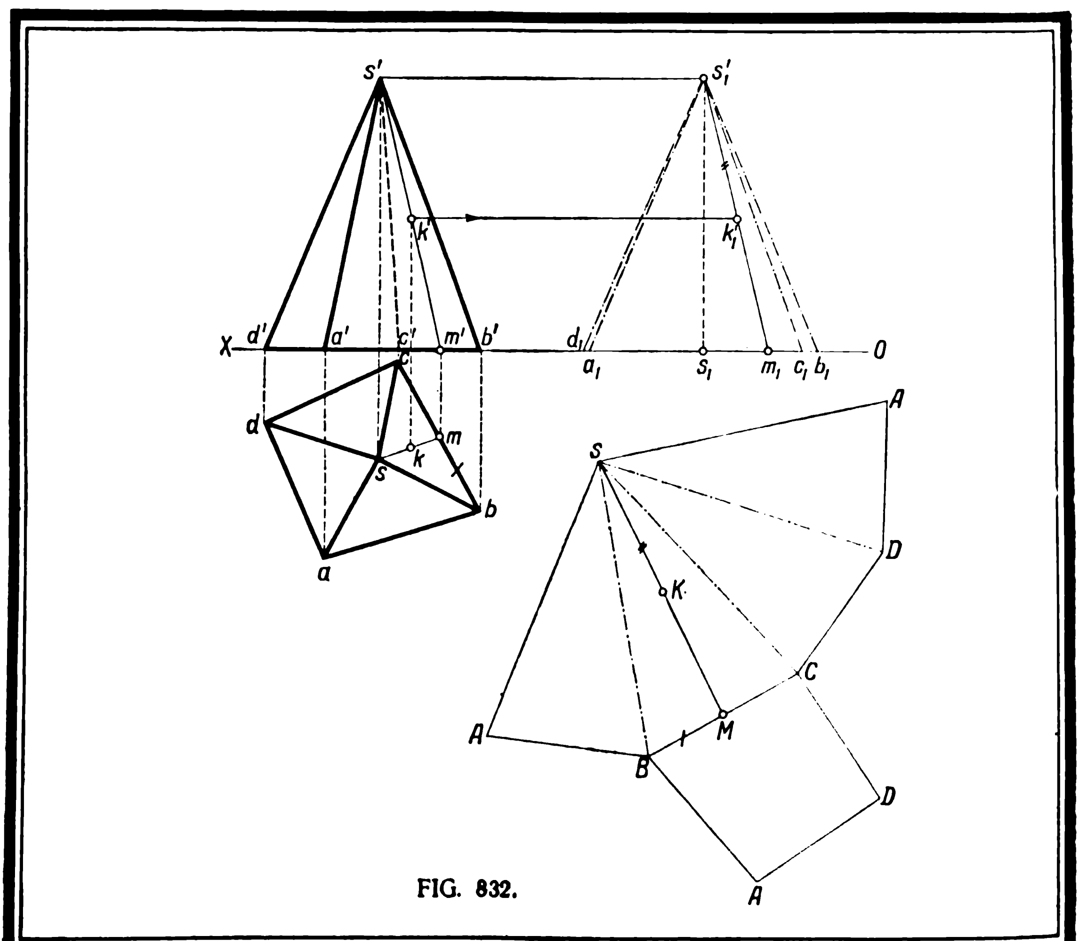



FIG. 832.

edges of the upper and lower prisms. Join the ends of the perpendiculars to obtain polygonal lines $ABCDA$ and $A_1B_1C_1D_1A_1$ with parallel sides, i.e. $AB \parallel A_1B_1$, $BC \parallel B_1C_1$, etc.

Then construct, at any face, the upper and lower bases of the prism, which are given true size.

Let us now see how the point $(p, p')$ given on the face $BB_1C_1C$ of the inclined prism is transferred to its development. Lay off on the side $BC$ a line-segment $BM = bm$; draw a straight line through the point $M$ and parallel to the edge and lay off on it a segment $MP = m'p'$.

*Note.* If the lateral edges of an inclined prism are not parallel to a projection plane, then displace them so that they become parallel to a plane of projection.

**Example 272**

Construct a complete development of the tetrahedral pyramid (Fig. 832).

*Solution.* To construct the faces of the pyramid true size, it is necessary to determine the true length of the lateral edges. Lay off on the coordinate axis, from an arbitrary point $s_1$, line-segments $s_1a_1$, $s_1b_1$, $s_1c_1$, $s_1d_1$ equal to the lengths of the horizontal projections of the lateral edges. Join the points $a_1$, $b_1$, $c_1$, $d_1$ to the point $s'_1$ to obtain the true lengths of these edges. Assume an arbitrary point $S$ and construct, in succession, the faces $SAB$, $SBC$, $SCD$, $SAD$, knowing the three sides of each face. Then construct the base $ABCD$ of the pyramid at any side, say, at $BC$. The figure thus obtained is a complete development of the pyramid.

Let us now see how the point $(k, k')$ given on the face $(sbc, s'b'c')$ of the pyramid is transferred to its development. Lay off on the side $BC$ a line-segment $BM = bm$ and then, joining $S$ to $M$, lay off on the line $SM$ a segment $SK = s'_1k'_1$.

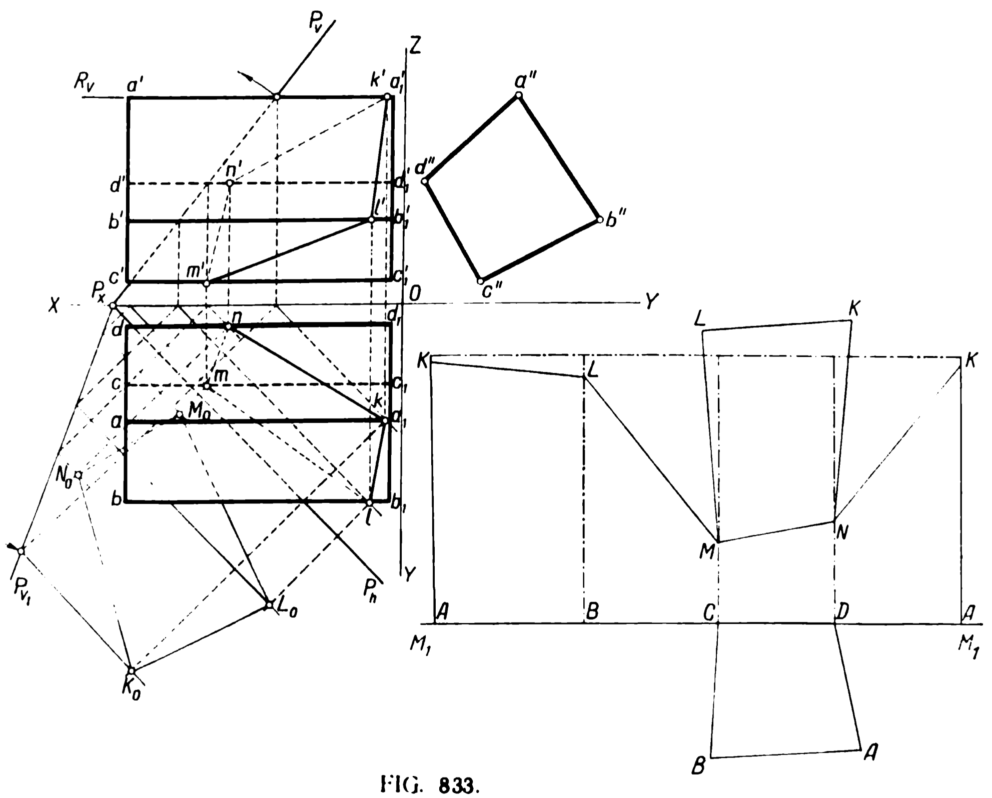



FIG. 833.

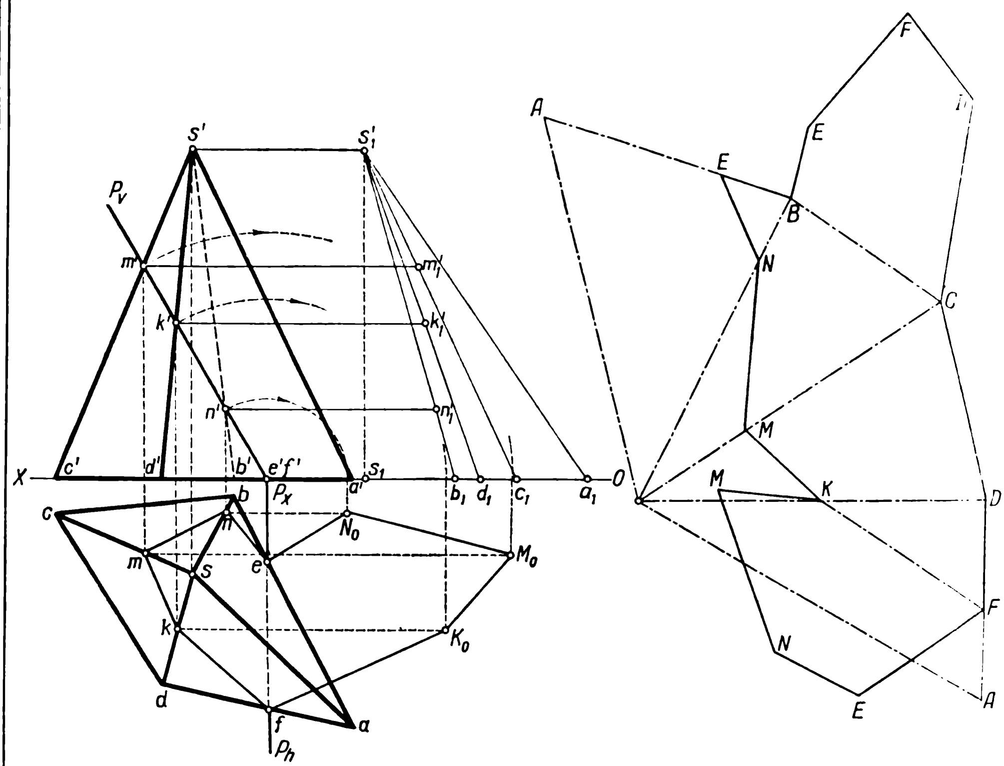



FIG. 834.

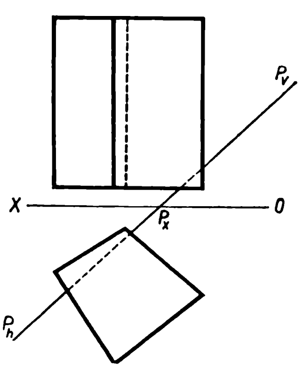



FIG. 835.

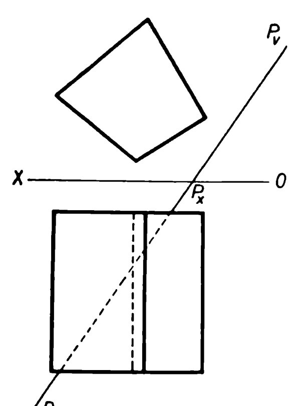



FIG. 836.

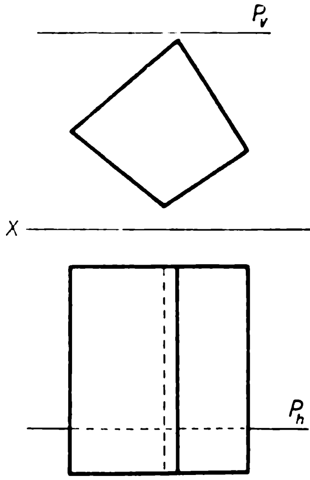



FIG. 837.

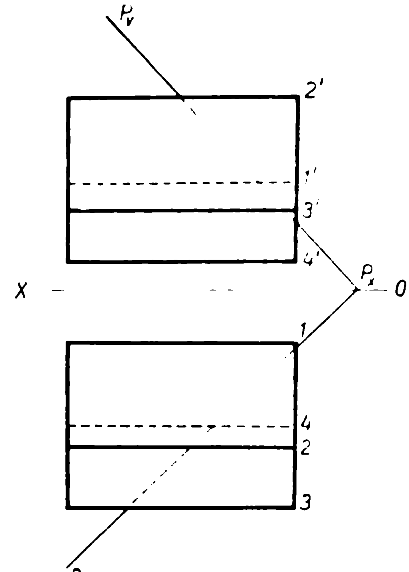



FIG. 838.

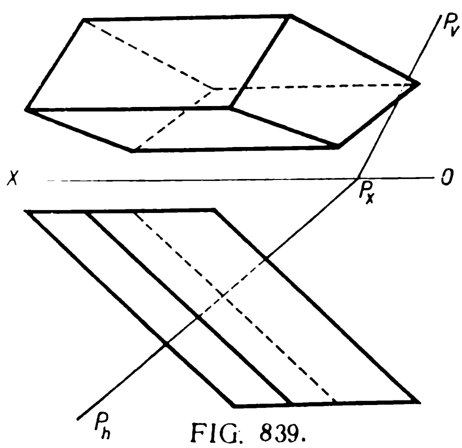



FIG. 839.

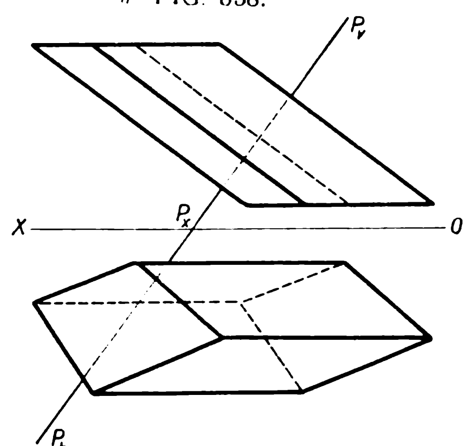



FIG. 840.

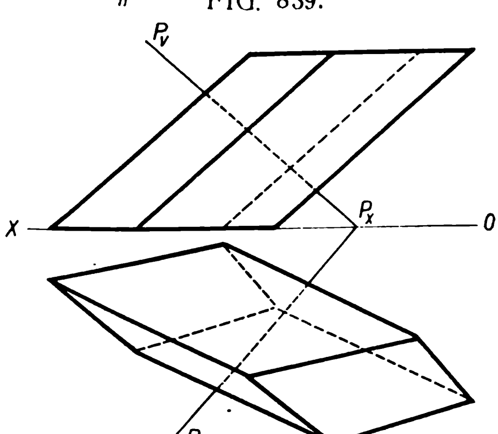



FIG. 841.

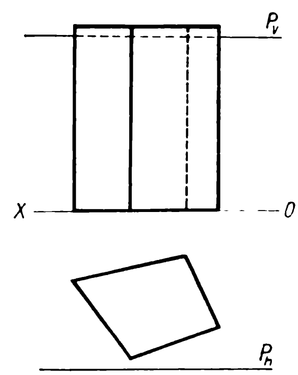



FIG. 842.

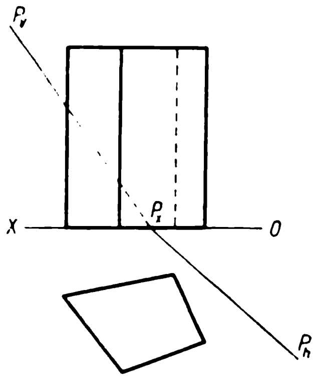



FIG. 843.

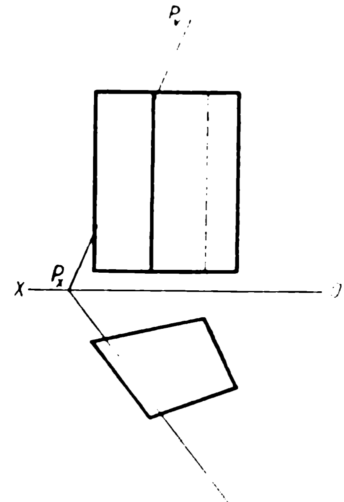



FIG. 844.

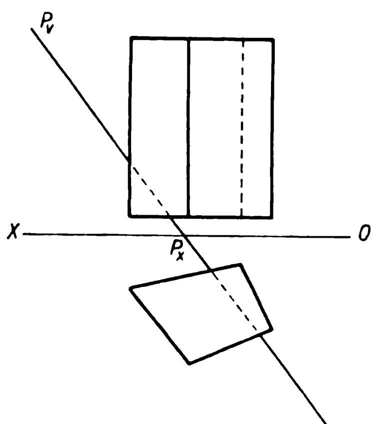



FIG. 845.

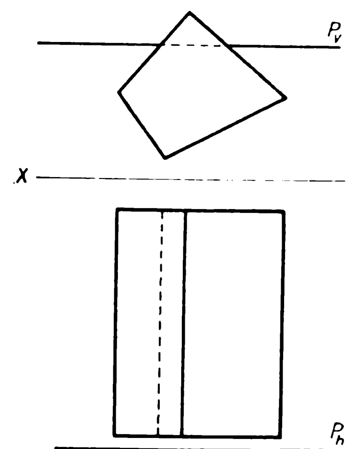



FIG. 846.

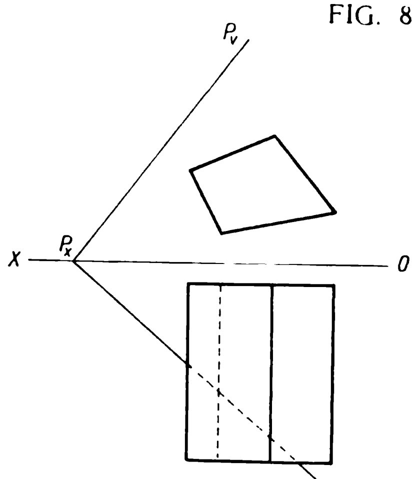



FIG. 847.

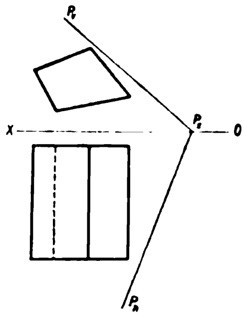



FIG. 848.

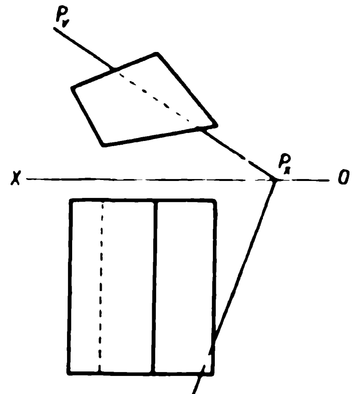



FIG. 849.

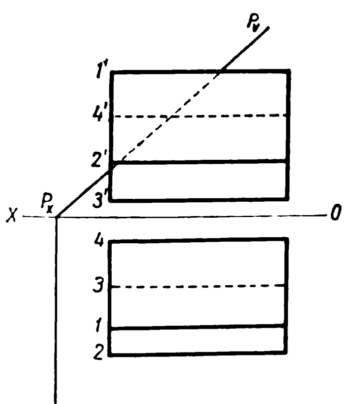



FIG. 850.

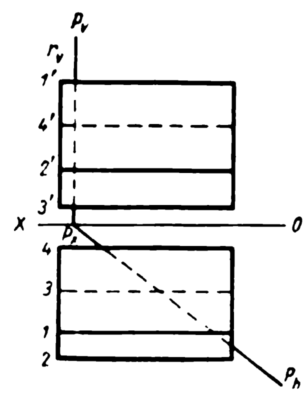



FIG. 851.

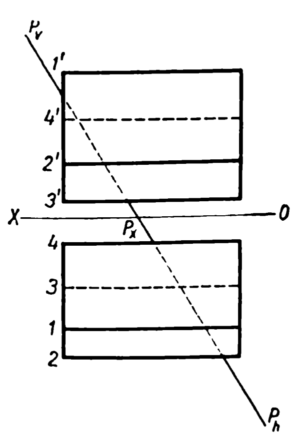



FIG. 852.

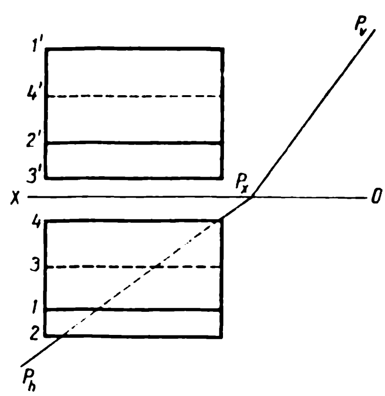



FIG. 853.

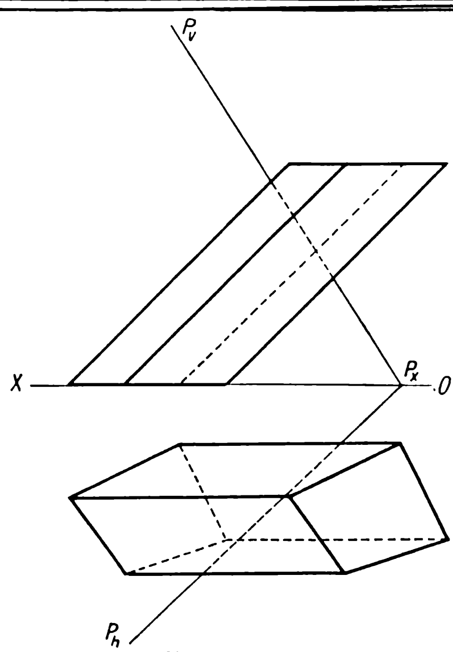



FIG. 854.

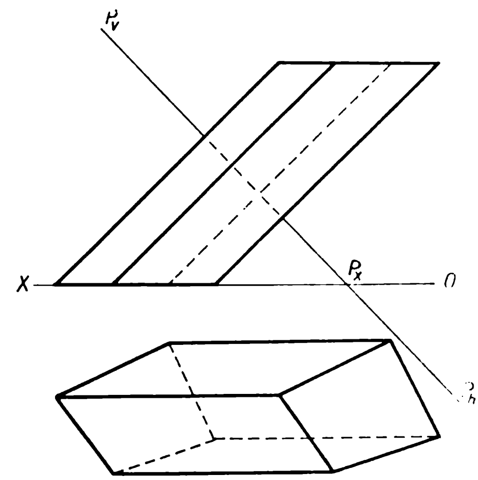



FIG. 855.

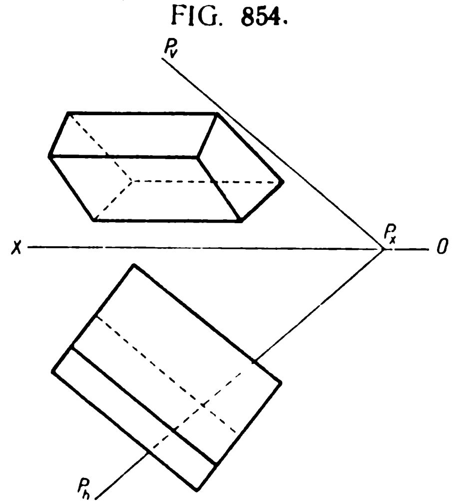



FIG. 856.

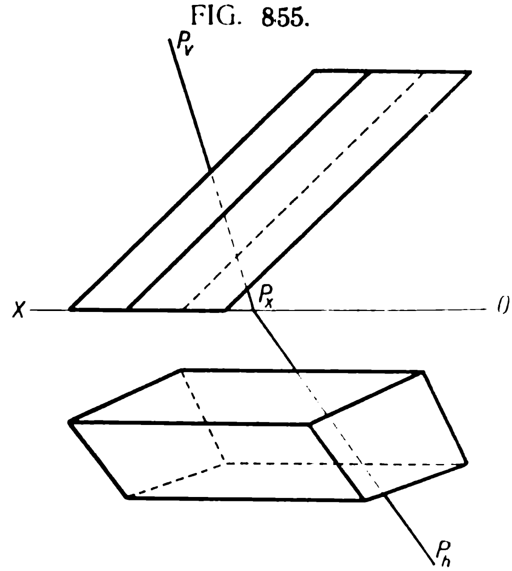



FIG. 857.

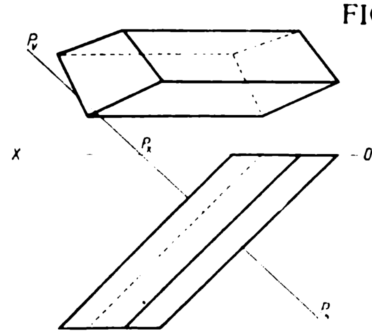



FIG. 858.

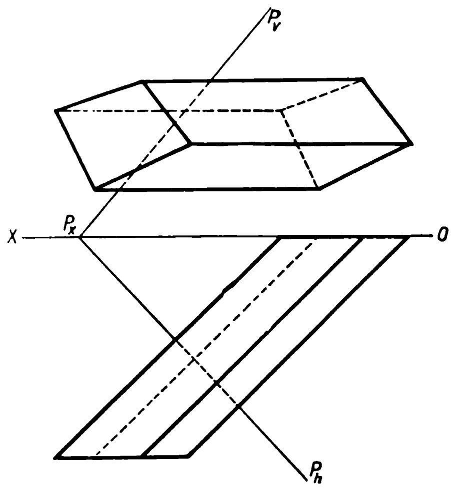



FIG. 859.

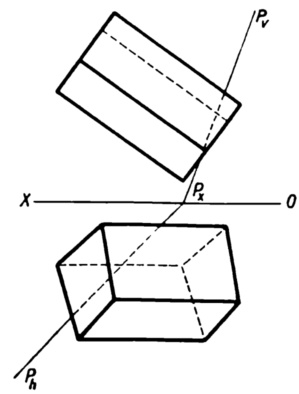



FIG. 860.

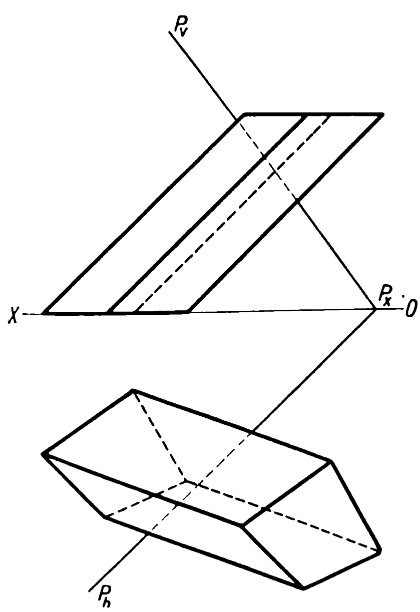



FIG. 861.

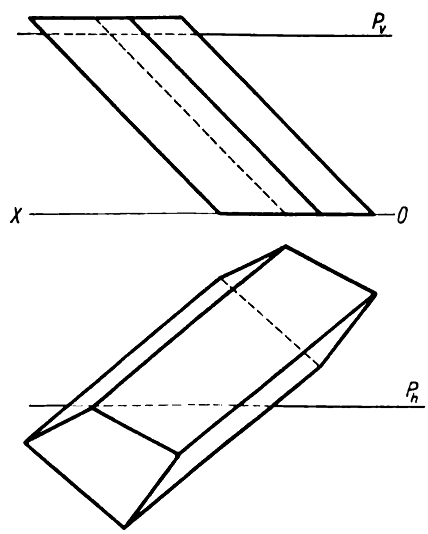



FIG. 862.

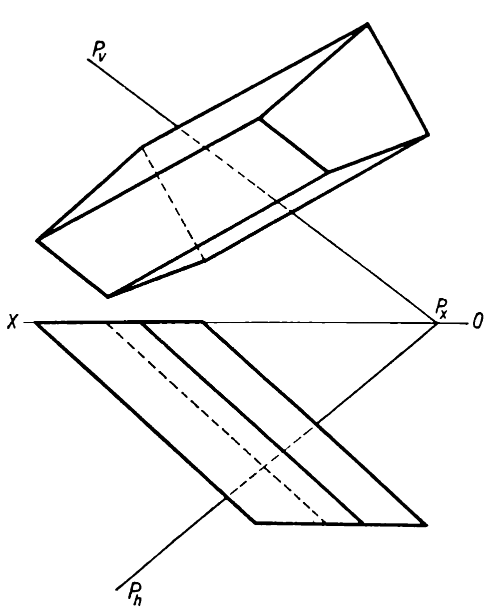



FIG. 863.

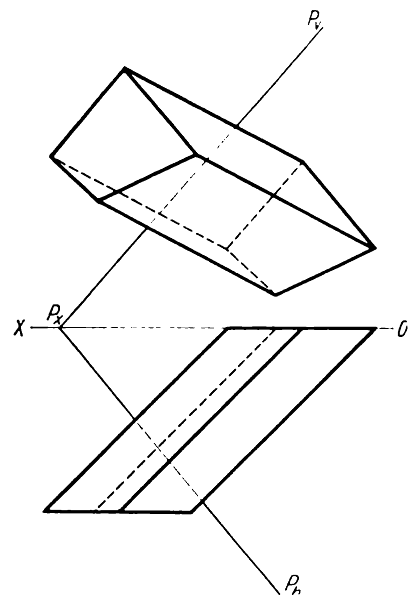



FIG. 864.

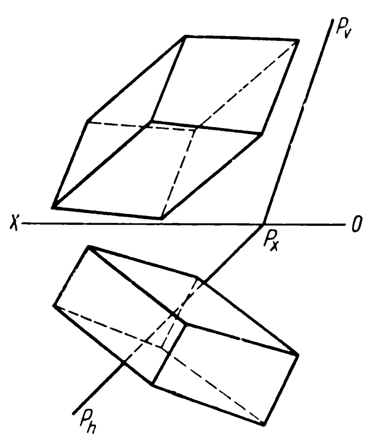



FIG. 865.

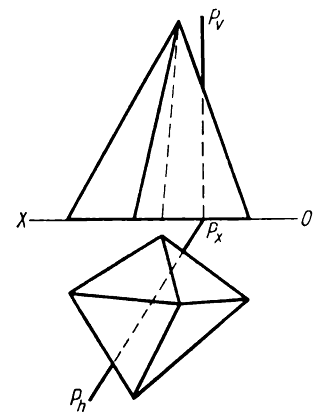



FIG. 866.

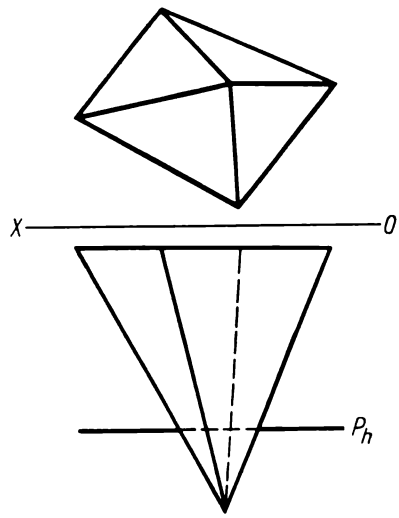



FIG. 867.

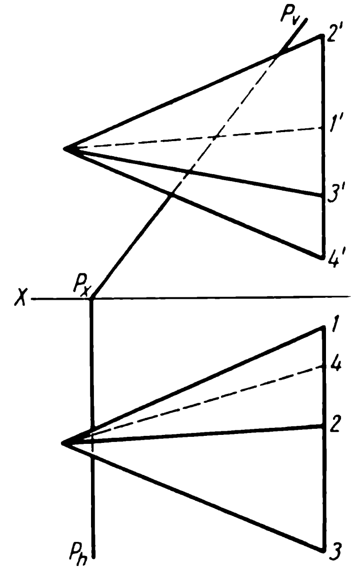



FIG. 868.

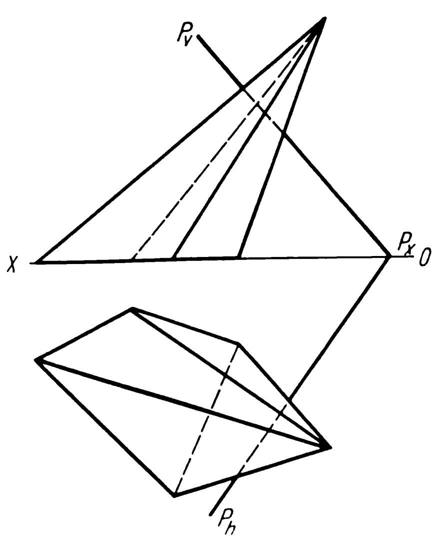



FIG. 869.

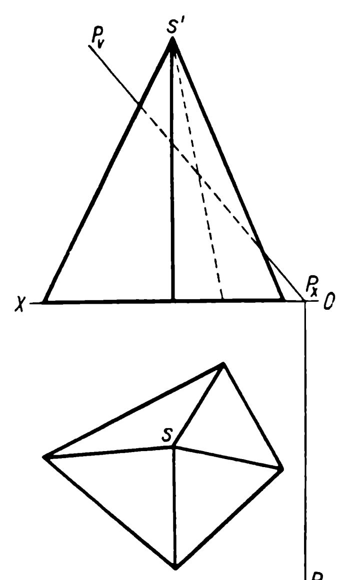



FIG. 870.

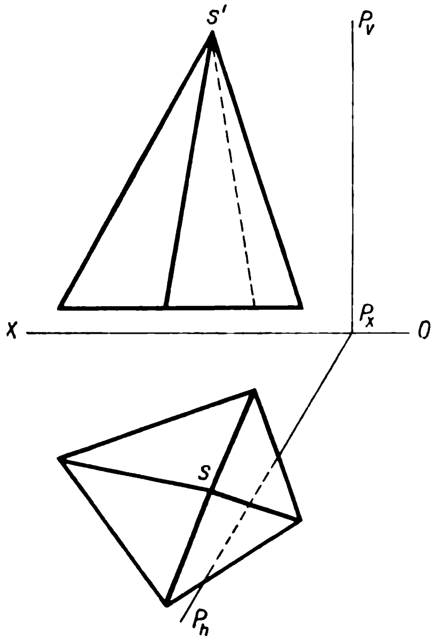



FIG. 871.

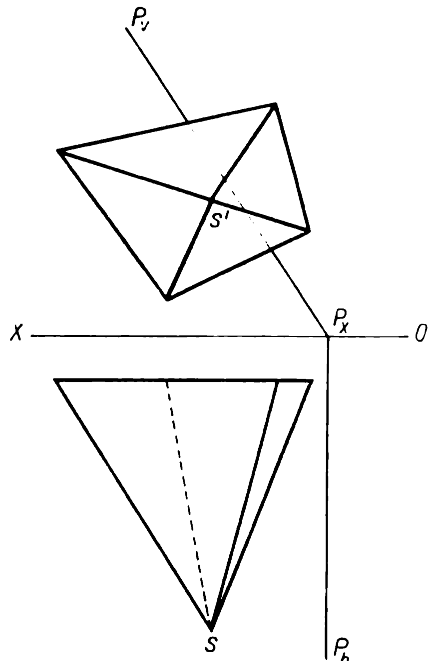



FIG. 872.

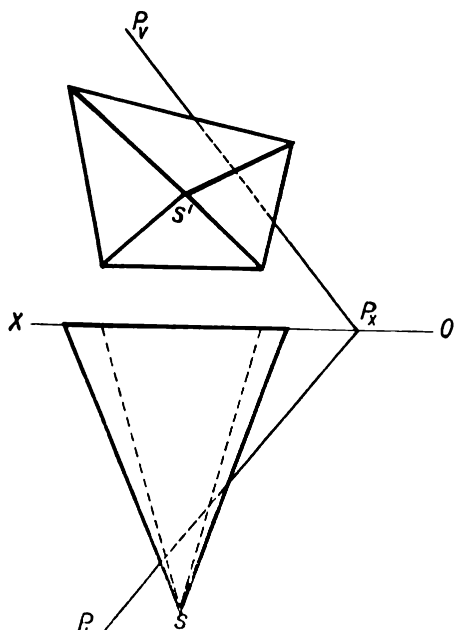



FIG. 873.

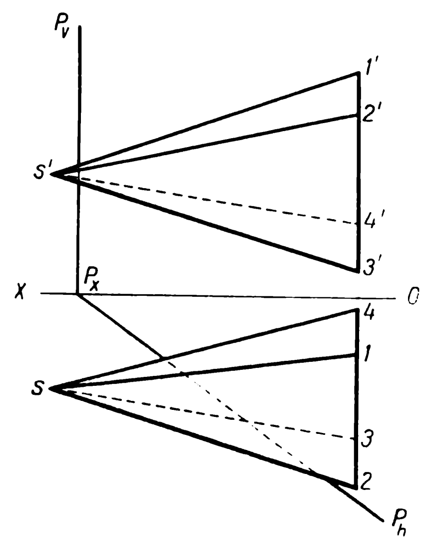



FIG. 874.

**Example 273**

Cut the prism with the plane $P$ and construct the development of either of its parts (Fig. 833).

*Solution.* Find the points of intersection of the edges of the prism with the plane $P$. For instance, to find the point $(k, k')$ of intersection of the edge $(aa_1, a', a_1')$

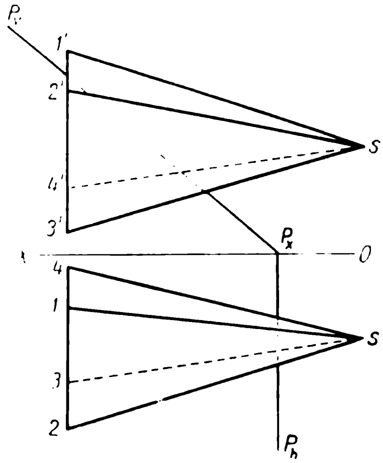



FIG. 875.

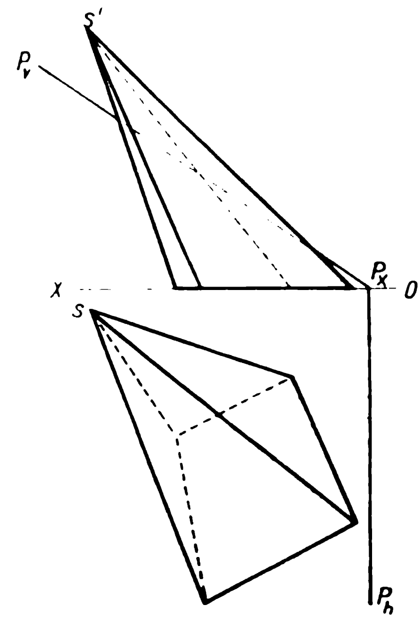



FIG. 876.

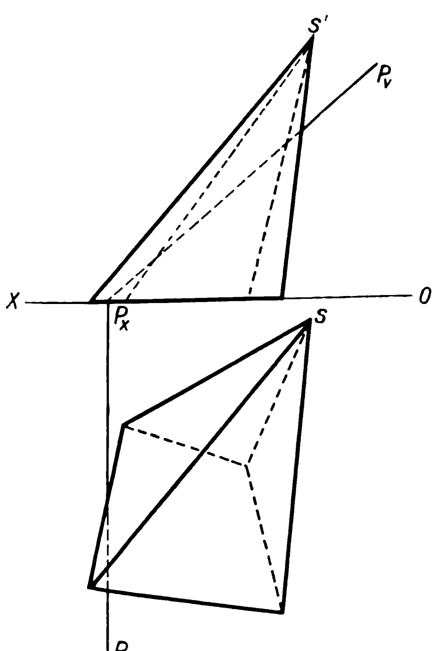



FIG. 877.

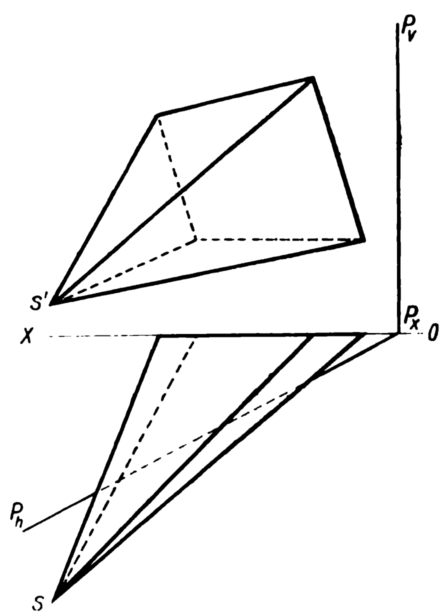



FIG. 878.

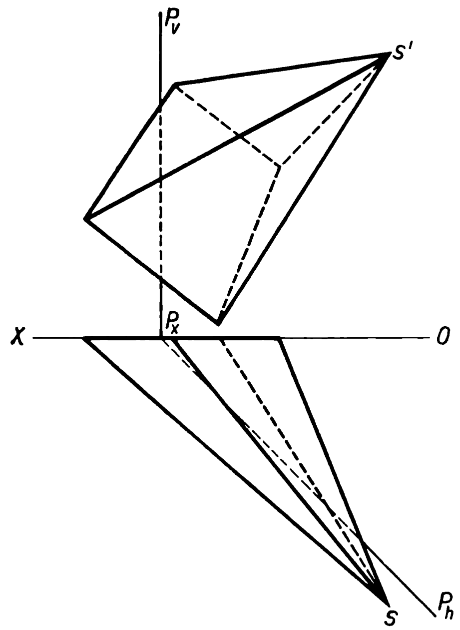


FIG. 879.

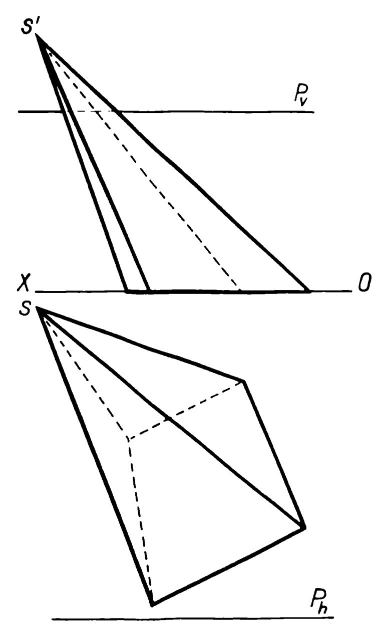


FIG. 880.

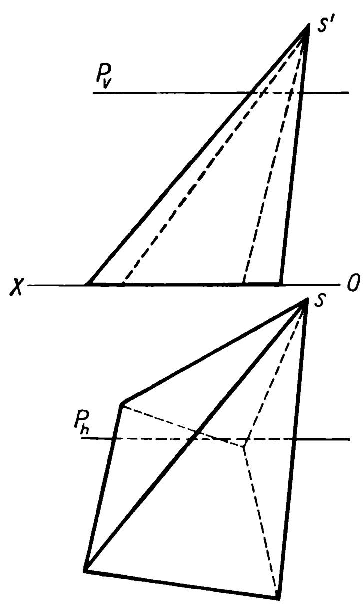


FIG. 881.

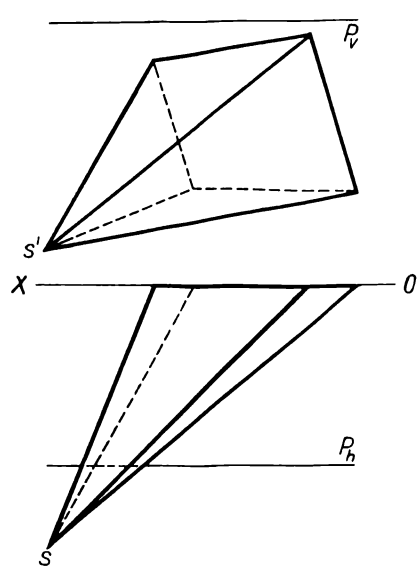


FIG. 882.

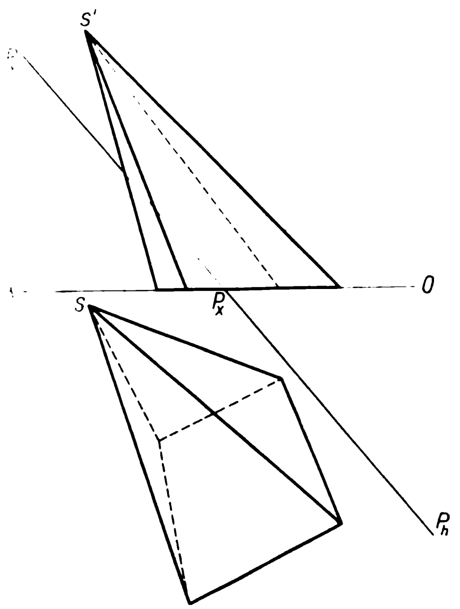



FIG. 883.

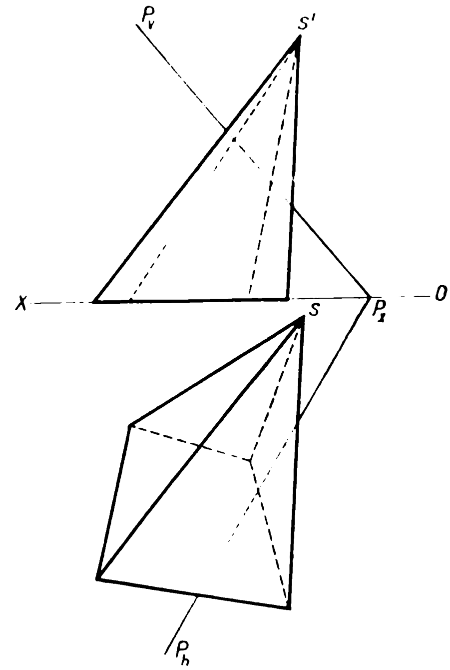



FIG. 884.

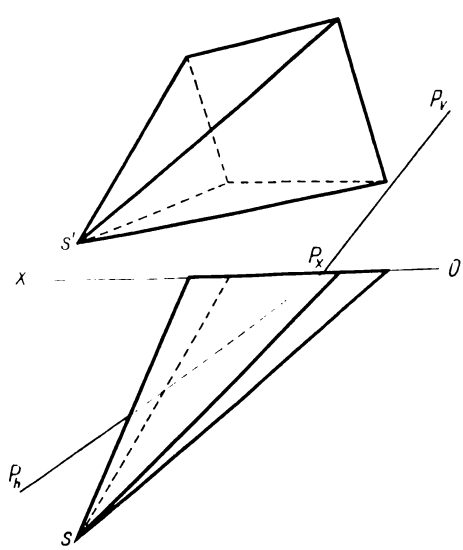



FIG. 885.

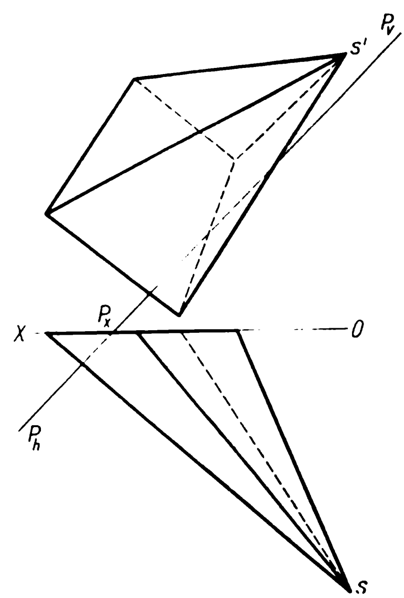



FIG. 886.

with the plane, pass through this edge a plane $R$ parallel to the $H$ plane and intersecting the plane $P$ along a horizontal line. The intersection of the horizontal projections of the edge and of the horizontal line yields the horizontal projection ($k$) of the point. Now find the vertical projection ($k'$) of the point on the vertical projection of the edge. Likewise find the points ($l$, $l'$), ($m$, $m'$), and ($n$, $n'$) of intersection of the other edges with the plane. Join these points *successively* to determine the projections of the line of intersection. The true size $K_0L_0M_0N_0$ is determined by bringing the plane $P$ into coincidence with the $H$ plane.

Prior to developing the truncated prism find the true size of the base $ABCD$ situated parallel to the $W$ plane. For this purpose find its profile projection ($a''b''c''d''$).

Draw an arbitrary line $M_1M_1$ and on it (from the point $A$) lay off line-segments $AB$, $BC$, $CD$, $DA$ equal to the respective sides of the base of the prism, i.e. $AB = a''b''$, $BC = b''c''$, and so on. Through the points, $A$, $B$, $C$, $D$, $A$ draw perpendiculars to the line $M_1M_1$ and on them mark off the lengths of the respective lateral edges, i.e. $AK = a'k'$, $BL = b'l'$, etc. Join the ends of the perpendiculars to obtain a polygonal line $KLMNK$. To complete the development, construct the base $KLMN$, say, at the side $MN$ and the base $ABCD$ at the side $CD$.

**Example 274**

Cut the pyramid in Fig. 834 with the plane $P$ and construct the development of either of its parts.

*Solution.* Find the points of intersection of the edges of the pyramid with the plane $P$. Since the cutting plane is a vertical projecting one, the intersections of the vertical projections ($s'b'$, $s'c'$, $s'd'$) of the edges and the vertical trace ($P_v$) of the plane yield the vertical projections ($k'$, $m'$, $n'$) of the points of intersection; find the horizontal projections ($k$, $m$, $n$) of these points. The edge ($sa$, $s'a'$) is not intersected by the cutting plane. The base of the pyramid intersects the cutting plane along the line ($ef$, $e'f'$). Find the true length of the lateral edges using the method of displacement and determine the true shape of the section figure by the coincidence method.

To obtain the development of the lateral surface of the truncated pyramid, construct the development of the side surface of the given pyramid and transfer to it the obtained points, $E$, $F$, $K$, $M$, $N$. Determine the true length of the line-segments ($sk$, $s'k'$), ($sm$, $s'm'$), ($sn$, $s'n'$) and lay them off on the lines $SA$, $SC$, $SD$. Then mark off $BE = be$ on $AB$, and $DF = df$ on $DA$. Now construct the upper and lower bases of the truncated pyramid at any of its faces. The figure thus obtained is a complete development of the surface of the truncated pyramid.

*Note.* If the cutting plane is oblique, the projections of the section are best found by the method of revolution (displacement) or by replacing the projection planes (see Examples 208 and 248).

PROBLEMS

467. Intersect a polyhedron (prism or pyramid) with the plane $P$ and construct a complete development of one of its parts (Figs. 835—886).

CHAPTER XXIII

# THE RELATIVE POSITIONS OF A PLANE AND A SURFACE

To construct the line of intersection of any surface with a plane, it is necessary to find a number of points belonging both to the surface and to the plane and then join these points in a smooth curve.

To find an arbitrary point on the line of intersection proceed as follows:

(1) construct an auxiliary plane;

(2) find the lines of intersection of this plane with the surface and with the given plane;

(3) the intersection of the obtained lines yields the desired points (usually two).

The required number of points can be found by consecutively constructing a number of auxiliary planes, the latter being chosen so that their lines of intersection with a given surface are projected onto the projection planes as simple lines—straight lines or circles.

If a given surface is generated by a straight line then the desired line of intersection can be determined in the following way: draw a number of generating lines on the given surface and find the points of their intersection with the plane; then join the points obtained. We have a smooth curve.

### Cutting a cylinder

A plane intersects the surface of a right circular cylinder:

(1) *in a circle* if the plane is perpendicular to the axis of the cylinder;

(2) *in an ellipse* if the plane is inclined at an arbitrary angle to the axis of the cylinder;

(3) *along two generating lines* if the plane is parallel to the axis of the cylinder and is separated from it by a distance ($l$) less than the radius $r$ of the cylinder;

(4) *along one generating line* if the plane is parallel to the axis of the cylinder and is separated from it by a distance ($l$) equal to the radius $r$ of the cylinder (in this case the plane is *tangent* to the surface of the cylinder).

*Note.* Any cylindrical surface can be intersected along straight lines by a plane parallel to its generating line.

### Cutting a cone

We will designate the angle of inclination of the generating line of the cone to its base as $\alpha$ and the angle of inclination of the cutting plane to the cone base as $\varphi$.

Any plane *passing through the vertex* of a right circular cone intersects its surface:

(1) *at a point* if $\varphi$ is less than $\alpha$;

(2) *along one generating line* if $\varphi = \alpha$, i.e. when the plane is *tangent* to the surface of the cone;

(3) *along two generating lines* if $\varphi$ is greater than $\alpha$ or $\varphi = 90°$, i.e. when the plane passes through the axis of the cone.

*Note.* Any conical surface can be intersected in straight lines by a plane passing through the vertex.

Any plane *not passing* through the vertex of a right circular cone intersects its surface:

(1) *in a circle* if the plane is perpendicular to the axis of the cone, i.e. $\varphi = 0$;

(2) *in an ellipse* if $\varphi$ is less than $\alpha$;

(3) *along a parabola* if $\varphi = \alpha$, i.e. the cutting plane is parallel to one of the generating lines of a cone;

(4) *along a hyperbola* if $\varphi$ is greater than $\alpha$ or $\varphi = 90°$, i.e. when the plane is parallel to the axis of the cone.

*Note.* When the cutting plane is inclined, the type of the line of intersection is determined by revolving the plane about the axis of the cone so that the plane becomes a vertical projecting plane if the axis of the cone is perpendicular to the horizontal plane of projection, and a horizontal projecting one if the axis of the cone is perpendicular to the vertical plane of projection.

### Cutting a sphere

Any cutting plane intersects a sphere in a *circle* if the distance $l$ from the plane to the centre of the sphere is less than the radius $R$ of the sphere.

In a particular case (when $l = R$) the plane is *tangent* to the surface of the sphere.

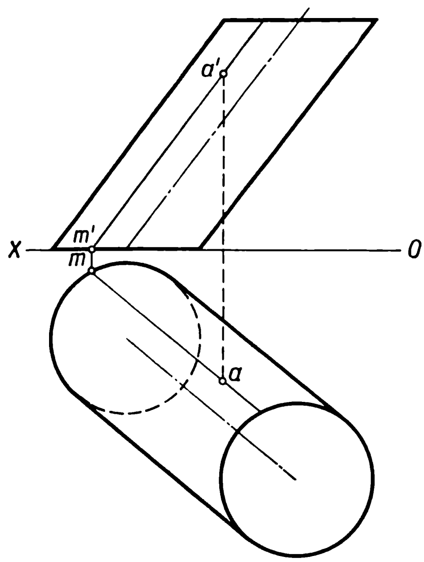



FIG. 887.

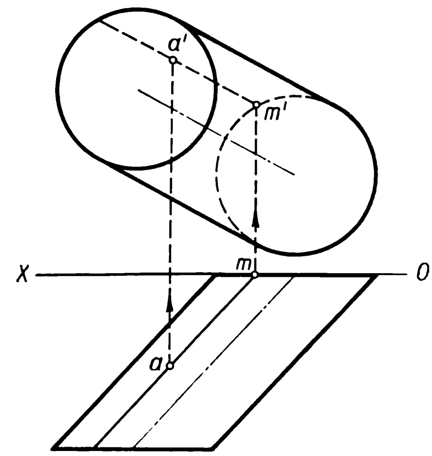



FIG. 888.

*Note.* Any surface of revolution can be intersected in a circle by a plane perpendicular to its axis.

In solving relevant problems we shall have to assume a point on the surface. This is done in the following way: draw an auxiliary line (a straight line or a circle) on the surface and take a point on this line (see the examples below).

EXAMPLES

**Example 275**

Assume an arbitrary point $A$ on the surface of an oblique cylinder (Fig. 887).

*Solution.* Assume an arbitrary point ($m$, $m'$) on the base of the cylinder and draw through it an auxiliary generating line. Take on this line a point ($a$, $a'$), which thus lies on the given surface.

**Example 276**

Find the vertical projection of a point $A$ lying on the surface of an oblique cylinder, given the horizontal projection (Fig. 888).

*Solution.* Through the point $a$ draw the horizontal projection ($am$) of the auxiliary generating line. Find the vertical projection ($m'$) of the point $M$ and draw through it the vertical projection of the generating line; then find $a'$ from the point $a$.

**Example 277**

Assume an arbitrary point $A$ on the surface of a cone.

*Alternative solutions*:

*First method* (Fig. 889). Assume an arbitrary point ($m$, $m'$) on the base of the cone and draw an auxiliary generating line through the points ($m$, $m'$) and ($s$, $s'$). On this line take a point ($a$, $a'$), which thus lies on the given surface.

*Second method* (Fig. 890). Draw an auxiliary circle on the surface of the cone; the vertical projection of the circle is a straight line parallel to the coordinate axis, the horizontal projection being a circle. On this circle take a point ($a$, $a'$), which thus lies on the given surface.

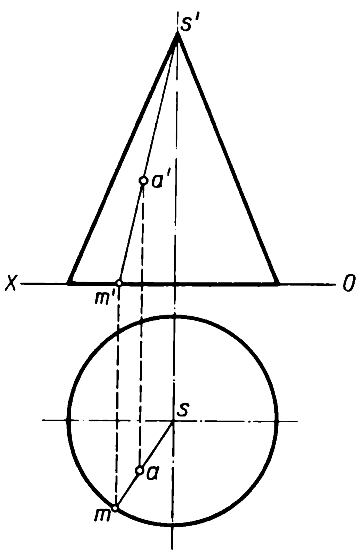



FIG. 889.

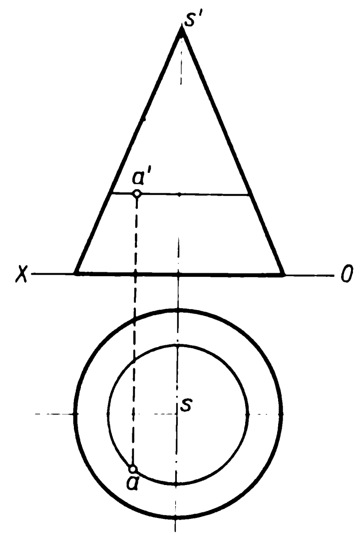



FIG. 890.

**Example 278**

Does the point $A$ lie on the surface of the truncated cone in the indicated drawings?

*Alternative solutions*:

*First method* (Fig. 891). Through the point $a$ draw the horizontal projection of the auxiliary circle situated on the surface of the cone, and then find the vertical projection. As is clear from the drawing, point $a'$ *does not lie* on the vertical projection of the auxiliary circle. Hence, the point $(a, a')$ does not lie on the cone. (We may also begin solving the problem by drawing the vertical projection of the auxiliary circle.)

*Second method* (Fig. 892). Through the point $a'$ draw the vertical projection $(m'n')$ of the auxiliary generating line and find its horizontal projection $(mn)$. As is clear from the drawing, the horizontal projection $(a)$ of the point *does not lie* on the horizontal projection $(mn)$ of the generating line. Hence, the point $(a, a')$ does not lie on the surface of the cone. (Can we begin solving the problem by drawing the horizontal projection of the generating line without completing the construction of the horizontal projection of the vertex of the cone?)

**Example 279**

Find the vertical projection of a point $A$ lying on the surface of a truncated cone, given the horizontal projection.

*Alternative solutions*:

*First method* (Fig. 893). Through the point $a$ draw the horizontal projection $(mn)$ of the auxiliary generating line and find its vertical projection $(m'n')$. Now find point $a'$ from $a$.

*Second method* (Fig. 894). Through the point $a$ draw the horizontal projection of the auxiliary circle located on the surface of the cone, and find its vertical projection. Now find $a'$ from $a$.

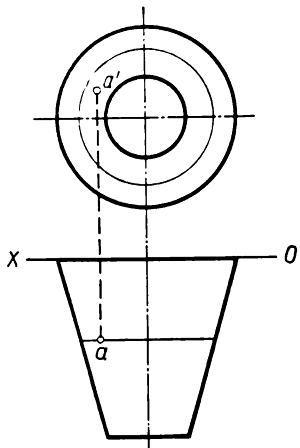



FIG. 891.

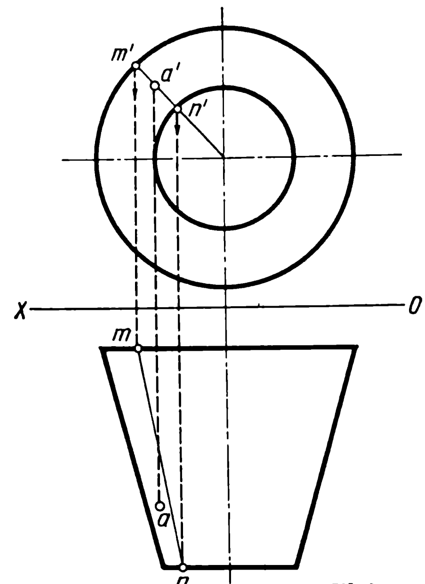



FIG. 892.

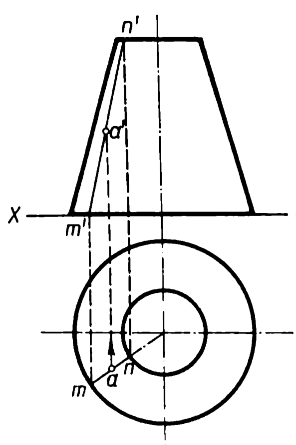



FIG. 893.

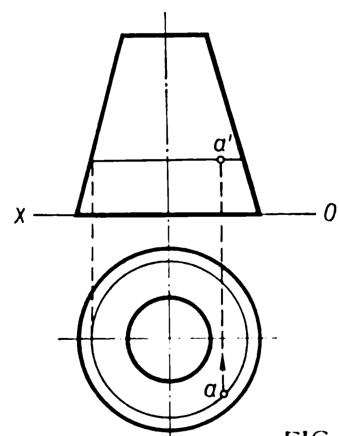



FIG. 894.

(Is it possible to solve the converse problem by the first method, i.e. to find the horizontal projection of the point from the vertical projection without completing the vertical projection of the vertex of the cone?)

**Example 280**

Find the vertical projection of a point $A$ lying on the surface of an oblique cone, given the horizontal projection (Fig. 895).

*Solution.* Through the point $a$ draw the horizontal projection ($sm$) of the auxiliary generating line. Determine the vertical projection ($m'$) of the point $M$ and draw through it the vertical projection ($s'm'$) of the generating line; then from $a$ find the point $a'$ on the line.

**Example 281**

Determine the horizontal projection of a point $A$ lying on the surface of a cone, given the vertical projection.

*Alternative solutions*:

*First method* (Fig. 896). Construct the profile projection of the cone. Through the point $a'$ draw the vertical projection of the auxiliary circle located on the cone, then find the profile projection. From $a'$ find $a''$ and then, having the two projections, determine the third one—point $a$.

*Second method* (Fig. 897). Construct the profile projection of the cone. Through the point $a'$ draw the vertical projection ($s'm'$) of the auxiliary generating line. Find the profile projection ($m''$) of the point $M$ and then, having $m'$ and $m''$, determine the point $m$. Draw the horizontal projection ($sm$) of the generating line and find point $a$ on it.

**Example 282**

Assume an arbitrary point $A$ on the surface of a sphere.

*Alternative solutions*:

*First method* (Fig. 898). Draw on the sphere an auxiliary circle situated parallel to the horizontal plane of projection; its vertical projection is a straight line parallel to the coordinate axis, the horizontal projection being a circle. Take an arbitrary point ($a$, $a'$) on the auxiliary circle; thus the point ($a$, $a'$) lies on the given surface.

*Second method* (Fig. 899). On the sphere draw an auxiliary circle parallel to the vertical plane of projection. Its horizontal projection is a straight line parallel to the coordinate axis, the vertical projection being a circle. Take an arbitrary point ($a$, $a'$) on the auxiliary circle; thus the point ($a$, $a'$) lies on the given surface.

**Example 283**

Find the horizontal projection of the point $A$ lying on the surface of the sphere, given the vertical projection.

*Alternative solutions*:

*First method* (Fig. 900). Through the point $a'$ draw the vertical projection of the auxiliary circle situated on the sphere and parallel to the $H$ plane. Find the horizontal projection and then determine point $a$ from $a'$.

*Second method* (Fig. 901). Through the point $a'$ draw the vertical projection of the auxiliary circle situated on the sphere and parallel to the $V$ plane. Find the horizontal projection and determine point $a$ from $a'$.

**Example 284**

Find the vertical projection of a point $A$ lying on the surface of a spherical segment, given the horizontal projection.

*Alternative solutions*:

*First method* (Fig. 902). Through the point $a$ draw the horizontal projection of an auxiliary circle located on the surface of the spherical segment and parallel to the $H$ plane. Find the vertical projection and then determine point $a'$ from $a$.

*Second method* (Fig. 903). Extend the segment to complete a hemisphere and through the point $a$ draw the horizontal projection of an auxiliary semicircle located on the surface of the hemisphere and parallel to the $V$ plane. Find the vertical projection and determine $a'$ from $a$.

**Example 285**

Assume an arbitrary point $A$ on the surface of a semitoroid (Fig. 904).

*Solution*. On the surface of the semitoroid draw an auxiliary semicircle situated parallel to the $V$ plane; the horizontal projection of the semicircle is a straight line parallel to the coordinate axis, the vertical projection being a semicircle. Take a point ($a$, $a'$) on the auxiliary semicircl e; thus the point ($a$, $a'$) lies on the given surface.

**Example 286**

Find the vertical projection of a point $A$ lying on the surface of a semitoroid, given the horizontal projection (Fig. 905).

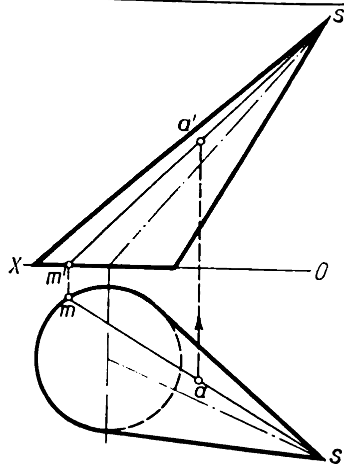



FIG. 895.

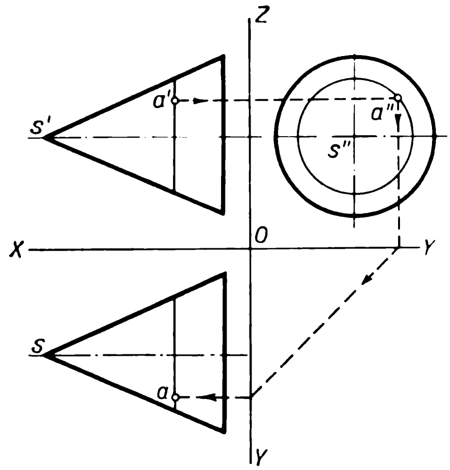



FIG. 896.

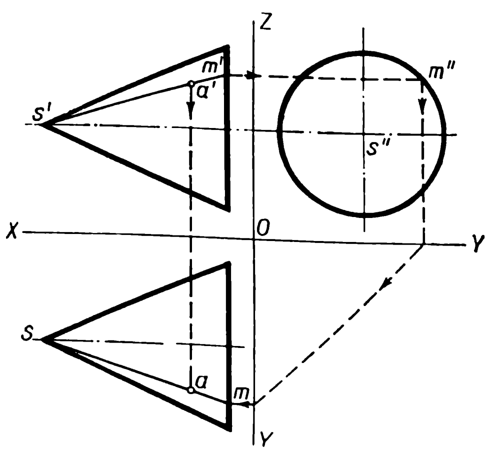



FIG. 897.

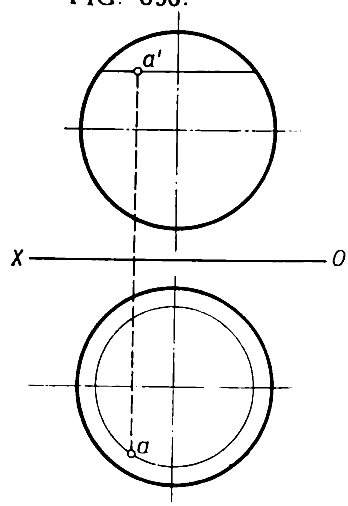



FIG. 898.

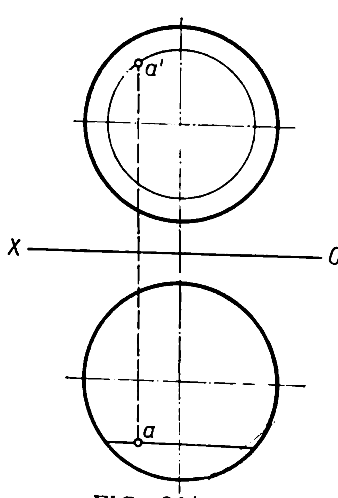



FIG. 899.

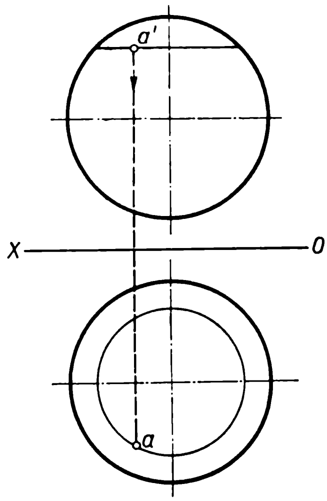



FIG. 900.

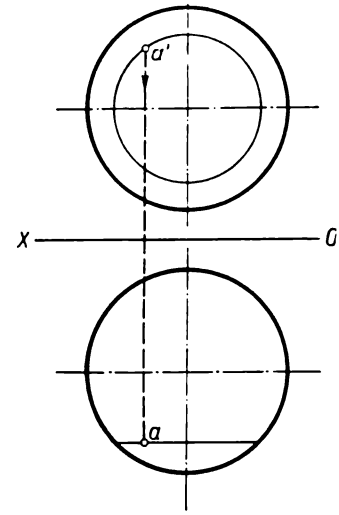



FIG. 901.

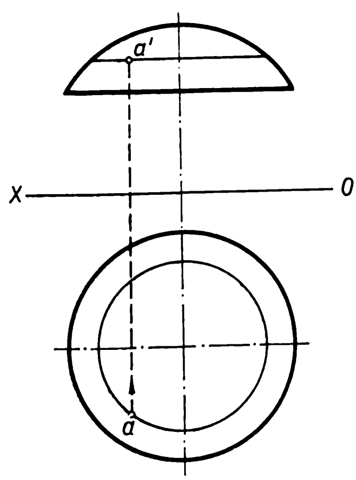



FIG. 902.

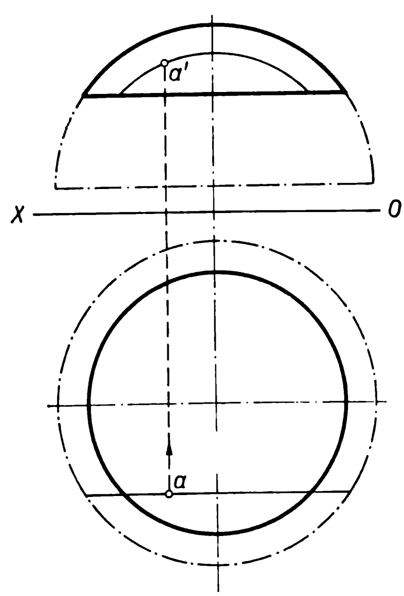



FIG. 903.

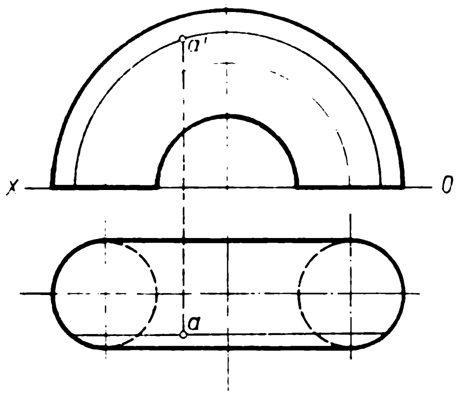



FIG. 904.

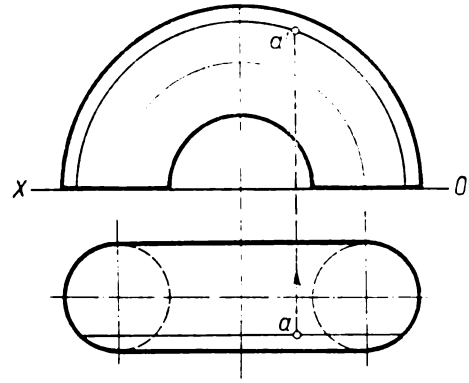



FIG. 905.

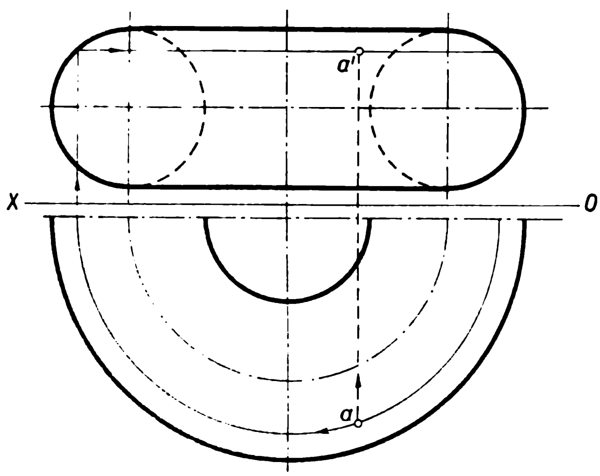



FIG. 906.

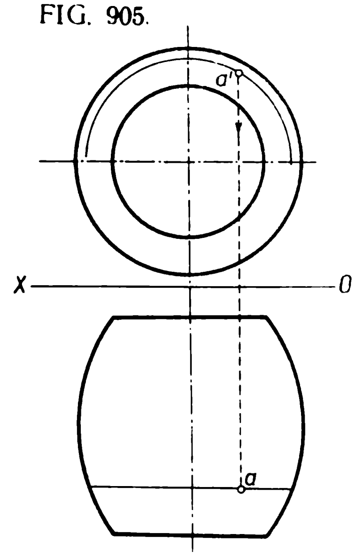



FIG. 907.

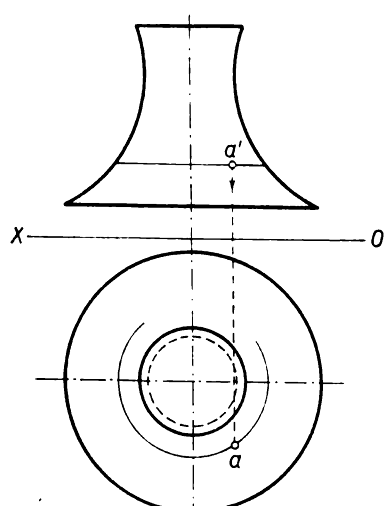



FIG. 908.

*Solution.* Through the point $a$ draw the horizontal projection of an auxiliary semicircle located on the surface of the semitoroid and parallel to the $V$ plane. Find the vertical projection and determine $a'$ from $a$.

**Example 287**

Find the vertical projection of a point $A$ lying on the surface of a toroid, given the horizontal projection (Fig. 906).

*Solution.* Through the point $a$ draw the horizontal projection of an auxiliary circle situated on the surface of the toroid and parallel to the $H$ plane. Find the vertical projection and determine point $a'$ from $a$.

**Example 288**

Find the horizontal projection of a point $A$ lying on the surface of revolution shown in Fig. 907, given the vertical projection.

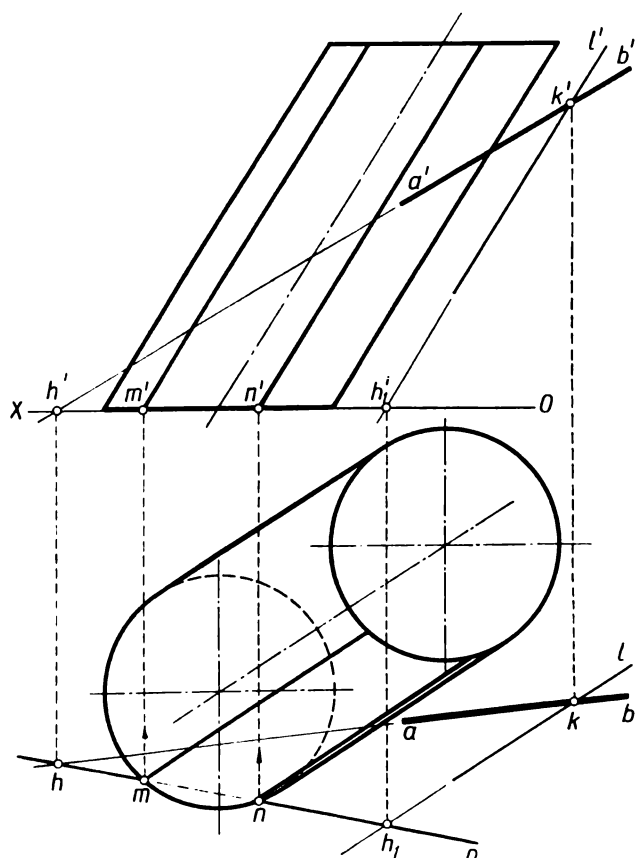


FIG. 909.

*Solution.* Through the point $a'$ draw the vertical projection of an auxiliary circle situated on the surface of revolution. Find the horizontal projection and determine point $a$ from $a'$.

**Example 289**

Determine the horizontal projection of a point $A$ lying on the surface of revolution shown in Fig. 908, given the vertical projection.

*Solution.* Through the point $a'$ draw the vertical projection of an auxiliary circle located on the surface of revolution. Find the horizontal projection and determine $a$ from $a'$.

**Example 290**

Through the line $AB$ pass a plane intersecting the surface of an oblique cylinder along the generating lines, and find the latter (Fig. 909).

*Solution.* Assume on the straight line ($ab$, $a'b'$) an arbitrary point ($k$, $k'$) and through it draw a straight line ($kl$, $k'l'$) parallel to the axis of the cylinder. The lines ($ab$, $a'b'$) and ($kl$, $k'l'$) specify the desired plane. Find the horizontal traces ($h$, $h'$) and ($h_1h_1'$) of these lines and draw, through the points $h$ and $h_1$, the horizontal trace ($P_h$) of the plane to intersect the base of the cylinder at points ($m$, $m'$) and ($n$, $n'$), through which draw the desired generating lines.

*Note.* In a particular case (when?) the horizontal trace ($P_h$) of the plane will be tangent to the base of the cylinder; then the plane is *tangent* to the surface of the cylinder.

**Example 291**

Through a point $A$ pass a plane tangent to the surface of the oblique cylinder shown in Fig. 910.

*Solution.* Through the point ($a$, $a'$) draw a straight line parallel to the axis of the cylinder and find the vertical trace ($v$, $v'$) of the line. Through the point $v'$ draw straight lines $v'k'$ and $v'k_1'$ tangent to the vertical projection of the base of the cylinder at points $m'$ and $m_1'$. Through the points ($m$, $m'$) and ($m_1$, $m_1'$) draw the generating lines ($mn$, $m'n'$) and ($m_1n_1$, $m_1'n_1'$) of the cylinder, which are tangent lines. The lines ($vk$, $v'k'$) and ($vk_1$, $v'k_1'$) together with the corresponding tangent lines specify the desired planes.

These planes can be represented by traces as well (how?).

**Example 292**

Pass an arbitrary plane intersecting the surface of a cone along the generating lines and find the latter (Fig. 911).

*Solution.* The cutting plane $P$ must pass through the vertex ($s$, $s'$) of the cone. Through this point draw an arbitrary horizontal line and determine its vertical trace ($v$, $v'$). Assume an arbitrary point $P_x$ on the coordinate axis and draw the traces $P_h$ and $P_v$ of the plane. The plane $P$ intersects the base of the cone along the chord ($mn$, $m'n'$). Join the vertex ($s$, $s'$) to the end points ($m$, $m'$) and ($n$, $n'$) of the chord to obtain the desired generating lines ($sm$, $s'm'$) and ($sn$, $s'n'$).

*Note.* The point $P_x$ can be assumed so that the horizontal trace ($P_h$) of the plane is tangent to the base of the cone; then the plane is *tangent* to the surface of the cone.

**Example 293**

Given: a cone and the vertical trace of a plane $P$ intersecting the cone along the generating lines. Find these lines (Fig. 912).

*Solution.* Since the plane $P$ must pass through the vertex ($s$, $s'$) of the cone lying in the $H$ plane, draw the horizontal trace ($P_h$) of the plane through the points $P_x$ and $s$. Find the line of intersection of the plane $P$ with the plane of the base of the cone and mark the points ($m$, $m'$) and ($n$, $n'$) of intersection of $P$ with the base circle. Join the vertex ($s$, $s'$) of the cone to the points ($m$, $m'$) and ($n$, $n'$) to obtain the desired generating lines ($sm$, $s'm'$) and ($sn$, $s'n'$).

**Example 294**

Pass a plane tangent to the surface of the cone in Fig. 913, given the vertical projection ($a'$) of the line of tangency.

*Solution.* Through the point $a'$ draw the vertical projection ($s'm'$) of a generating element and determine the horizontal projection ($sm$). Through the point ($m$, $m'$)

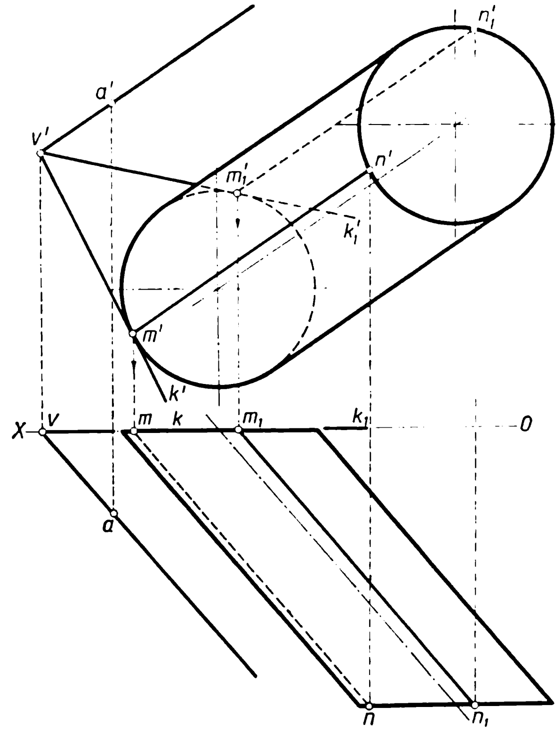



FIG. 910.

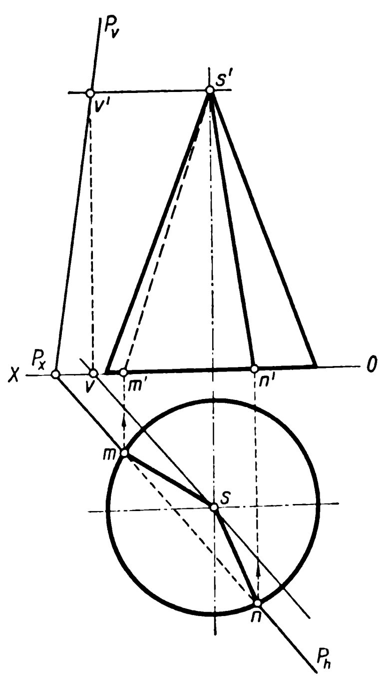



FIG. 911.

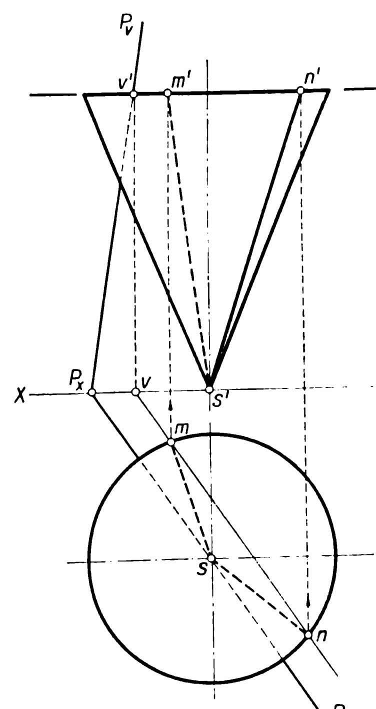



FIG. 912.



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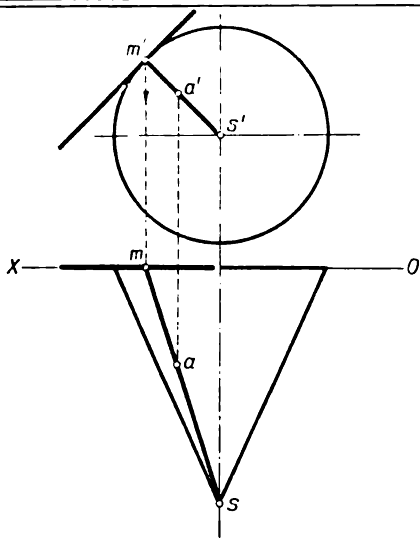



FIG. 913.

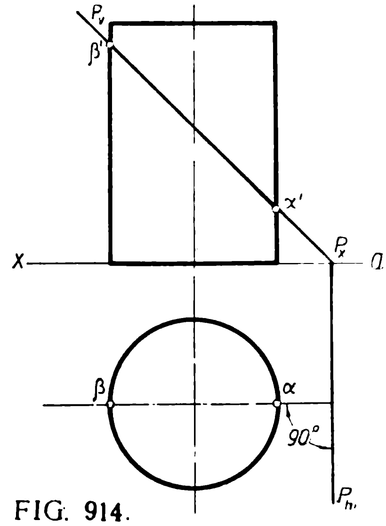



FIG. 914.

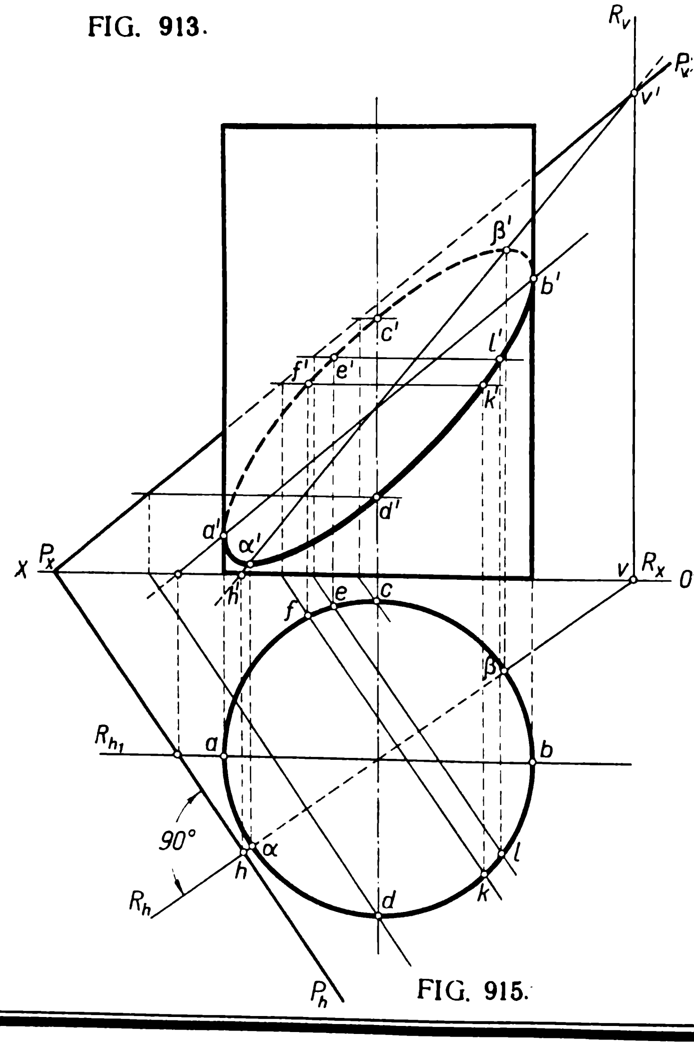



FIG. 915.

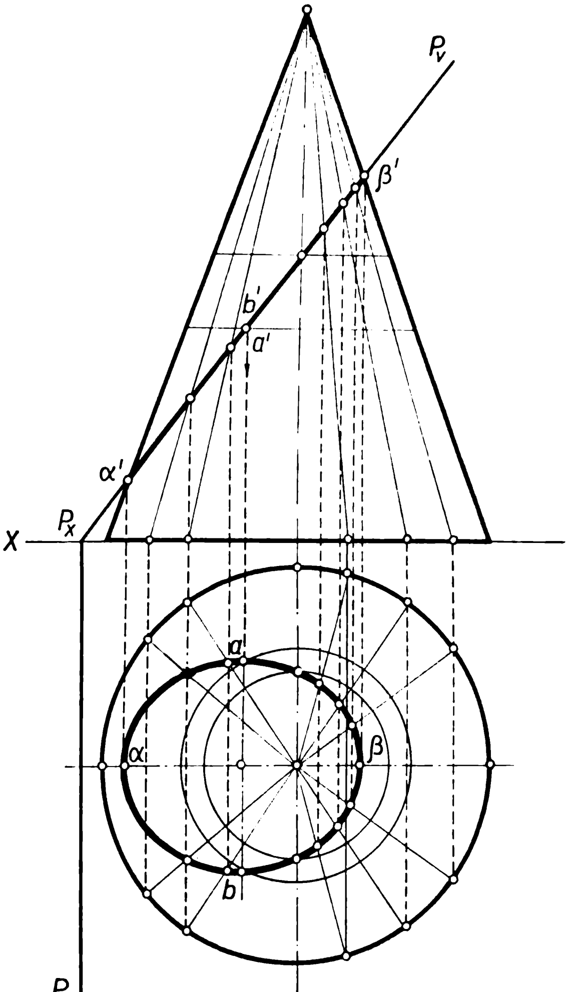


FIG. 916.

draw a frontal line tangent to the base circle. The frontal line and the generating line ($sm$, $s'm'$) specify the desired plane. The plane can be represented by traces as well (how?).

**Example 295**

Construct the projections of the line of intersection of plane $P$ with the surface of the cylinder shown in Fig. 914.

*Solution.* The plane $P$ intersects the cylinder in an ellipse, whose vertical projection coincides with the vertical trace ($P_v$) of the plane and whose horizontal projection coincides with the horizontal projection of the cylinder. The true size of the ellipse can be constructed from its principal axes: the major axis is equal to the line-segment $\alpha'\beta'$ and the minor axis, to the diameter of the cylinder. The point ($\alpha$, $\alpha'$) is the lowest point of the line of intersection, and the point ($\beta$, $\beta'$), the highest one.

**Example 296**

Construct the projections of the line of intersection of the plane $P$ with the surface of the cylinder (Fig. 915).

*Solution*. Through the axis of the cylinder draw a horizontal projecting plane $R$ *perpendicular* to the plane $P$. The plane $R$ intersects the surface of the cylinder along its generating lines, and the plane $P$ intersects the cylinder along the straight line ($hv$, $h'v'$). Their intersection yields the lowest ($\alpha$, $\alpha'$) and the highest ($\beta$, $\beta'$) points of the line of intersection. Through the axis of the cylinder pass a plane $R_1$ parallel to the $V$ plane. The plane $R_1$ intersects the cylinder along the extreme generating lines, and the plane $P$ intersects the cylinder along a frontal line. Their intersection yields the points ($a$, $a'$) and ($b$, $b'$) of the line of intersection.

Find the points of intersection of the profile generating lines of the cylinder with the plane $P$. The horizontal projections ($c$) and ($d$) of these points are known (why?). Using horizontal lines, determine the vertical projections ($c'$) and ($d'$). Likewise find the points of intersection of some more generating lines of the cylinder with the plane. Joining in a smooth curve the vertical projections of all the points obtained, we get the vertical projection of the line of intersection, which is an ellipse. The true size of the ellipse can be constructed from its principal axes: the major axis is equal to the line-segment $\alpha\beta$, $\alpha_1\beta_1$ and the minor axis, to the diameter of the cylinder.

**Example 297**

Construct the projections of the line of intersection of the plane $P$ with the surface of the cone (Fig. 916).

*Solution*. The plane $P$ intersects the cone in an *ellipse* whose vertical projection coincides with the vertical trace ($P_v$) of the plane. The horizontal projection of the

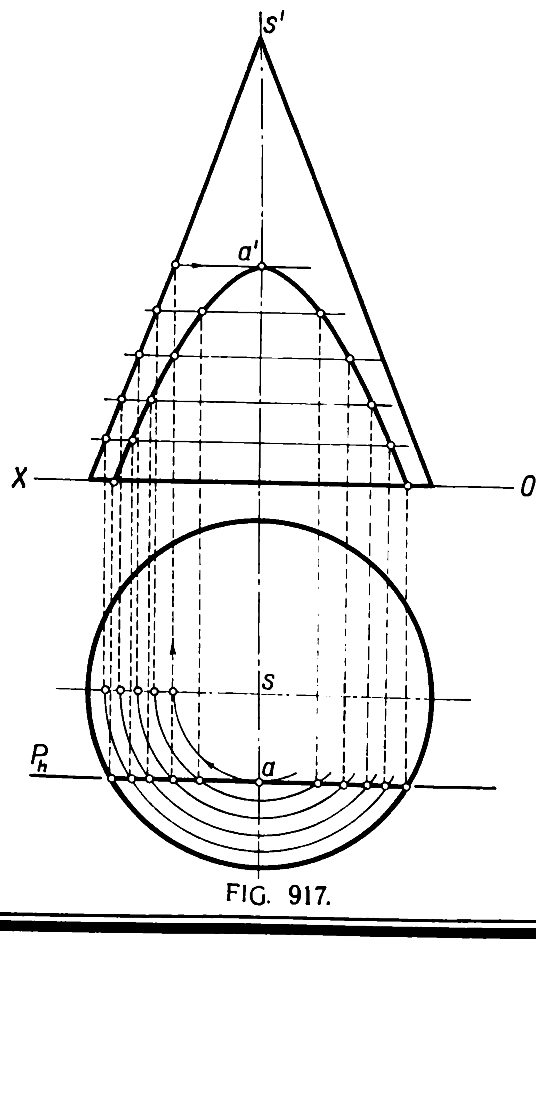



FIG. 917.

ellipse should be constructed point by point: assume the vertical projections of a number of its points and find their horizontal projections (see Examples 279 to 281). Then join the horizontal projections of the points in a smooth curve (ellipse). The horizontal projection of the line of intersection—ellipse—can be constructed also from its principal axes: the major axis $\alpha\beta$ and minor axis $ab$. The true size of the ellipse can be determined from its principal axes: $\alpha'\beta'$ (major) and $ab$ (minor), which is found with the aid of the vertical projection ($a'b'$).

**Example 298**

Construct the projections of the line of intersection of the plane $P$ with the surface of the cone (Fig. 917).

*Solution.* The plane $P$ intersects the cone along a *hyperbola* with the vertex at the point ($a$, $a'$) whose horizontal projection coincides with the horizontal trace ($P_h$) of the plane. The vertical projection of the line of intersection is constructed by points: assume the horizontal projections of a number of arbitrary points on the line of intersection and find their vertical projections (see Examples 279 to 281). Then join the vertical projections of the points in a smooth curve (hyperbola).

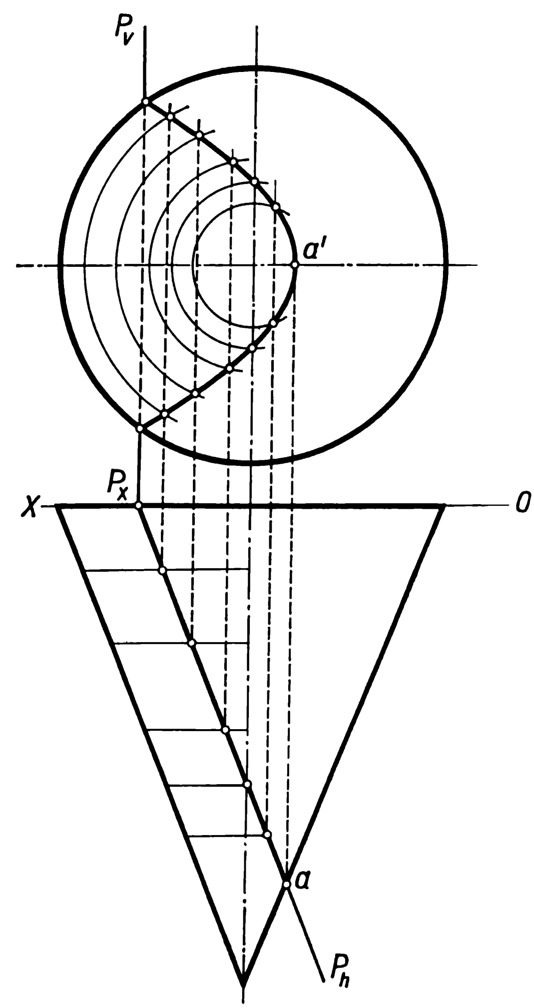



FIG. 918.

**Example 299**

Construct the projections of the line of intersection of the plane $P$ with the surface of a cone (Fig. 918).

*Solution.* The plane $P$ intersects the cone along a *parabola* with the vertex at the point ($a$, $a'$) whose horizontal projection coincides with the horizontal trace ($P_h$) of the plane. The vertical projection of the parabola is constructed by points: assume

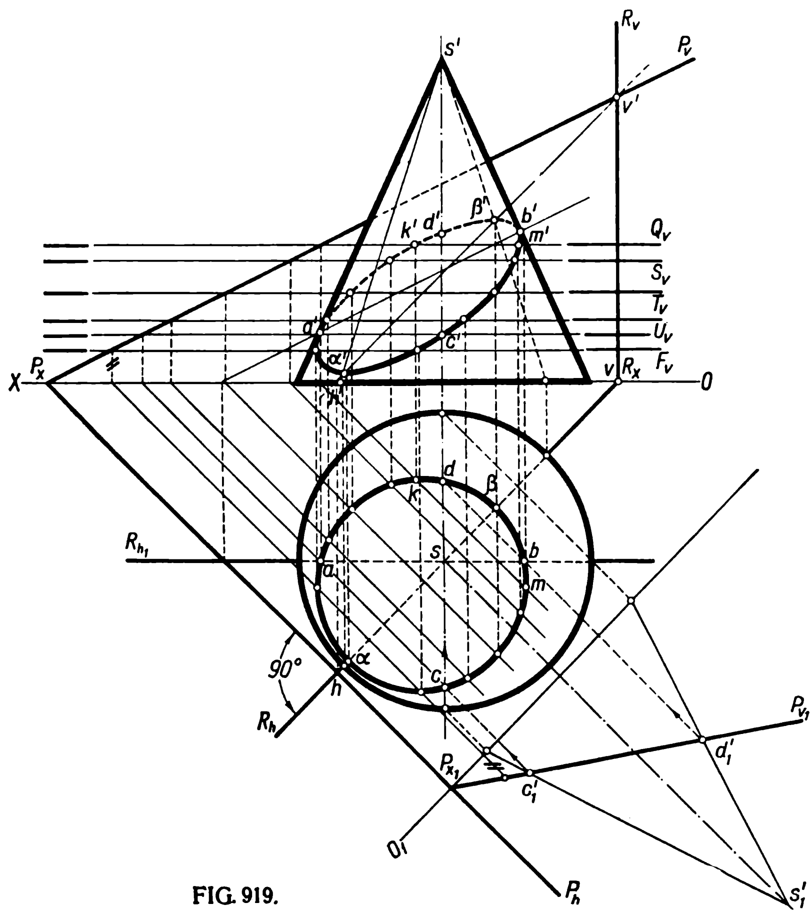


FIG. 919.

the horizontal projections of a number of arbitrary points on the line of intersection and find their vertical projections. Then join the vertical projections of the points in a smooth curve (parabola).

**Example 300**

Construct the projections of the line of intersection of the plane $P$ with the surface of a cone.

*Alternative solutions*:

*First method* (Fig. 919). Through the axis of the cone pass a horizontal projecting plane $R$ *perpendicular* to the plane $P$. The plane $R$ intersects the cone along the generating lines, and the plane $P$ intersects it along the straight line ($hv$, $h'v'$). Their intersection yields the lowest ($\alpha$, $\alpha'$) and the highest ($\beta$, $\beta'$) points of the desired line. Through the axis of the cone pass a plane $R_1$ parallel to the $V$ plane. The plane $R_1$ intersects the cone along the extreme generating lines, and the plane $P$ intersects it along a frontal line. Their intersection yields two more points $(a, a')$ and $(b, b')$ of the line of intersection. To find the points belonging to the line of intersection and lying on the profile generating lines, replace the $V$ plane with $V_1$ so that the plane $P$ becomes a vertical projecting one, and determine the points $(c, c_1')$ and $(d, d_1')$ of intersection of these generating lines with the plane $P_1$. Then find the points $(c, c')$ and $(d, d')$. To find intermediate points of the line of intersection proceed in the following way: introduce between the points $(\alpha, \alpha')$ and $(\beta, \beta')$ an auxiliary plane $Q$

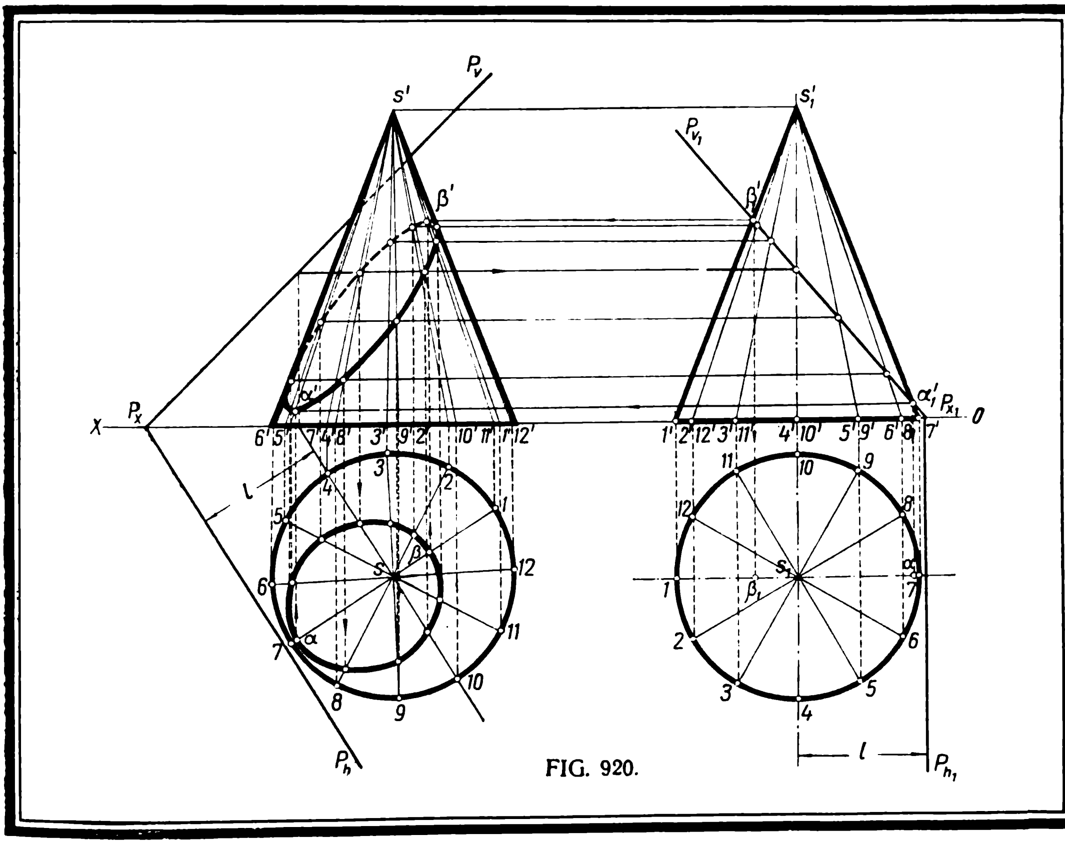



FIG. 920.

parallel to the $H$ plane. The plane $Q$ intersects the cone in a circle, whereas the plane $P$ intersects it along a horizontal line; their intersection yields two points $(k, k')$ and $(m, m')$. In similar fashion find some more points and then join the like projections of the obtained points in smooth curves (ellipses).

*Second method* (Fig. 920). Displace the given system parallel to the $H$ plane so that the plane $P$ becomes a vertical projecting plane. Find the projections of the line of intersection and then, by reverse displacement, determine the projections of the points of intersection in the initial system (for details see the accompanying drawing).

*Third method* (Fig. 921). The points $(a, a')$ and $(b, b')$ on the extreme generating lines and the points $(c, c')$ and $(d, d')$ on the profile generating lines of the cone are found as in the first method. Intermediate points are determined by using auxiliary planes passing through the vertex $(s, s')$ of the cone whose horizontal traces are parallel to the trace $P_h$. Each auxiliary plane intersects the cone along two generating lines, and the plane $P$ intersects it along a horizontal line. The intersection of each pair of lines obtained yields two points. To find points $(a, a')$ and $(\beta, \beta')$, pass auxiliary planes *tangent* to the cone. The intersection of the lines of tangency (which are

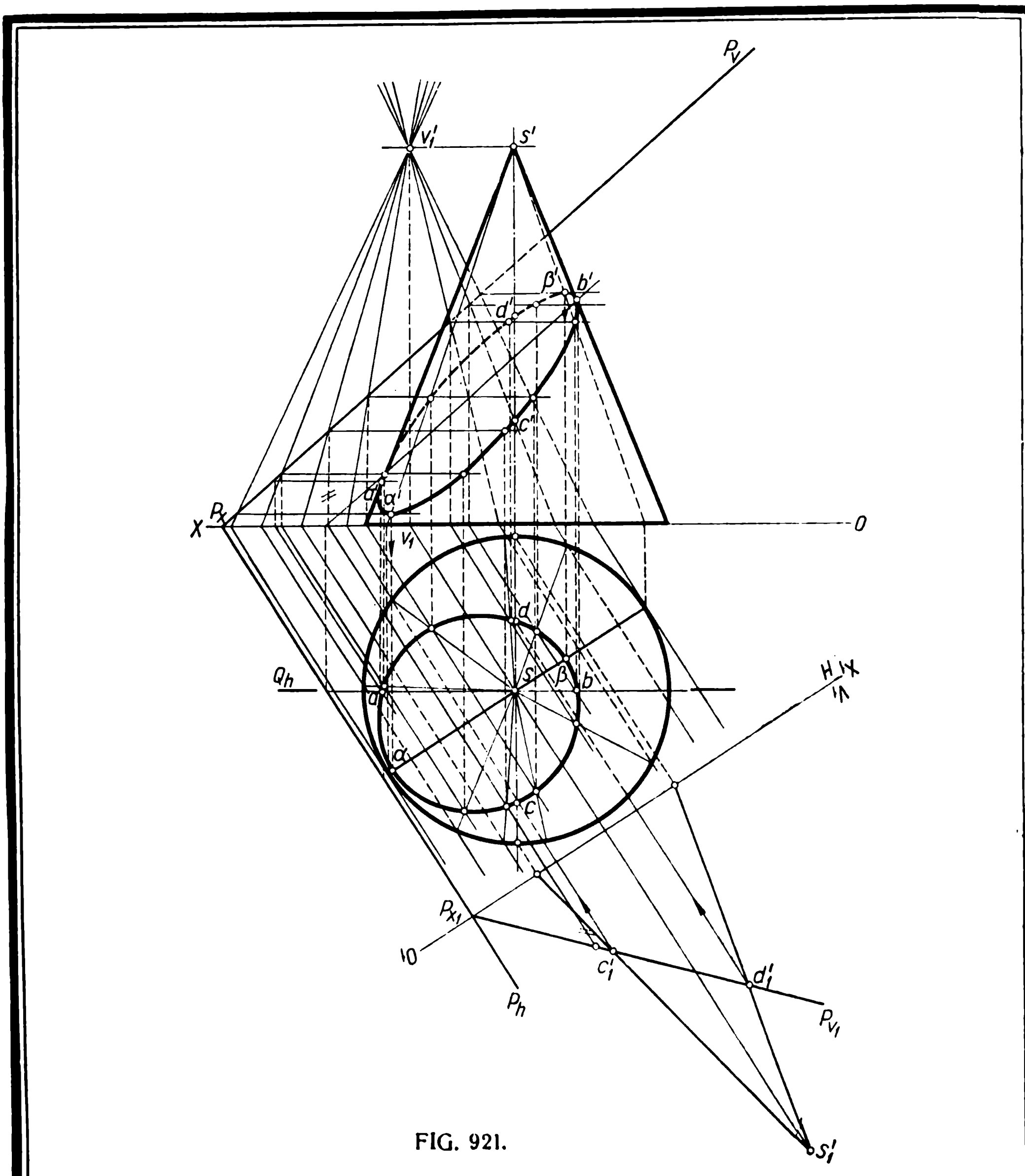


FIG. 921.

at the same time the generating lines of the cone) with the respective horizontal lines yields the required points. Then join in smooth curves (ellipses) the like projections of all the points obtained.

**Example 301**

Construct the projections of the line of intersection of the plane $P$ with a sphere (Fig. 922).

*Solution.* The plane $P$ intersects the sphere in *a circle*, whose horizontal projection ($cd$) coincides with the horizontal trace ($P_h$) of the plane. The vertical projection of the circle—*ellipse*—is constructed from its principal axes: the major axis is the vertical projection ($a'b'$) of the diameter perpendicular to the $H$ plane, while the minor axis is the vertical projection ($c'd'$) of the diameter parallel to the $H$ plane.

The vertical projection of the circle can be constructed point by point as well. Assume the horizontal projections of a number of points of the circle and find their vertical projections (see Examples 283, 284). Then join these points in an ellipse.

To determine the visible and invisible parts on the vertical projection of the curve, find the vertical projections ($m'$) and ($n'$) of its points located on the principal meridian.

**Example 302**

Construct the projections of the line of intersection of the plane $P$ with a sphere (Fig. 923).

*Solution.* The plane $P$ intersects the sphere in a circle, whose vertical projection ($c'$, $d'$) coincides with the vertical trace ($P_v$) of the plane. The horizontal projection

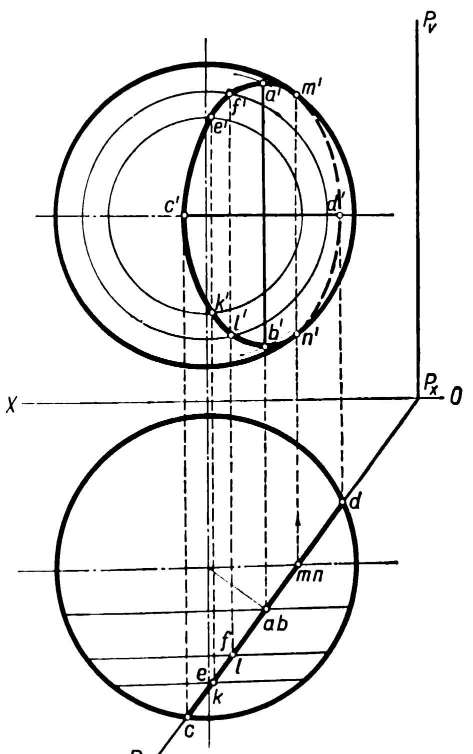



FIG. 922.

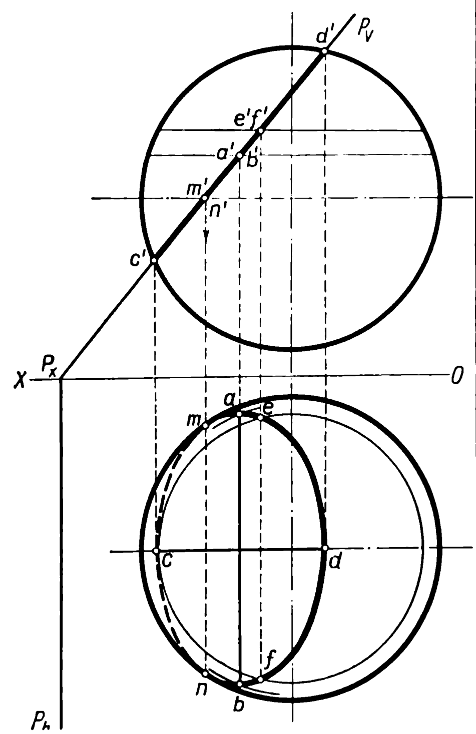



FIG. 923.

of the sphere (ellipse) is constructed from the principal axes: the major axis is the horizontal projection ($ab$) of the diameter perpendicular to the $V$ plane, the minor axis being the horizontal projection ($cd$) of the diameter parallel to the $V$ plane. The horizontal projection of the circle can be constructed by points as well: assume the vertical projections of a number of points on the circle and find their horizontal projections (see Examples 283 and 284). Then join these points in a smooth curve (ellipse).

To determine the visible and invisible parts on the horizontal projection of the curve, find the horizontal projections ($m$) and ($n$) of its points lying on the equator.

**Example 303**

Construct the projections of the line of intersection of the plane $P$ with a sphere.

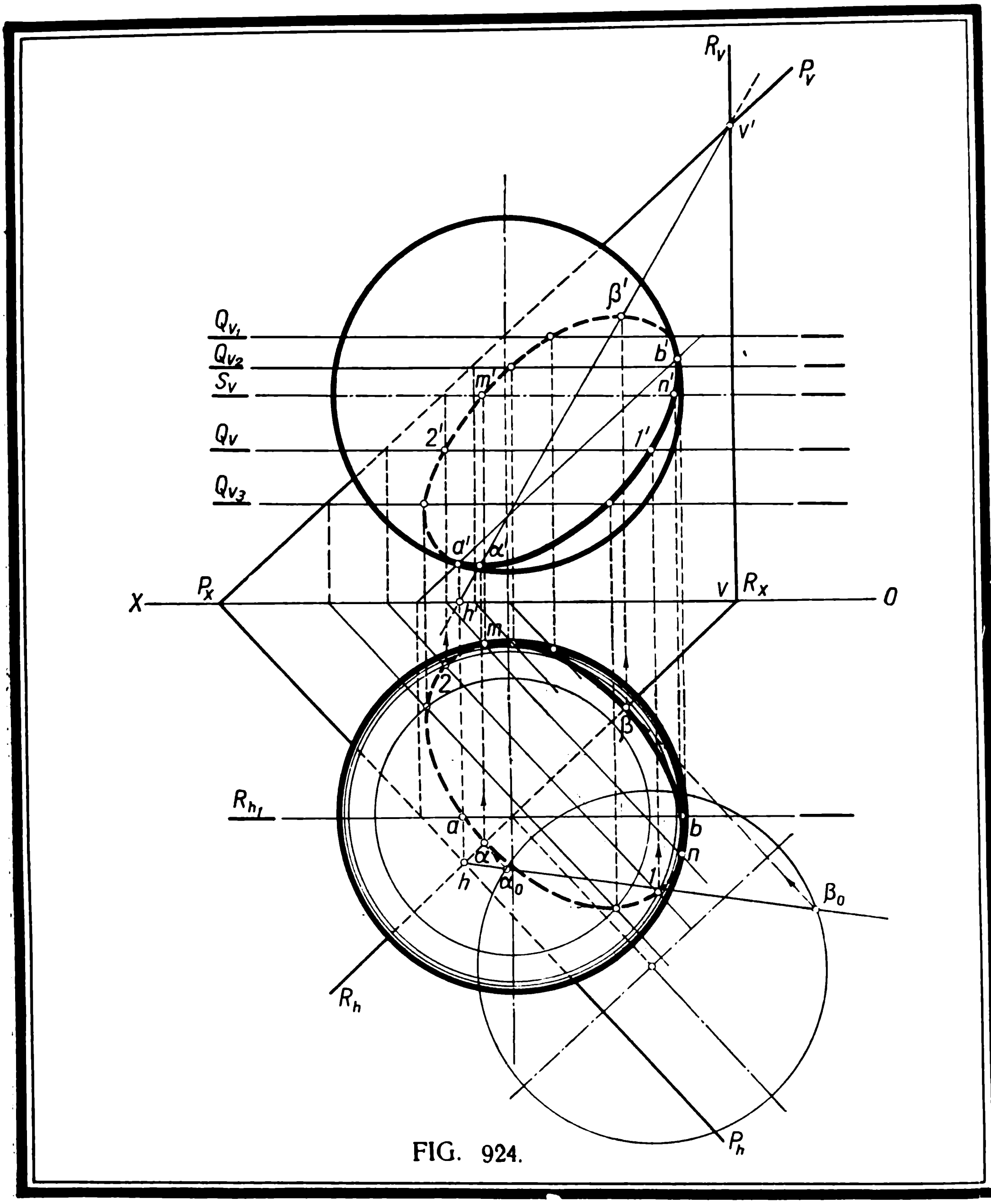



FIG. 924.

*Alternative solutions*:

*First method* (Fig. 924). Through the centre of the sphere pass a horizontal projecting plane $R$ *perpendicular* to the plane $P$. The plane $R$ intersects the sphere in a circle, whereas the plane $P$ intersects it along a straight line ($hv$, $h'v'$). Their intersection yields the lowest ($a$, $a'$) and the highest ($\beta$, $\beta'$) points of the desired line. To avoid constructing an ellipse in the $V$ plane, bring the plane $R$ to coincidence with the $H$ plane and first find these points in the coincident position (see the drawing).

To determine intermediate points of the line of intersection, pass a number of auxiliary planes ($Q$, $Q_1$, etc.), parallel to the $H$ plane, between the points ($\alpha$, $\alpha'$) and ($\beta$, $\beta'$). For instance, the plane $Q$ intersects the sphere in a circle, and the plane $P$ intersects it along a horizontal line. Their intersection yields two points (*1*, *1'*) and

(2, 2′). To determine the visible and invisible parts on the vertical projection of the curve, pass a plane $R_1$ through the centre of the sphere and parallel to the $V$ plane. The plane $R_1$ intersects the sphere along the ***principal meridian***, and the plane $P$ intersects it along a frontal line. Their intersection yields the points ($a$, $a'$) and ($b$, $b'$). To determine the visible and invisible parts on the horizontal projection of the curve pass a plane $S$ through the centre of the sphere and parallel to the $H$ plane to intersect the sphere along the *equator*; the plane $P$ intersects it along a horizontal line. Their intersection yields points ($m$, $m'$) and ($n$, $n'$). Then join the like projections of all the points obtained in smooth curves (ellipses).

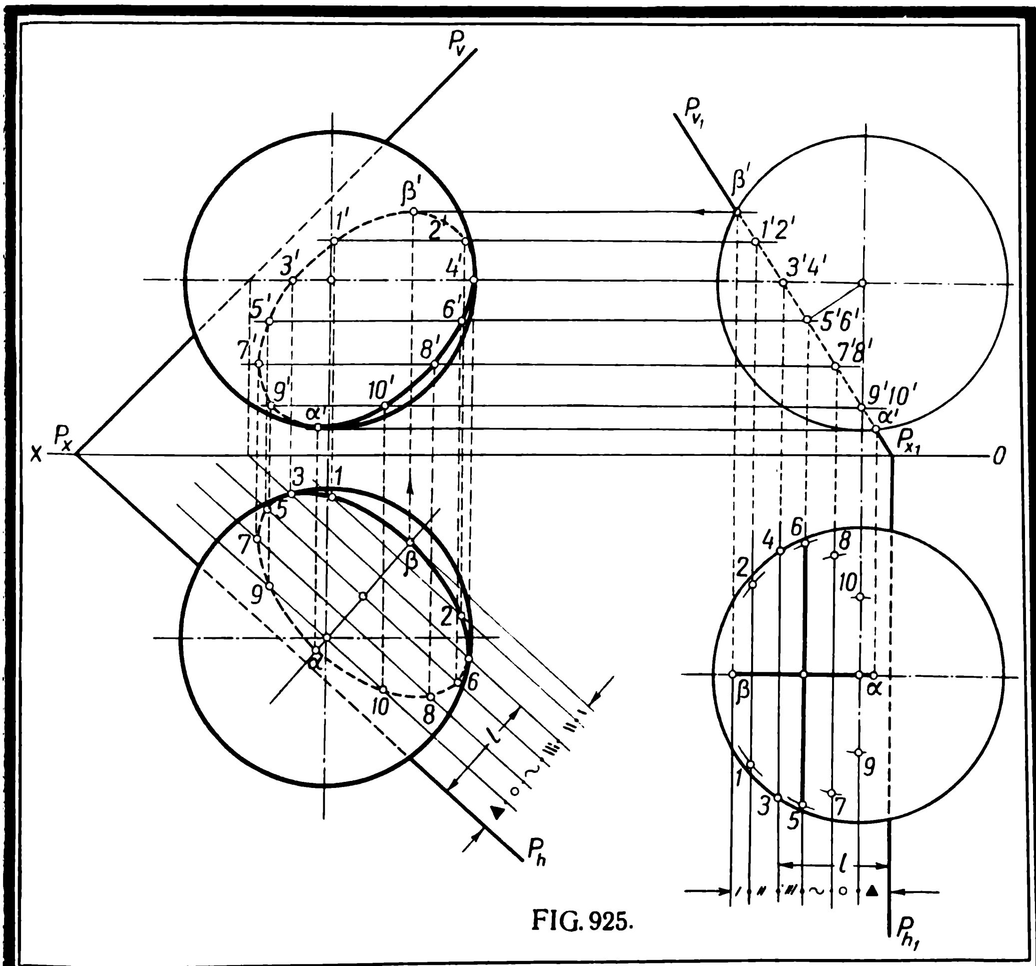



FIG. 925.

*Second method* (Fig. 925). Displace the given system parallel to the $H$ plane so that the plane $P$ becomes a vertical projecting plane. Then find the projections of the line of intersection and by reverse displacement find its projections (ellipses) in the initial system (the construction is clear from the drawing).

*Third method* (Fig. 926). Drop a perpendicular from the centre of the sphere to the plane $P$ and find its foot ($c$, $c'$), which is the centre of a circle. Using auxiliary constructions (see the drawing), determine the radius $r$ of the circle from the true length of the distance $l$ from the centre of the sphere to the plane $P$, and from the radius $R$ of the sphere. Bring the plane $P$ to coincidence with the $H$ plane and determine the position ($C_0$) of the centre of the circle. Construct the coincident position of the circle and then find its projections (see Example 229).

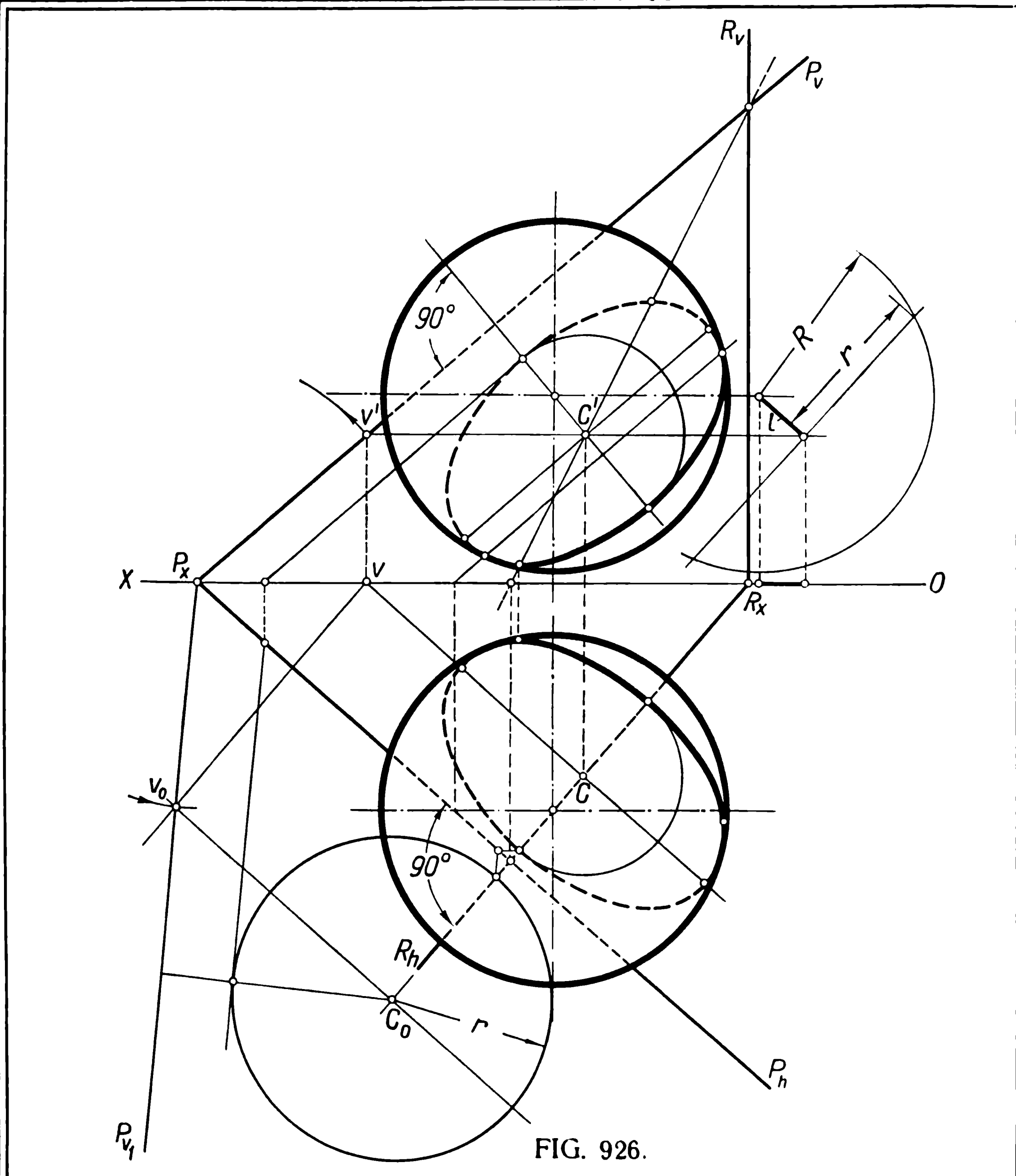


FIG. 926.

**Example 304**

On the plane $P$ draw a straight line making an angle $\varphi$ with the coordinate axis (Fig. 927).

*Solution.* The locus of straight lines in space forming the given angle with the coordinate axis is the surface of a right circular cone of arbitrary altitude and with an angle $2\varphi$ at the vertex (point $P_x$). The desired straight lines are the generating lines of the cone along which the plane $P$ intersects this surface (the construction is obvious from the drawing).

**Example 305**

Given: a point $S$ and a plane $P$. Through the point $S$ draw a straight line parallel to the plane $P$ and making an angle $\varphi$ with the $V$ plane (Fig. 928).

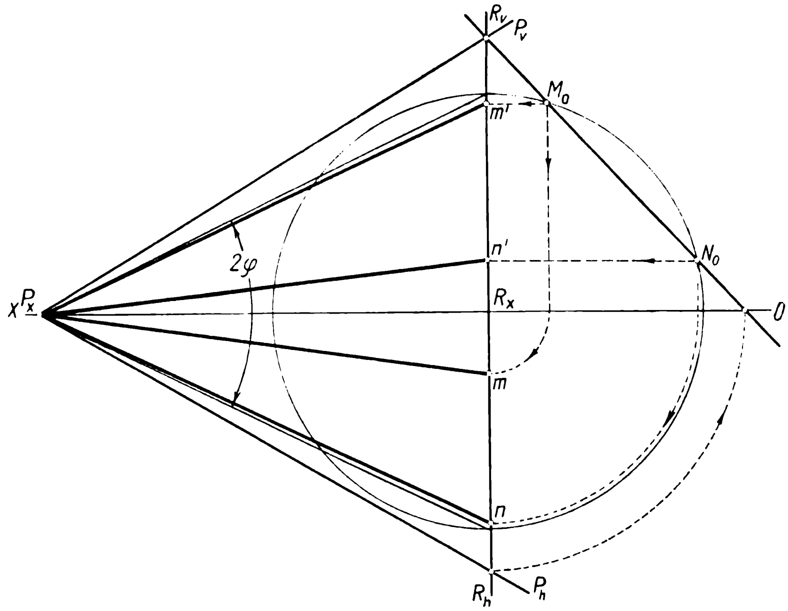



FIG. 927.

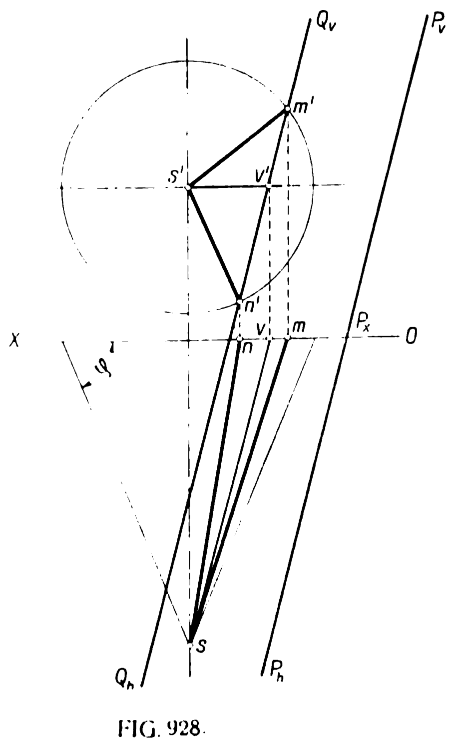



FIG. 928.

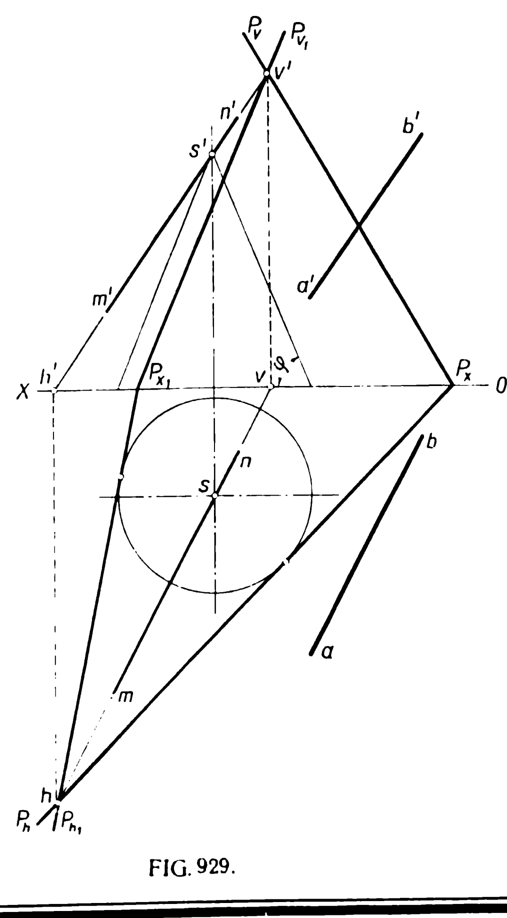



FIG. 929.

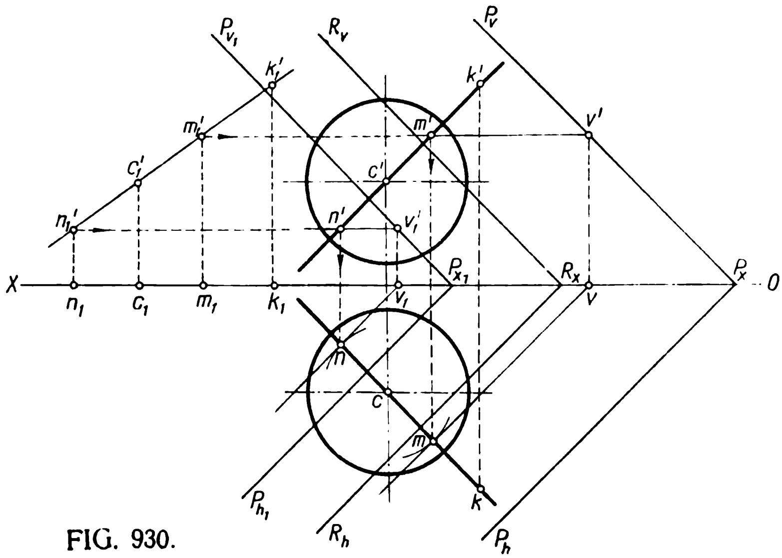



FIG. 930.

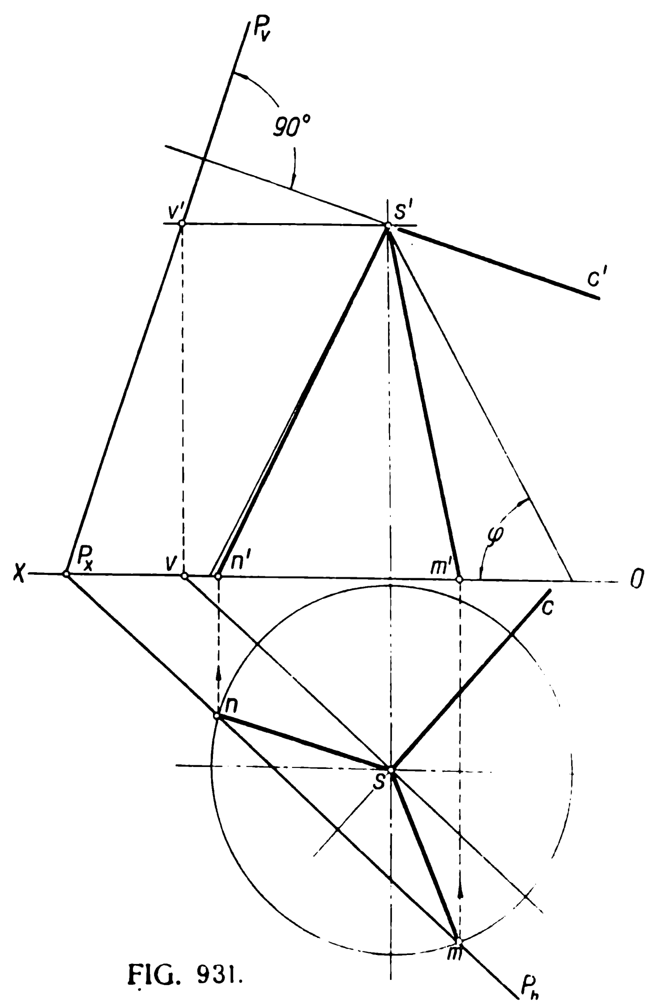



FIG. 931.

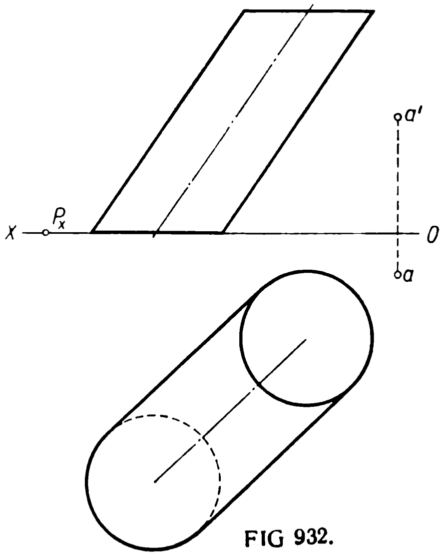



FIG 932.

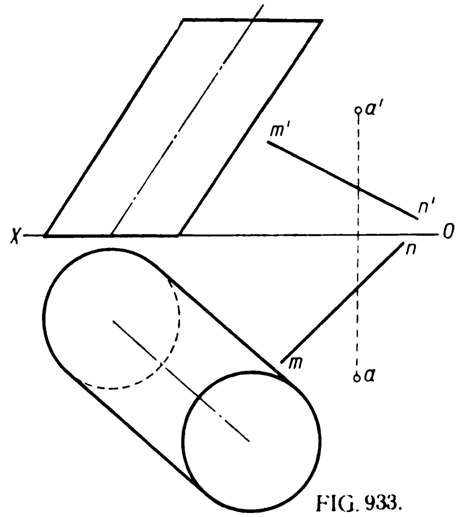



FIG. 933.

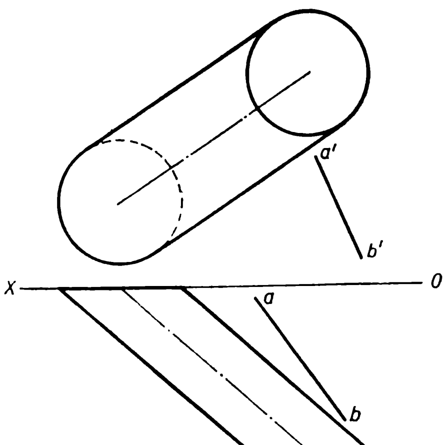



FIG. 934.

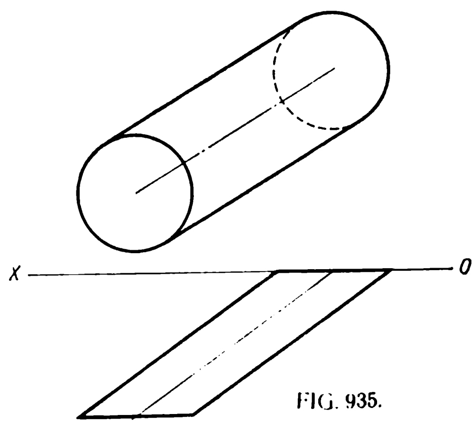



FIG. 935.

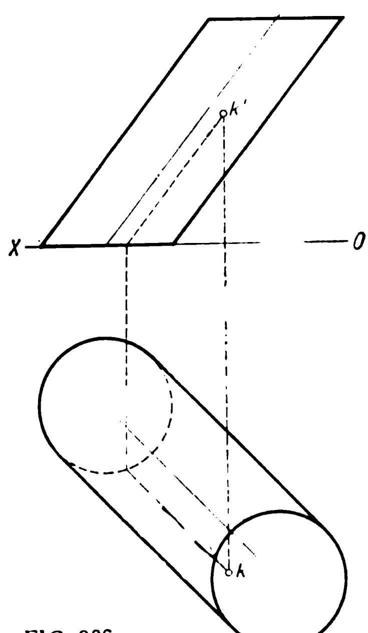



FIG. 936.

*Solution.* The locus of straight lines in space passing through the point $S$ and inclined to the $V$ plane at an angle $\varphi$ is the surface of a right circular cone with vertex at the point $S$ whose generating lines are inclined at an angle $\varphi$ to the $V$ plane. The desired straight lines are those generating lines of the cone along which the plane $Q$ passing through the point $S$ parallel to the plane $P$ intersects the surface of the cone (the construction is obvious from the drawing).

**Example 306**

Given: a point $S$ and a line $AB$. Through the point $S$ pass a plane $P$ inclined at a given angle $\varphi$ to the $H$ plane and parallel to the straight line $AB$ (Fig. 929).

*Solution.* Any plane that is *tangent* to the surface of a right circular cone with a vertex at the point $S$ whose generating lines make the same angle $\varphi$ with the $H$ plane satisfies the stipulated conditions. Of these planes, the desired plane is the one containing a straight line parallel to $AB$. Hence, through the point $S$ draw a straight line $MN$ parallel to $AB$, and through this line pass planes *tangent* to the surface of the cone. The problem has two alternative solutions (the construction is obvious from the drawing).

**Example 307**

Pass a plane $P$ tangent to a sphere and parallel to the plane $R$ (Fig. 930).

*Solution.* Through the centre of the sphere draw a straight line *perpendicular* to the plane $R$ and find the points $(m, m')$ and $(n, n')$ at which it pierces the sphere. The simplest way to do this is to lay off, on the drawn line, segments $(cm, c'm')$ and $(cn, c'n')$ equal to the radius of the sphere. Through the points $(m, m')$ and $(n, n')$ thus obtained pass planes $P$ and $P_1$ *parallel* to the plane $R$, or, what is the same thing, planes perpendicular to the straight line $(mn, m'n')$ (the construction is obvious from the drawing).

**Example 308**

Through point $S$ draw a straight line at an angle $\varphi$ to the $H$ plane and perpendicular to the line $CS$ (Fig. 931).

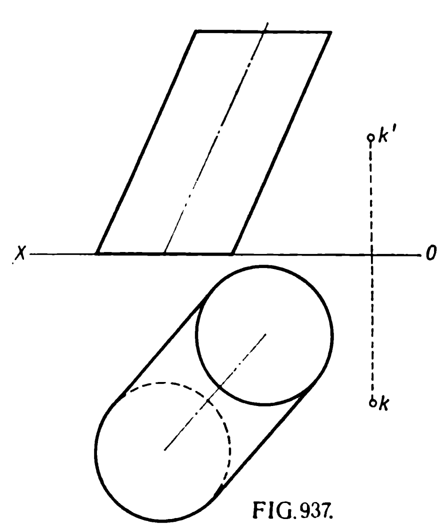


FIG. 937.

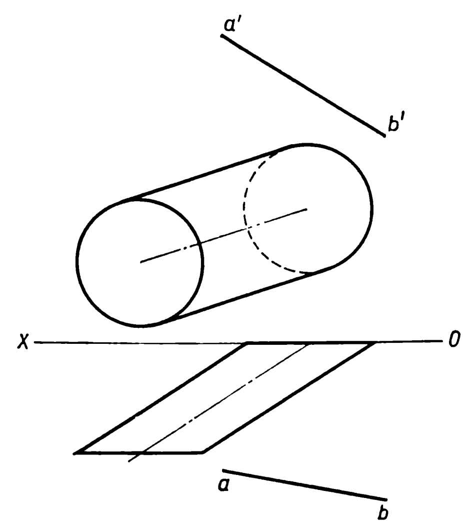


FIG. 938.

*Solution.* The locus of straight lines passing through the point $S$ and making an angle $\varphi$ with the $H$ plane is the surface of a right circular cone with vertex at $S$ whose generating lines are inclined at an angle $\varphi$ to the $H$ plane. On the other hand, the locus of straight lines in space passing through the point $S$ and perpendicular to the straight line $CS$ is a plane perpendicular to $CS$. Hence, the desired straight lines are obtained as a result of intersection of the cone with a plane perpendicular to the line $CS$ (the construction is obvious from the drawing).

## PROBLEMS

468. Through the point $A$ pass a plane $P$ intersecting the surface of an inclined cylinder along the generating lines and find these lines (Fig. 932).

469. Through the point $A$ pass a plane $P$ parallel to the straight line $MN$ and intersecting a cylinder along the generating lines; find these lines (Fig. 933).

470. Through the straight line $AB$ pass a plane intersecting an inclined cylinder along the generating lines and find these lines (Fig. 934).

471. Pass an arbitrary plane $P$ intersecting an inclined cylinder along the generating lines and find these lines (Fig. 935).

472. Construct the traces of a plane tangent to the surface of a cylinder and passing through the point $K$ lying on its surface (Fig. 936).

473. Construct the traces of a plane tangent to the surface of a cylinder and passing through the point $K$ (Fig. 937).

474. Construct the traces of a plane tangent to the surface of an inclined cylinder and parallel to the line $AB$ (Fig. 938).

475. Through the point $A$ pass a plane $P$ intersecting the surface of a cone along the generating lines and find these lines (Fig. 939).

476. Pass a plane $P$ intersecting the surface of a cone along the generating lines and parallel to the line $AB$, and find these lines (Fig. 940).

477. Pass a plane $P$ intersecting the surface of a cone along the generating lines and parallel to the plane $Q$; find the generating lines (Fig. 941).

478. Construct the traces of a plane tangent to the surface of a cone and passing through the point $K$ lying on its surface (Fig. 942).

479. Through the point $K$ pass a plane tangent to the surface of a cone (Fig. 943).

480. Pass a plane parallel to the line $AB$ and tangent to the surface of a cone (Fig. 944).

481. Construct the projections of the line of intersection of a right circular cylinder with the plane $P$ (Figs. 945 to 963).

482. Construct the projections of the line of intersection of a right circular cylinder with the plane $P$ (Figs. 964 to 982).

483. Construct the projections of the line of intersection of a sphere with the plane $P$ (Fig. 983).

484. Construct the traces of a plane tangent to a sphere and passing through the line $AB$ (Fig. 984).

485. Construct the traces of a plane tangent to a sphere, given the vertical projection ($k'$) of the point of tangency (Fig. 985).

486. Pass a plane $P$ parallel to the plane $Q$ and tangent to a sphere (Fig. 986).

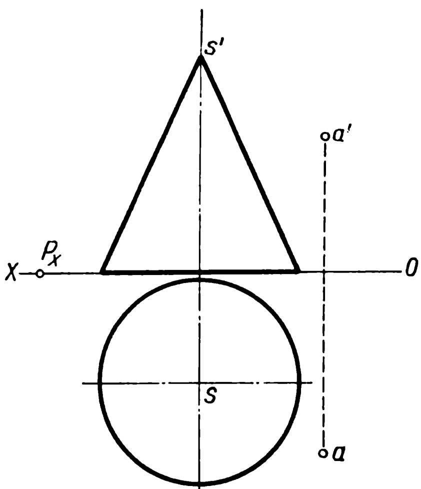



FIG. 939.

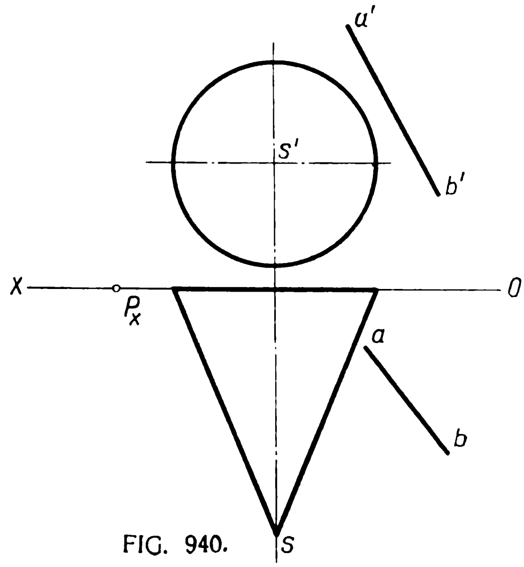



FIG. 940.

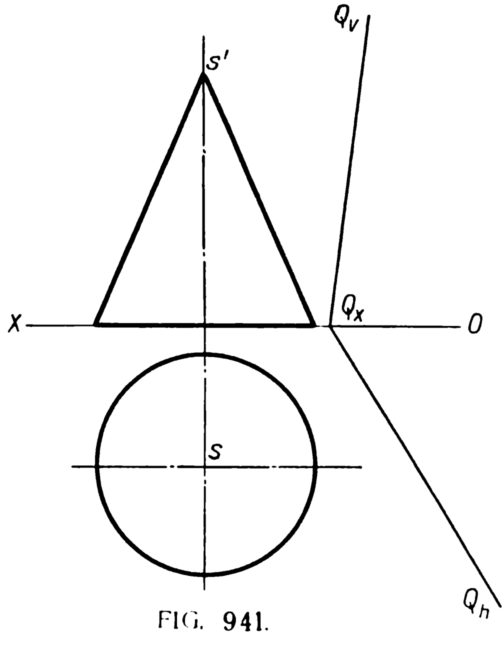



FIG. 941.

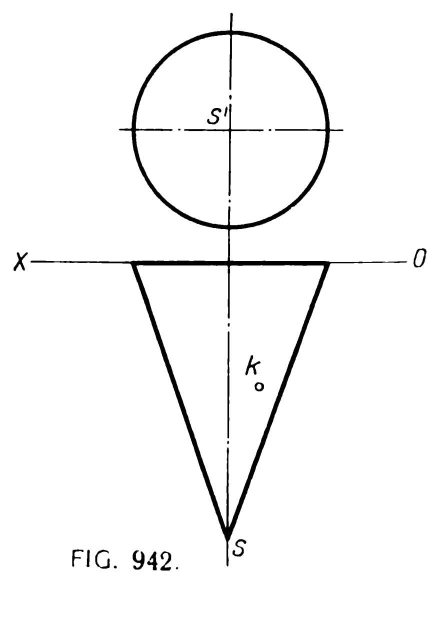



FIG. 942.

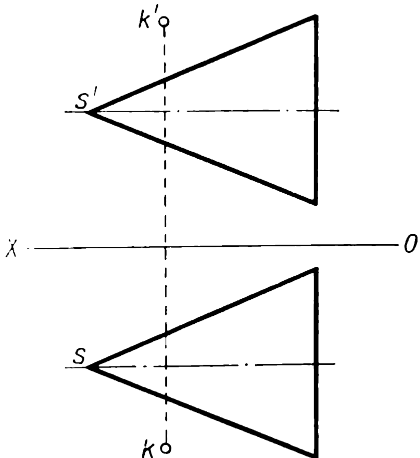


FIG. 943.

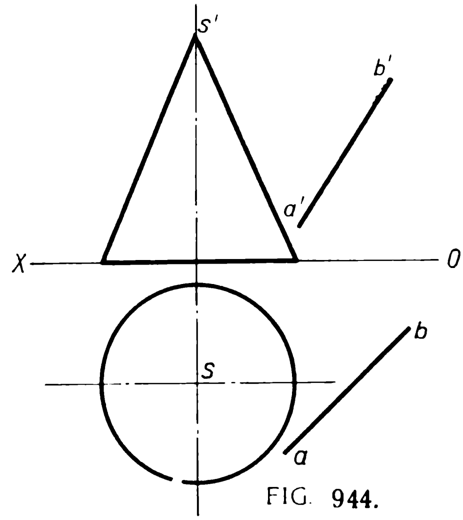


FIG. 944.

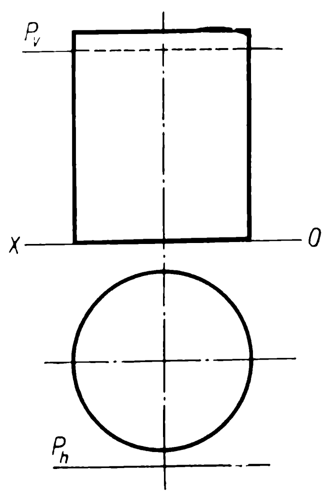


FIG. 945

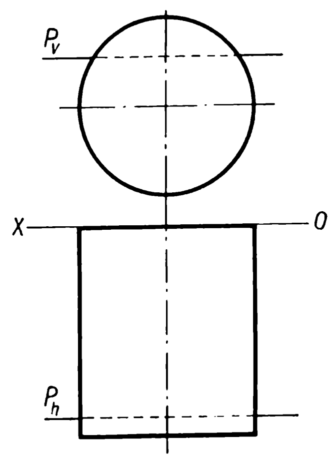


FIG. 946.

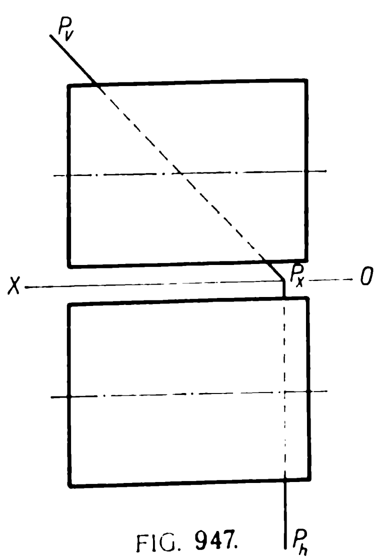


FIG. 947.

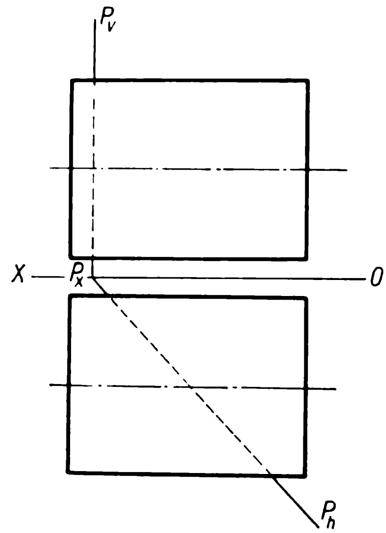



FIG. 948.

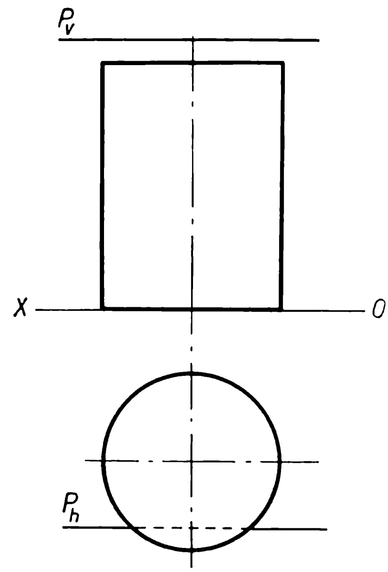



FIG. 949.

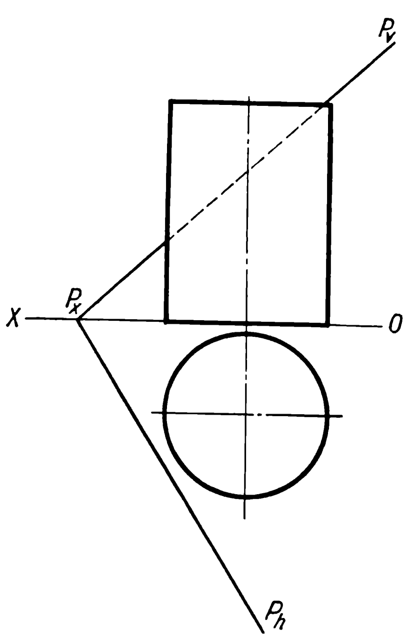



FIG. 950.

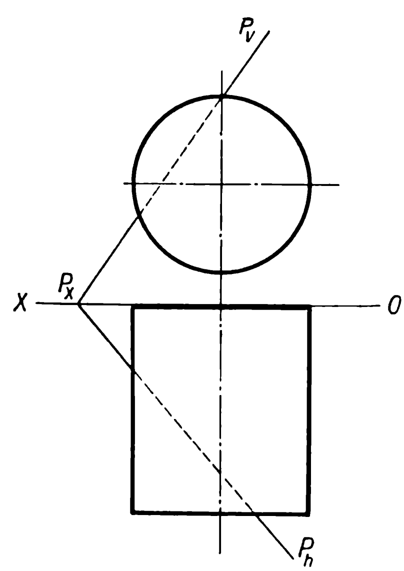



FIG. 951.

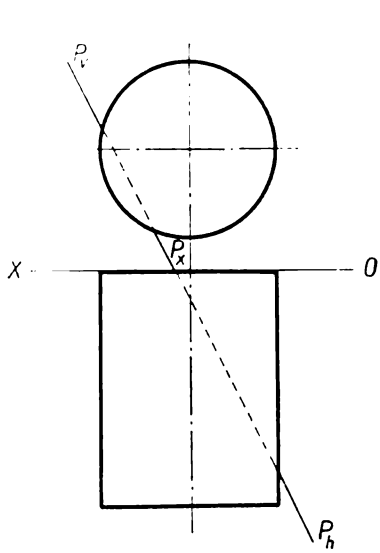



FIG. 952.

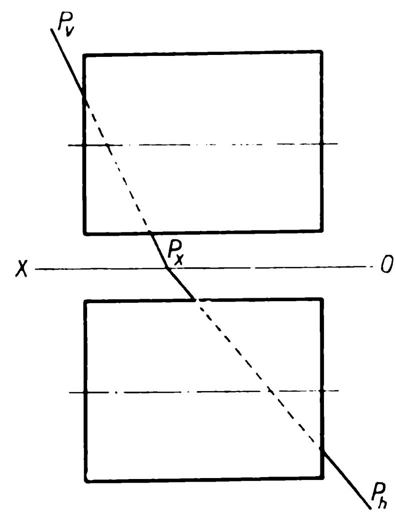



FIG. 953.

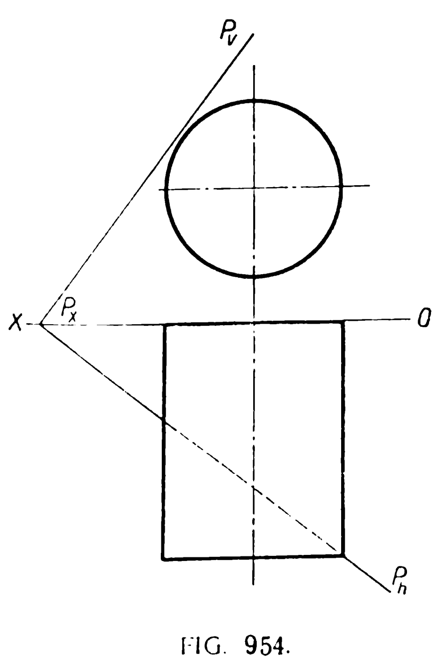



FIG. 954.

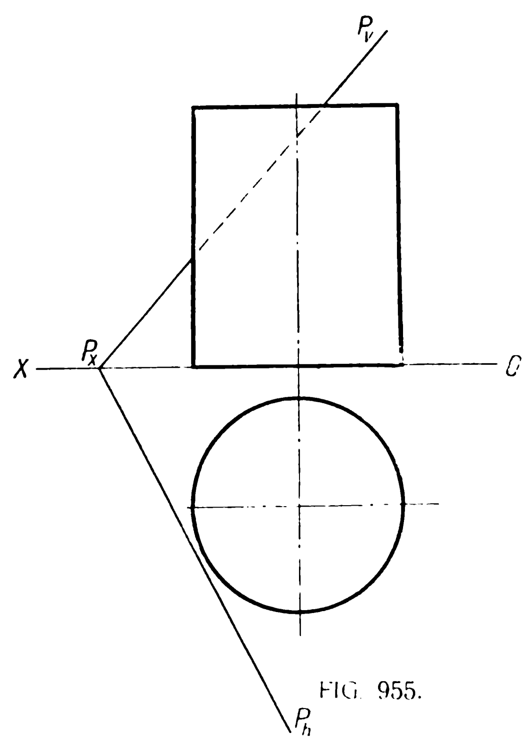



FIG. 955.

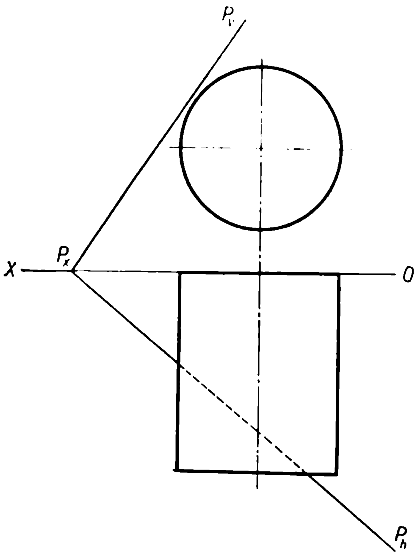



FIG. 956.

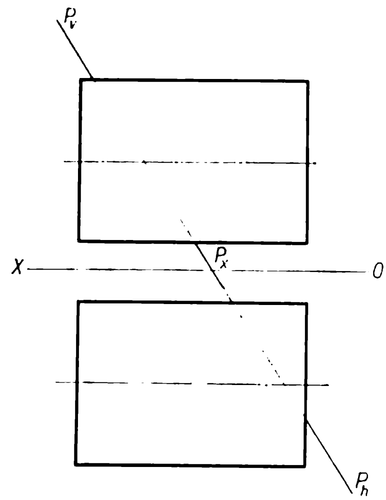



FIG. 957.

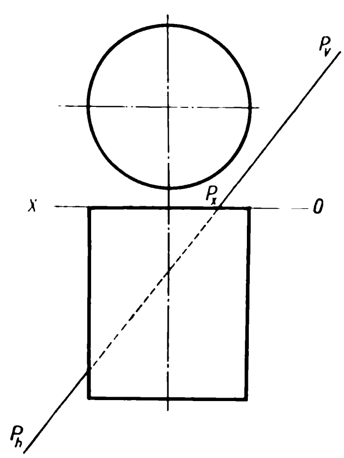



FIG. 958.

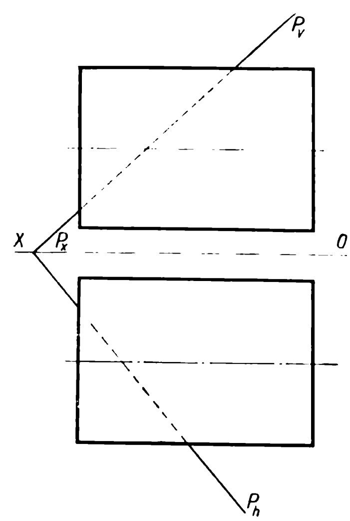



FIG. 959.

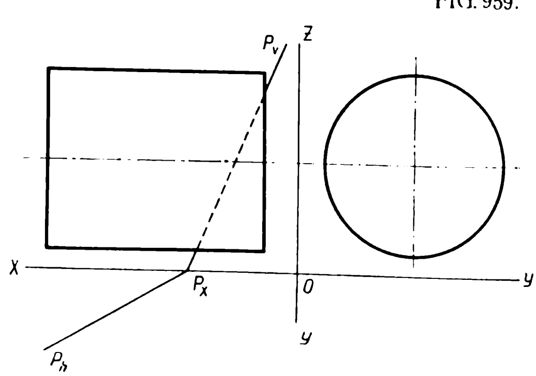



FIG. 960.

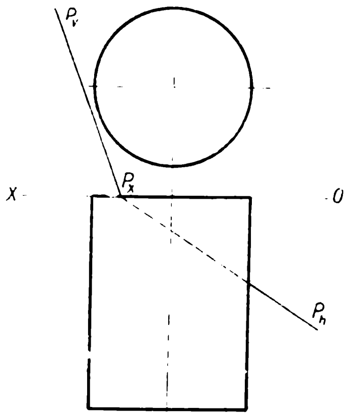



FIG. 961.

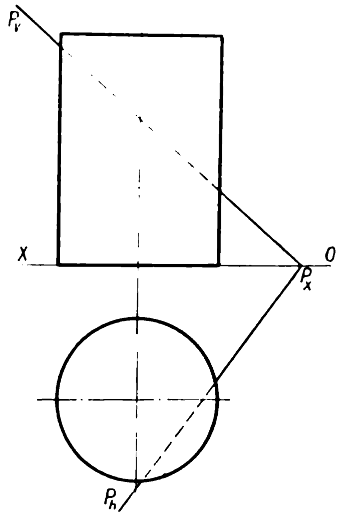



FIG. 962.

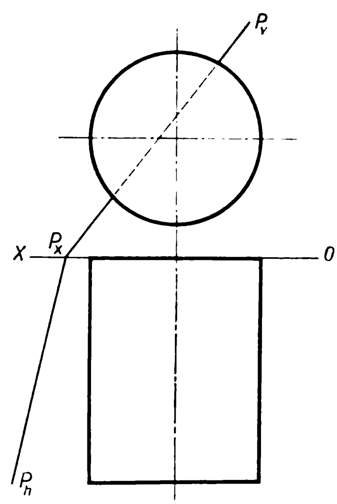



FIG. 963.

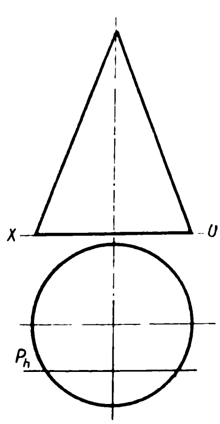



FIG. 964.

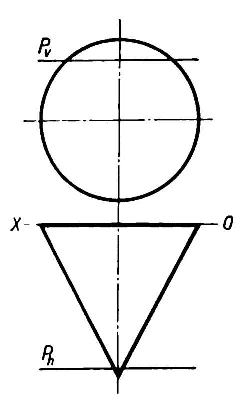



FIG. 965.

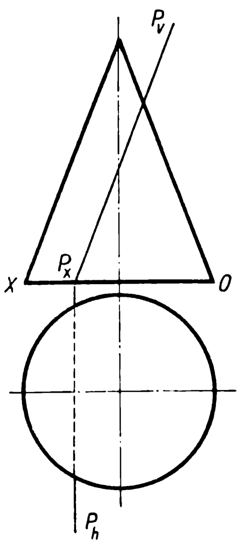



FIG. 966.

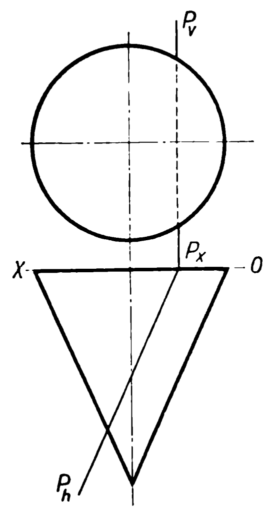



FIG. 967.

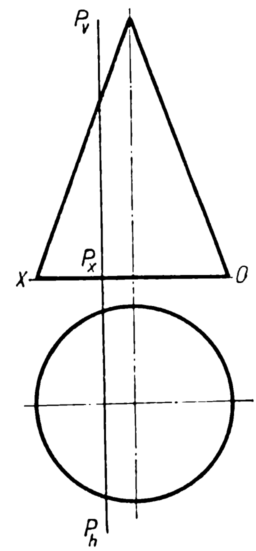



FIG. 968.

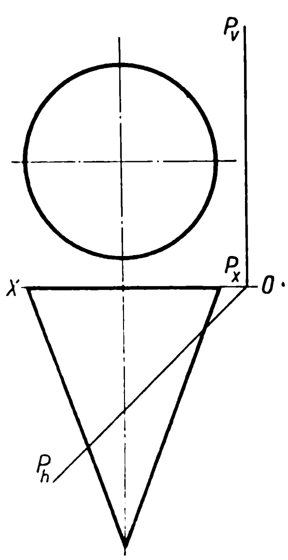



FIG. 969.

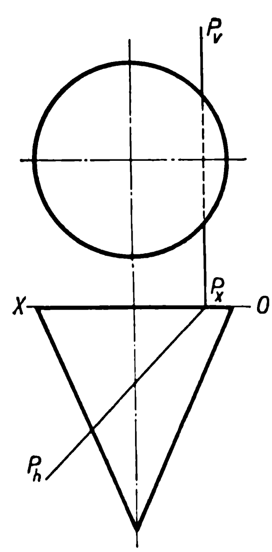



FIG. 970.

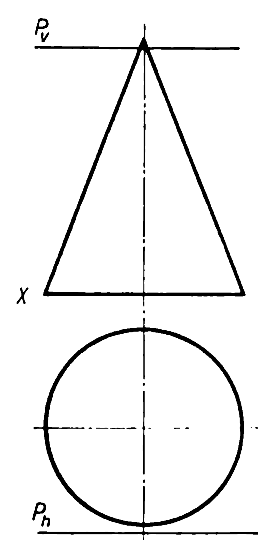



FIG. 971.

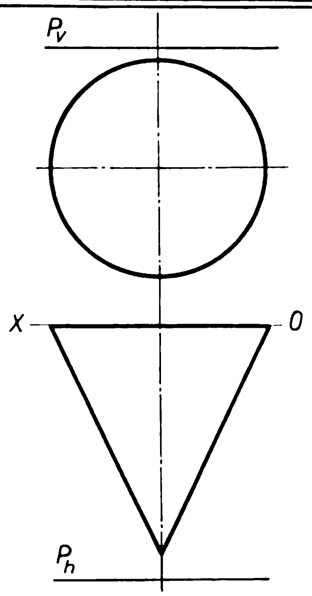



FIG. 972.

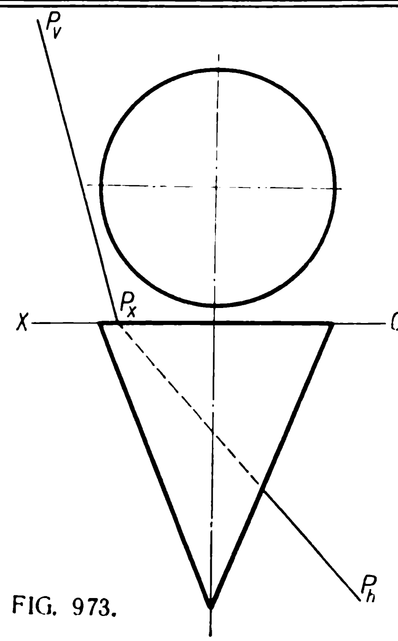



FIG. 973.

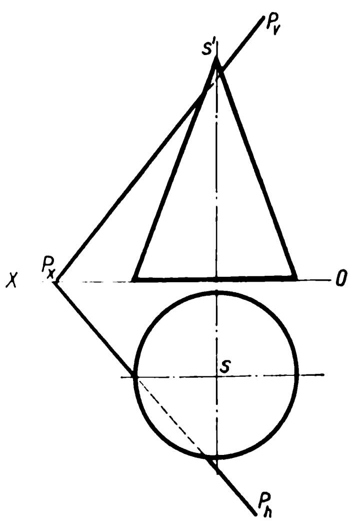



FIG. 974.

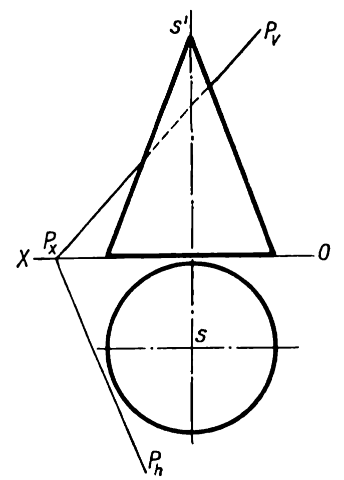



FIG. 975.

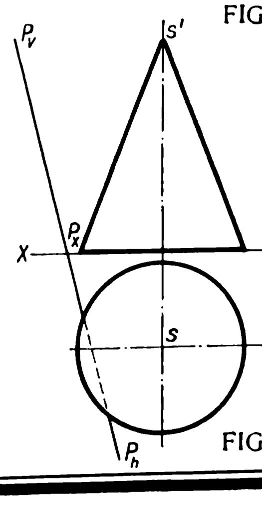



FIG. 976.

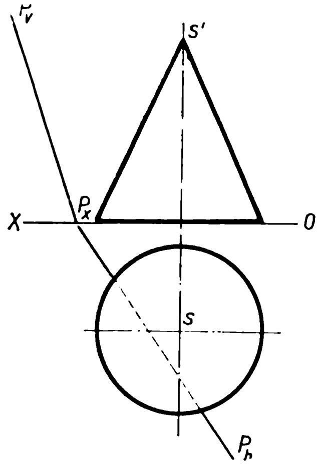



FIG. 977.

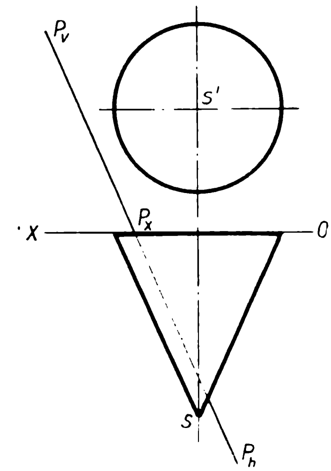



FIG. 978.

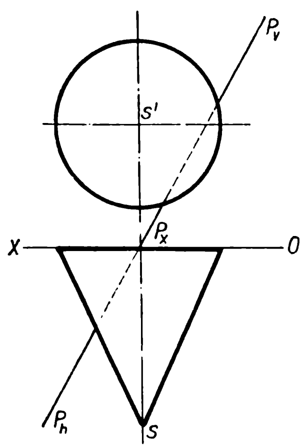



FIG. 979.

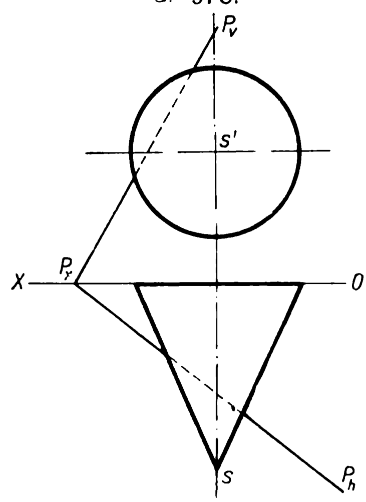



FIG. 980.

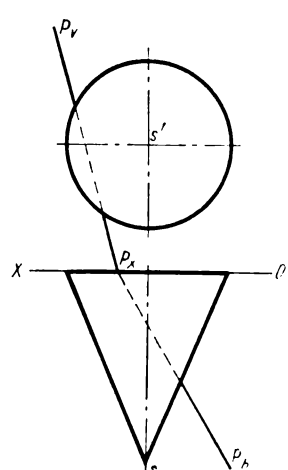



FIG. 981.

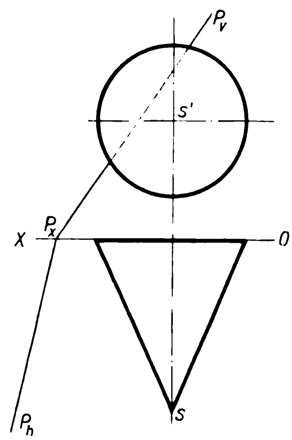



FIG. 982.

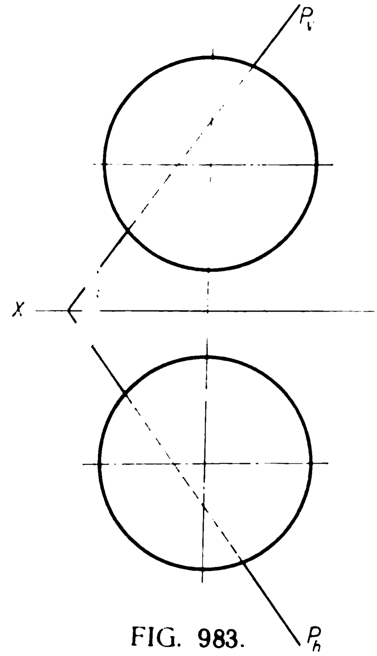



FIG. 983.

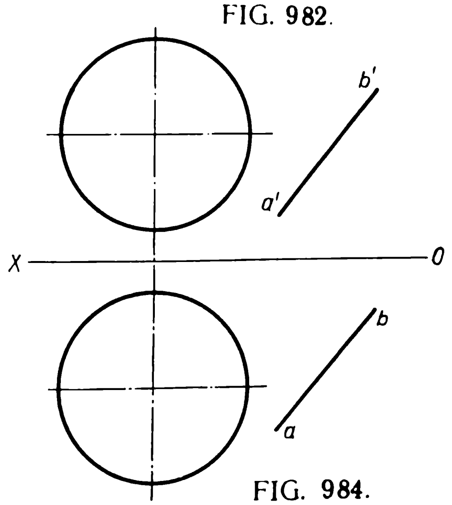



FIG. 984.

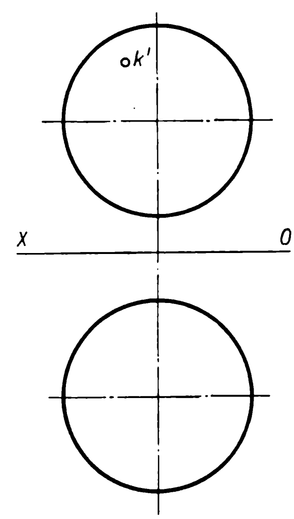



FIG. 985.

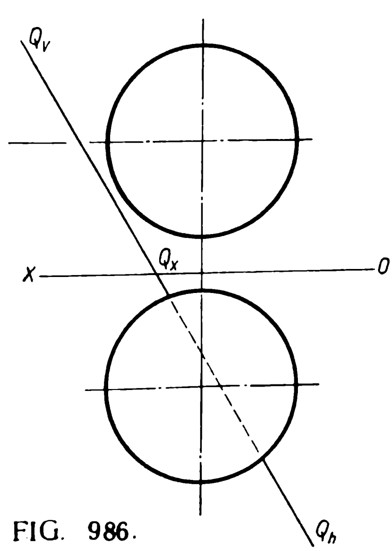



FIG. 986.

CHAPTER XXIV

# THE DEVELOPMENT OF CURVED SURFACES

EXAMPLES

**Example 309**

Construct the development of the lateral surface of a right circular cylinder (Fig. 987).

*Solution.* The development of the lateral surface of a right circular cylinder with the radius of the base circle equal to $r$ and an altitude equal to $h$, is a *rectangle* with an altitude equal to $h$ and a length equal to $2\pi r$. To avoid calculations in determining the length of the circumference of the base circle, a regular dodecagon is *usually* inscribed in the base circle of the cylinder (the drawing shows only the vertices *0*, *1*, *2*,

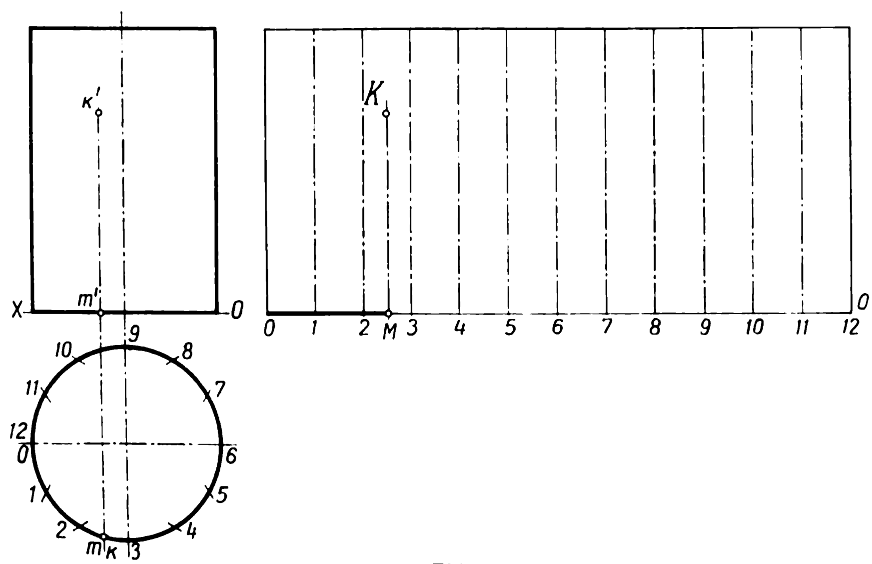


FIG. 987.

etc. of the dodecagon) and its perimeter is taken for the length of the rectangle base. The development of the lateral surface of a right circular cylinder is thus *replaced* by the development of the lateral surface of a right regular dodecagonal *prism* inscribed in the given cylinder. This approximate method is suitable for practical purposes.

We will now show how to transfer the *a* point ($k$, $k'$) from the surface of the cylinder to its development. Lay off on the base of the rectangle a line-segment $OM$ equal to the length of the rectified arc $Om$, erect a perpendicular at the point $M$, and mark off on it a segment $MK$ equal to $m'k'$.

**Example 310**

Construct the development of the lateral surface of a truncated right circular cylinder (Fig. 988).

*Solution.* Divide the base of the cylinder into 12 equal parts and draw generating lines through the division points. Draw a straight line separately, take an arbitrary zero point on it and lay off the sides of a regular dodecagon inscribed in the base of the cylinder. Erect at points *0*, *1*, *2*, etc. (including point *12*) perpendiculars to the

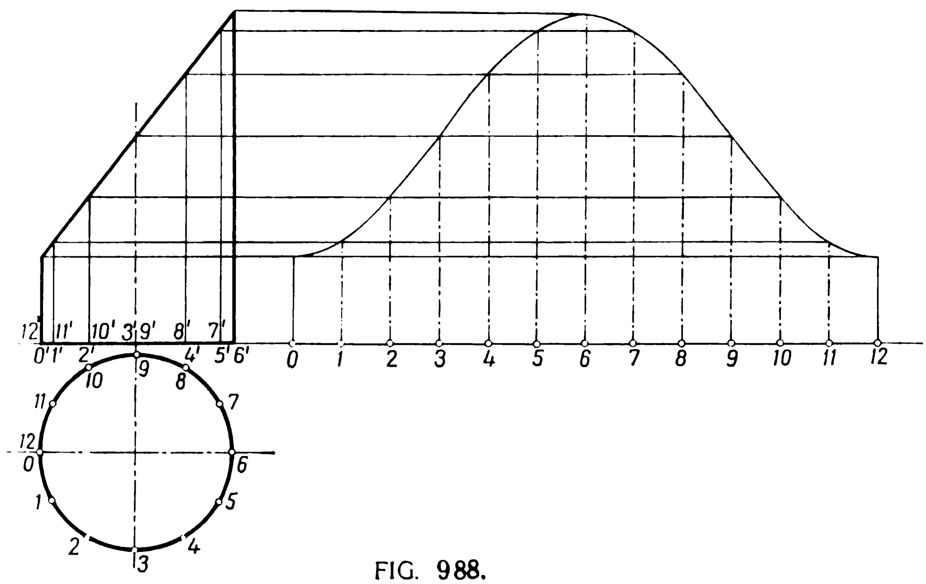



FIG. 988.

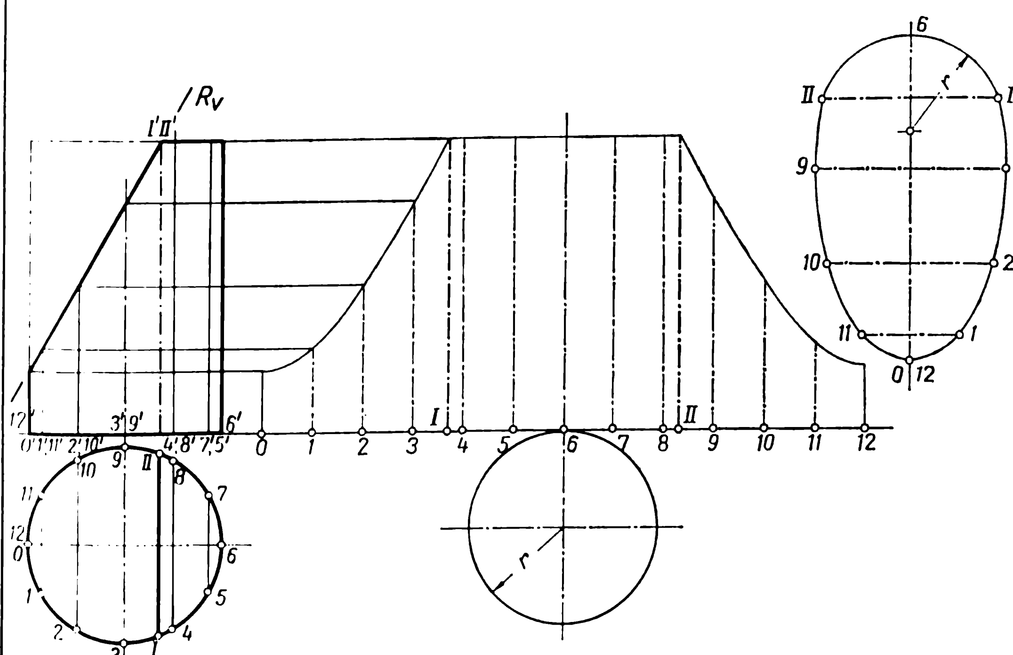



FIG. 989.

base line and mark off on them the lengths of the corresponding generating lines. Join the ends of the generating lines in a smooth curve to obtain the desired development.

**Example 311**

Construct the complete development of the lower part of a right circular cylinder truncated by the plane $R$ (Fig. 989).

*Solution.* Divide the base of the cylinder into 12 equal parts and draw generating lines through the division points. As is evident from the drawing the generating lines of the cylinder intersect the plane $R$ *only* along the arc $I\ 0\ II$. Draw a straight line separately (as shown in the drawing), take an arbitrary zero point on it and mark off from this point equal segments $01$, $12$, etc., and two equal intermediate segments $0\ I$ and $12\ II$. Erect at the points $0, 1, \ldots, 12, I, II$ perpendiculars to the base line and mark off on them the lengths of the respective generating lines. Join the ends of the generating lines located within the segment $I$-$II$ with a straight line and those outside this segment, with a smooth curve to obtain the desired development.

To obtain the complete development, affix to the development of the lateral surface of the cylinder the lower base, which is a circle, part of the upper base—a segment, and a section figure, which is a portion of an ellipse, whose true size should be determined previously (see the drawing).

*Note.* If the cutting plane is oblique, it is advisable first to displace it so that it becomes a projecting plane by rotating the whole system about the axis of the cylinder through an angle $\varphi$.

**Example 312**

Construct the complete development of the lateral surface of an inclined cylinder with a circular base (Fig. 990).

*Solution.* The development of the lateral surface of the inclined cylinder is constructed according to the rule for developing an inclined prism (see Example 271). Pass an auxiliary plane $R$ perpendicular to the cylinder axis, dividing the latter into two right cylinders with a common base. Determine the true size of the normal section. Divide the base of the cylinder into 12 equal parts and draw generating lines of the cylinder through the division points, which divide the perimeter of the normal section into 12 *unequal* parts: $I$-$II$, $II$-$III$, etc. Draw a straight line separately and, from an arbitrary point, lay off *straight*-line segments $I$-$II$, $II$-$III$, etc., equal to the sides of the polygon $I, II, III, \ldots I$ inscribed in the curve of the normal section. Through the obtained points $I, II, III, \ldots I$ draw perpendiculars to the line $I$-$I$ and lay off on each of them the lengths of the generating lines of the upper and lower cylinders, which are given true size (why?). Join the ends of the generating lines in a smooth curve to obtain the desired development.

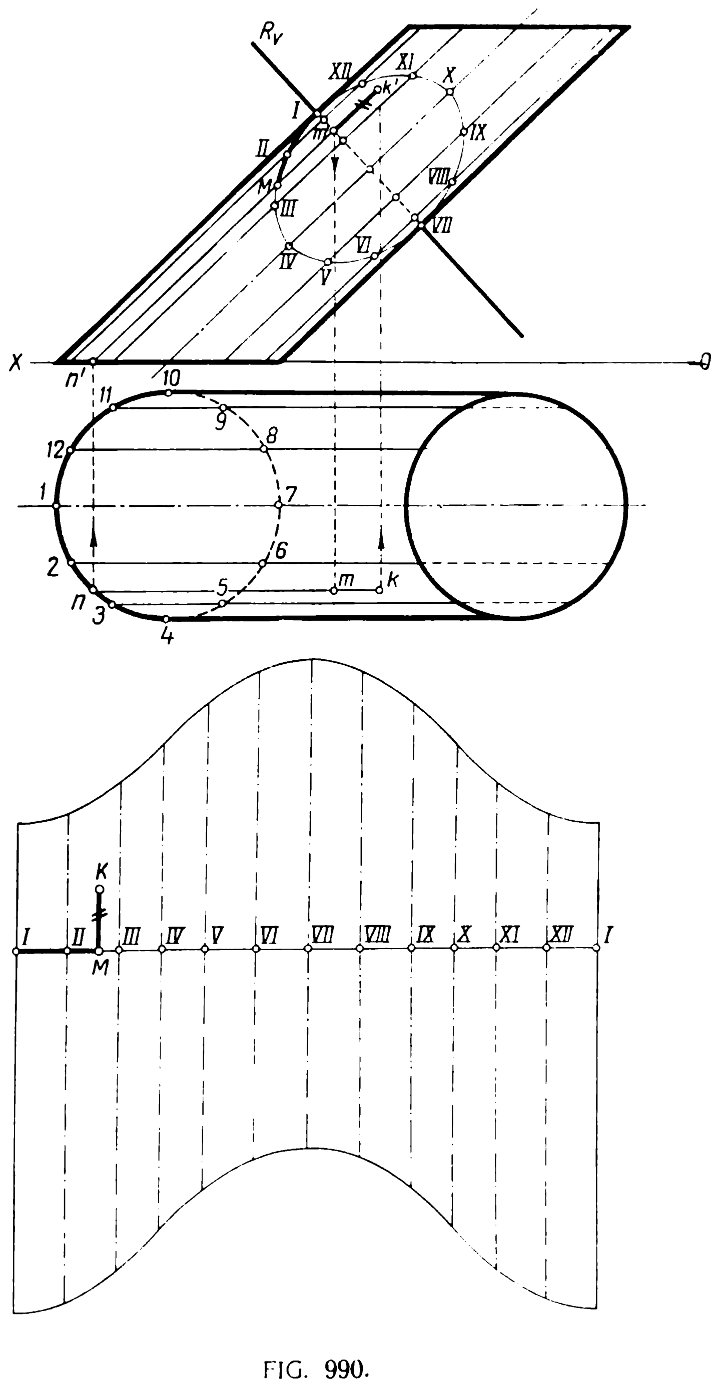
We will now show how to transfer a point $K$ lying on the surface of the cylinder to the development, given the horizontal projection ($k$) of the point. Through the point $k$ draw the horizontal projection ($kn$) of an auxiliary generating line, and then on its vertical projection find point $k'$ from the point $k$. Transfer the point ($m$, $m'$) of the generating line ($kn$, $k'n'$) lying on the line of the normal section onto the development and draw a perpendicular to the line $I$-$I$ at the point $M$. On the perpendicular lay off *upwards* a line-segment $MK = m'k'$.

*Note.* If the axis of the cylinder is not parallel to a projection plane, then first displace the cylinder so that its axis becomes parallel to the $H$ or $V$ plane (why?).

**Example 313**

Construct the development of the lateral surface of the right circular cone (Fig. 991).

*Solution.* The development of the lateral surface of a right circular cone whose radius of the base circle is equal to $r$ and the length $l$ of the generating line is a *sector* of radius $l$ with central angle $\varphi = \frac{2\pi r}{l}$. To avoid involved calculations in determin-



$R_V$
XII
XI
X
k'
I
m'
IX
II
M
VIII
III
VII
VI
IV
V
X
n'
10
0
11
9
8
12
1
7
6
2
5
m
k
n
3
4
K
M
I
II
III
IV
V
VI
VII
VIII
IX
X
XI
XII
I


FIG. 990.

ing the length of the sector arc or the angle $\varphi$, the following procedure is used: inscribe a regular dodecagon in the base of the cone (the drawing shows only its vertices *0*, *1*, *2*, etc.); then, from an arbitrary point $S$ as centre, strike an arc of radius $l$, mark off from any point twelve small arcs whose chords are equal in length to the sides of the dodecagon. Thus, the development of the lateral surface of the right circular cone is replaced (with sufficient accuracy) by the development of a regular dodecahedral pyramid inscribed in the given cone.

Now we will show how to transfer a point $K$ from the surface of the cone to its development, given the horizontal projection ($k$) of the point. Through the point $k$ draw the horizontal projection ($sm$) of an auxiliary generating line, and find its vertical projection ($s'm'$); from $k$ determine $k'$ on $s'm'$. Find the generating line $SM$ on the development and lay off on it the true length of the segment ($sk$, $s'k'$) of the generating line.

**Example 314**

Construct the development of the lateral surface of an oblique circular cone (Fig. 992).

*Solution.* Divide the base of the cone into twelve equal parts and draw generating lines through the division points. Thus, the whole lateral surface of the cone is divided into twelve *curvilinear* triangles, which for practical purposes may be replaced by *plane* triangles. Consequently, the lateral surface of the cone is replaced by the lateral surface of a dodecahedral *pyramid* inscribed in the cone. Find the true length of all the generating lines of the cone (see the drawing) and construct consecutively plane triangles *S01*, *S12*, etc. Join the ends of the generating lines in a smooth curve to obtain the desired development. The drawing also shows the transfer of an arbitrary point $K$ from the cone surface to its development.

**Example 315**

Construct the development of the lateral surface of an oblique truncated cone without using its vertex (Fig. 993).

*Solution.* Divide the upper and lower bases of the cone into twelve equal parts and join by generating lines the points *I-I*, *2-II*, etc., i.e. inscribe a dodecahedral truncated pyramid in the truncated cone. Replace each *curvilinear* trapezoid with a *plane* one, divide them into triangles and find the true length of all the generating lines and diagonals (see the drawing). Construct consecutively plane triangles *1 II I*, *1 2 II*, etc. Join the end points *I*, *II*, *III*, . . . and *1*, *2*, *3* of the generating lines in smooth curves to obtain the desired development with an accuracy sufficient for practical purposes.

*Note.* This "triangular" method can be successfully used in constructing more complicated developments.

**Example 316**

Construct a complete development of the lower part of a right circular cone cut by the plane $P$ (Fig. 994).

*Solution.* Divide the base of the cone into twelve equal parts, pass generating lines through the division points and find the points of their intersection with the plane $P$. Construct the development of the whole cone (see Example 313) and lay off on each generating line the true length of the line-segment of the respective generating line of the cone from its vertex to the point of intersection with the given plane. Then join the ends of these line-segments in a smooth curve. To obtain a complete development of the lower portion of the cone, affix true size to the development of its lateral surface the base (a circle) and the section figure (an ellipse) (see the drawing).

*Note.* If the cutting plane is oblique, then it is advisable first to displace the whole system so that the plane becomes a projecting one by rotating it about the axis of the cone through an appropriate angle $\varphi$.

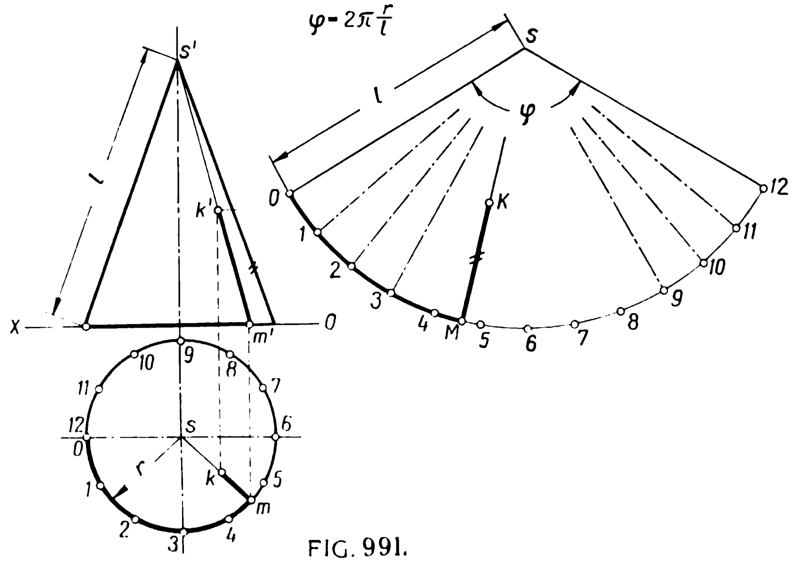



FIG. 991.

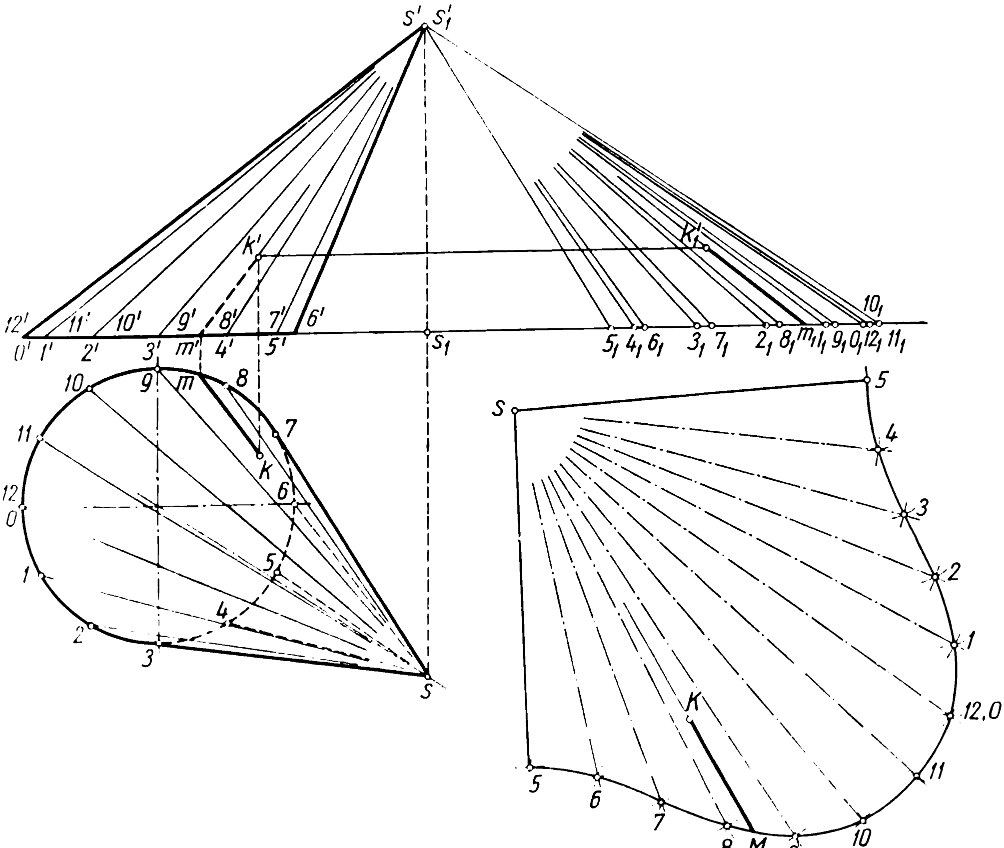



FIG. 992.

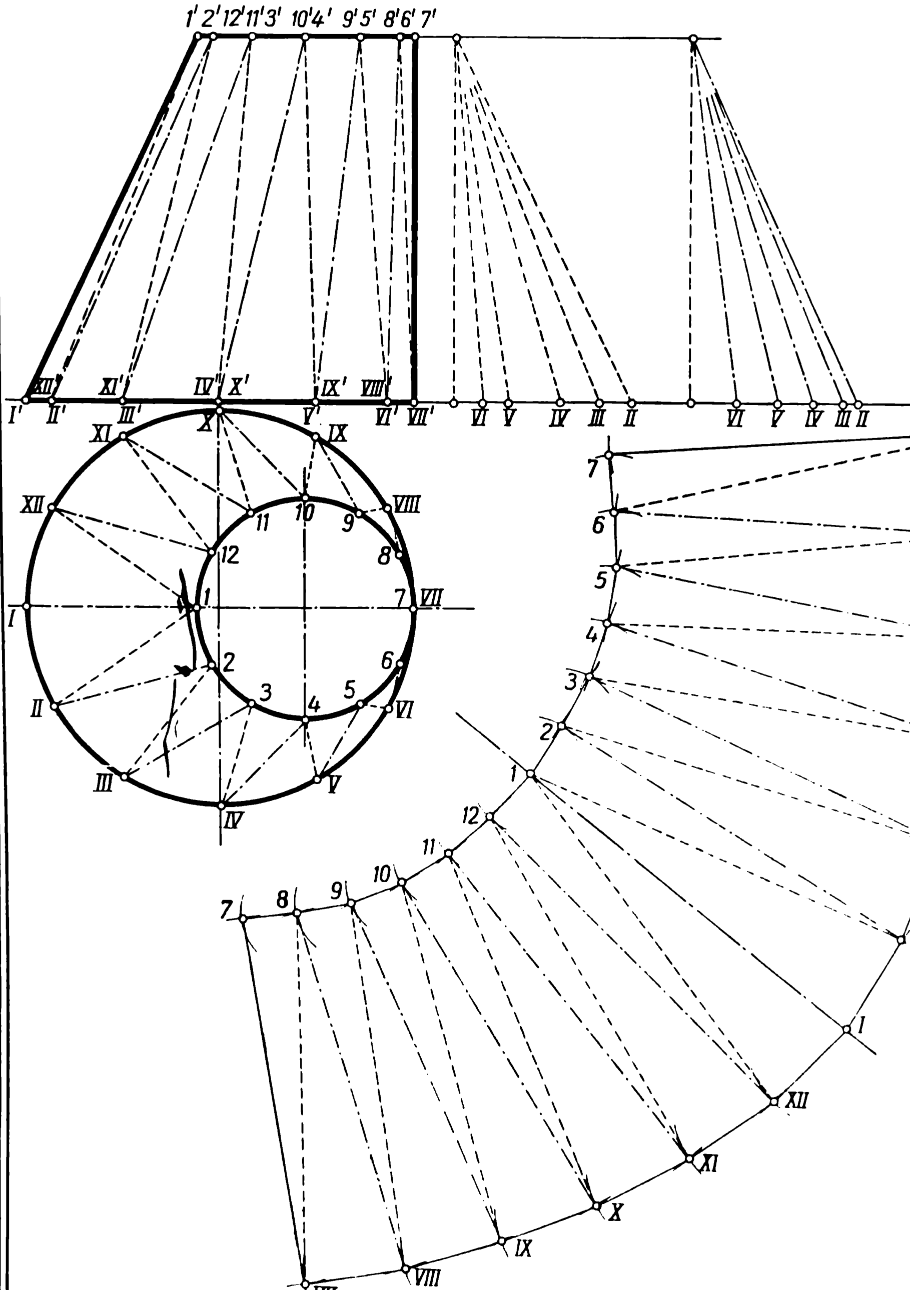

1' 2' 12' 11' 3' 10' 4' 9' 5' 8' 6' 7'
XII' XI' IV' X' IX' VIII'
I' II' III' X V' VI' VII' VI V IV III II VI V IV III II
XI IX XII 11 10 9 VIII 12 8 I 1 7 VII 2 6 II 3 4 5 VI III V IV
VII 7 6 VI 5 V 4 3 IV 2 1 III 12 11 10 9 8 7 II I XII XI X IX VIII VII

FIG. 993.

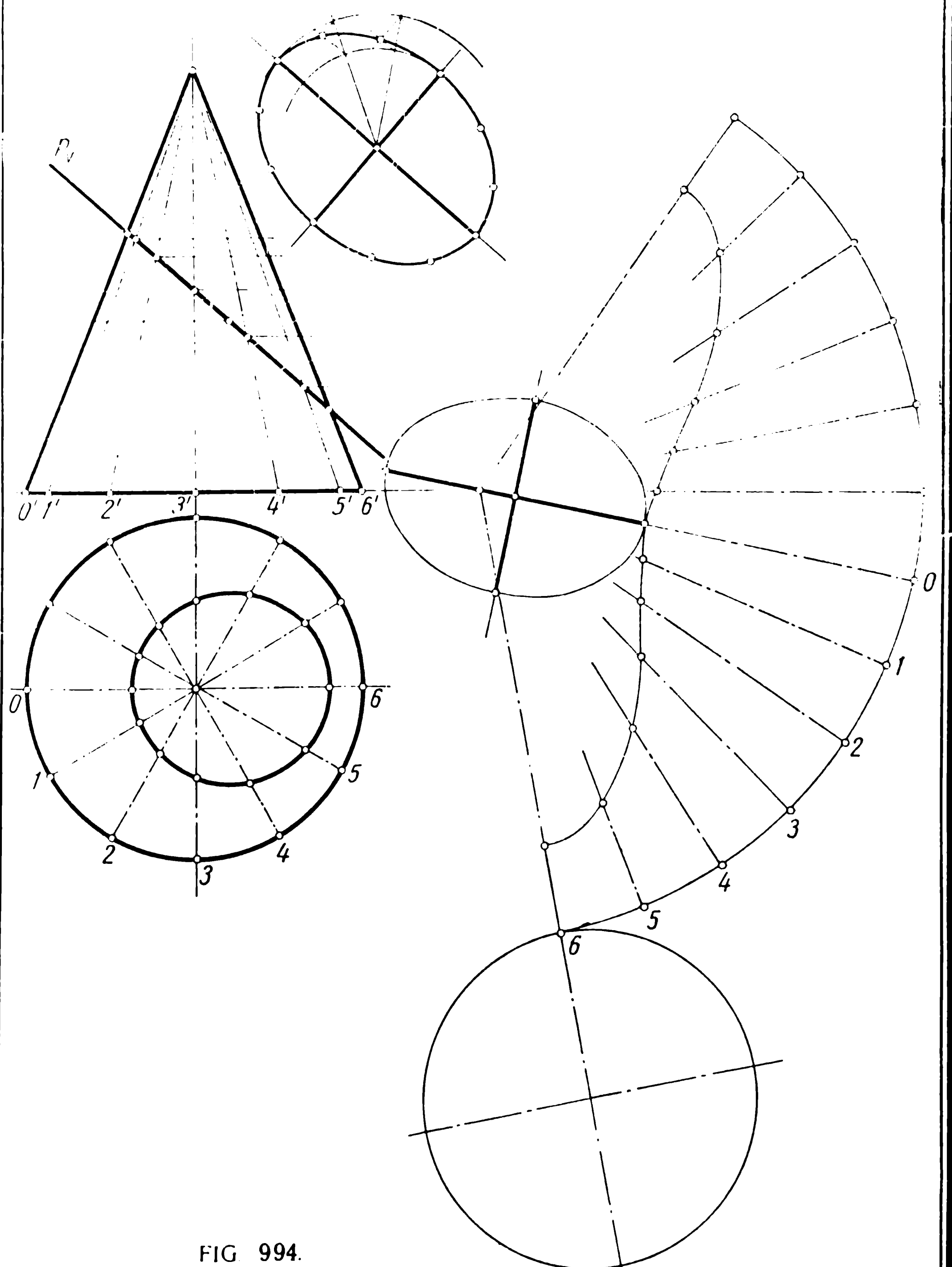



FIG. 994.

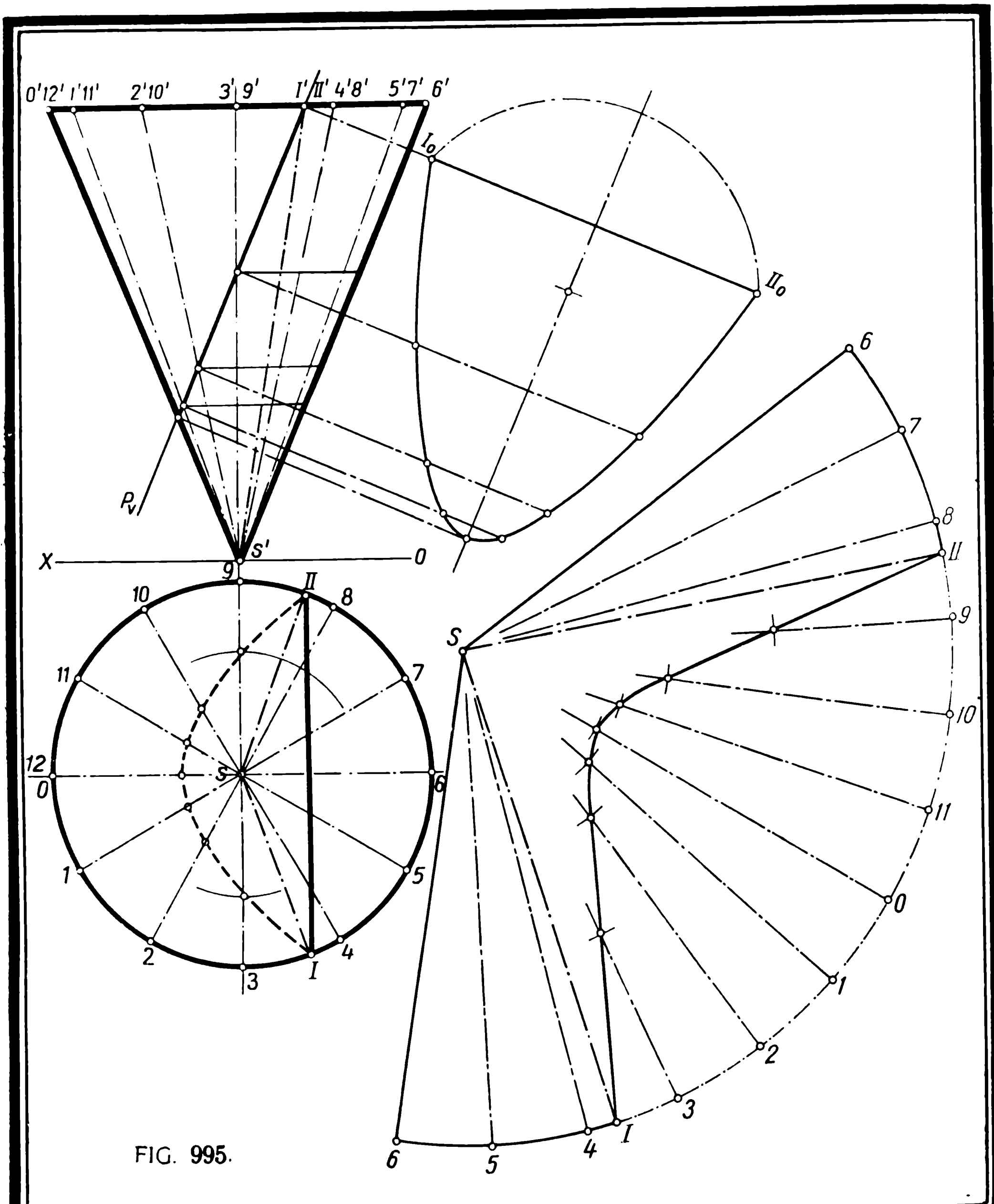



FIG. 995.

**Example 317**

Construct the complete development of the lower portion of a right circular cone cut by the plane $P$ (Fig. 995).

*Solution.* Divide the base of the cone into twelve equal parts and draw generating lines through the division points. As is evident from the construction, the generating lines of the cone intersect with the plane $P$ *only* within the portion $I\ 0\ II$ of the arc. Construct the development of the whole cone and draw on it two intermediate generating lines $SI$ and $SII$. On each generating line intersecting the plane $P$, lay off the true length of the line-segment of the respective generating line of the cone from its vertex to the point of intersection with the plane. Now join the ends of these line-segments in a smooth curve. To obtain the desired development, affix true size to the development of the lateral surface the appropriate portion of the base (a segment) and the section figure (a parabola) (see the drawing).

PROBLEMS

487. Intersect a right circular cylinder with the plane $P$ and construct the complete development of one of its parts (Figs. 945 to 963).

488. Intersect a right circular cone with the plane and construct the complete development of one of its parts (Figs. 964 to 982).

CHAPTER XXV

# INTERSECTION OF A STRAIGHT LINE WITH A CURVED SURFACE

The problem of determining the points of intersection of a straight line with the surface of any geometrical solid (prism, pyramid, cylinder, cone, sphere, etc.) is solved in the same sequence as was suggested for determining the point of intersection of a straight line with a plane, viz.

(1) through the given straight line draw an auxiliary plane;

(2) find the line (straight or curved) of intersection of the given plane with the auxiliary plane;

(3) find the desired points at the intersection of the given line with the line of intersection.

In a particular case the straight line may be *tangent* to the surface.

*Note.* When passing an auxiliary plane through a given straight line, the former should be chosen so that its line of intersection with the given surface *is projected* onto the planes of projection as simple lines—a straight line or a circle.

EXAMPLES

**Example 318**

Determine the points of intersection of the surface of a prism with the straight line $AB$ (Fig. 996).

*Solution.* Pass through the line $AB$ a vertical (or horizontal) projecting plane $R$ which intersects the prism in a quadrilateral. Find the horizontal projections ($m$) and ($n$) of the desired points at the intersection of the horizontal projections of the quadrilateral and the given straight line. Then find the vertical projections ($m'$) and ($n'$) of the points on the line $a'b'$. In this particular case the desired points can be found without introducing the plane $R$ (why?).

**Example 319**

Determine the points of intersection of the surface of a pyramid with the straight line $AB$ (Fig. 997).

*Solution.* Through the given straight line $AB$ pass a horizontal projecting plane $R$ which intersects the pyramid in a quadrilateral. Find the vertical projections ($m'$) and ($n'$) of the desired points at the intersection of the vertical projections of the quadrilateral and the given straight line. Then find the horizontal projections ($m$) and ($n$) of the points on the line $ab$.

**Example 320**

Determine the points of intersection of the straight line $AB$ with the surface of a pyramid (Fig. 998).

*Solution.* Through the given straight line $AB$ pass a profile plane $R$, which intersects the pyramid in a triangle $CDE$. Find the profile projections ($m''$) and ($n''$)

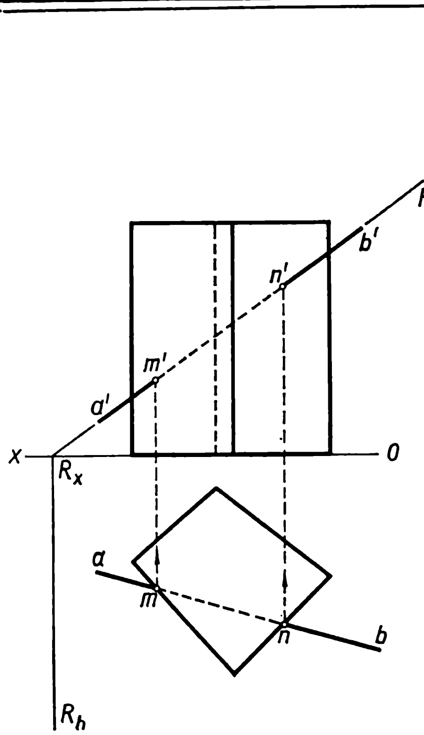


FIG. 996.

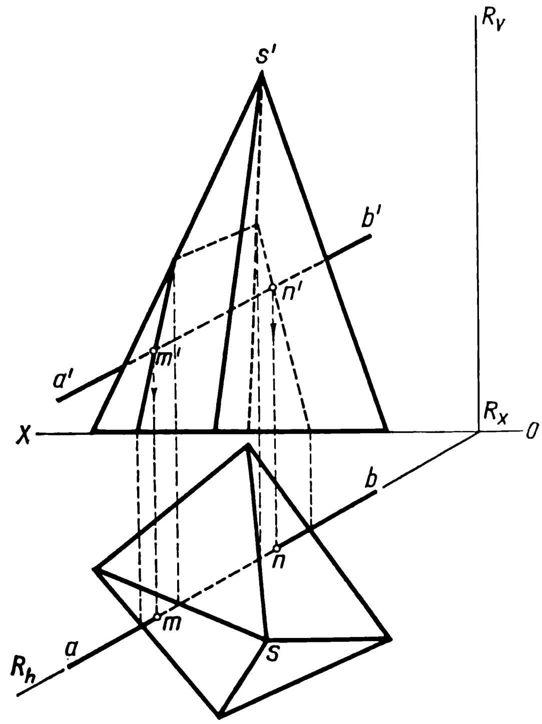


FIG. 997.

of the desired points at the intersection of the profile projections of the triangle obtained and the given straight line. Then determine the points *m* and *n* on the line *ab* and the points *m'* and *n'* on the line *a'b'*.

**Example 321**

Find the points of intersection of the straight line *AB* with the surface of a cylinder (Fig. 999).

*Solution.* Through the given straight line *AB* pass a horizontal projecting plane *R*, which intersects the cylinder along two generatrices. The intersection of the vertical projections of the generatrices and the given straight line yields the vertical projections (*m'*) and (*n'*) of the desired points. Now find the points *m* and *n* on the line *ab* (we could alternatively pass a vertical projecting plane through *AB*).

**Example 322**

Find the points of intersection of the straight line *AB* with the surface of an oblique cylinder (Fig. 1000).

*Solution.* Through the given line *AB* pass a plane *R* parallel to the axis of the cylinder. To this end, assume an arbitrary point (*c*, *c'*) on the straight line (*ab*, *a'b'*) and draw through it a line (*cd*, *c'd'*), parallel to the cylinder axis. This plane, specified by two intersecting lines, intersects the cylinder along two generatrices. Find the horizontal traces (*h*, *h'*) and ($h_1$, $h_1'$) of the straight lines (*ab*, *a'b'*) and (*cd*, *c'd'*) and, through the points *h* and $h_1$, draw the horizontal trace ($R_h$) of the plane (the vertical trace of the plane is not needed for solving this problem—why?). The plane *R* intersects the base of the cylinder along a chord (*12*, *1'2'*). Through the points (*1*, *1'*) and (*2*, *2'*) draw generating lines of the cylinder. The intersection of the vertical projections of these lines with the vertical projection (*a'b'*) of the given straight line

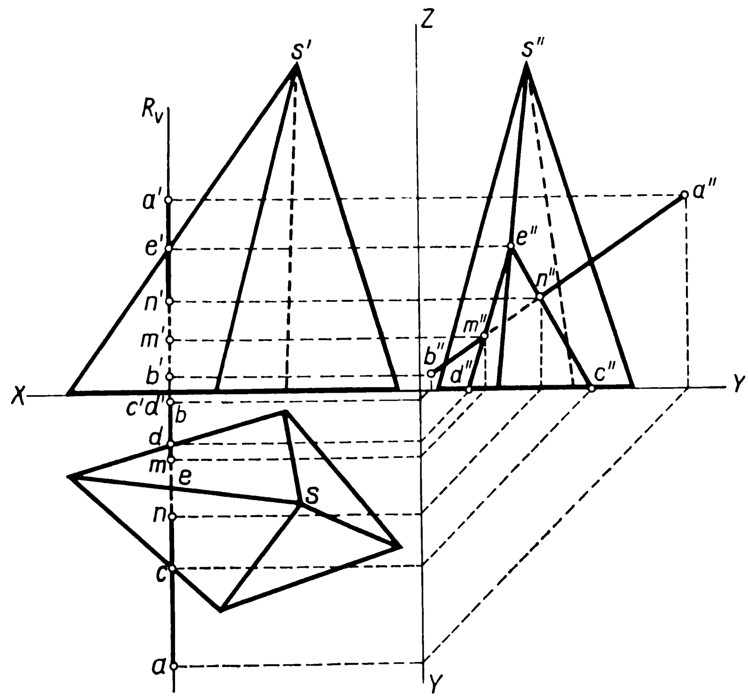



FIG. 998.

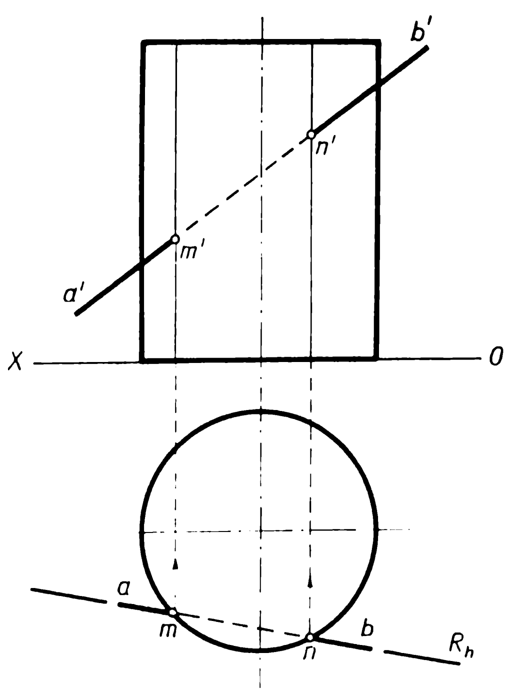



FIG. 999.

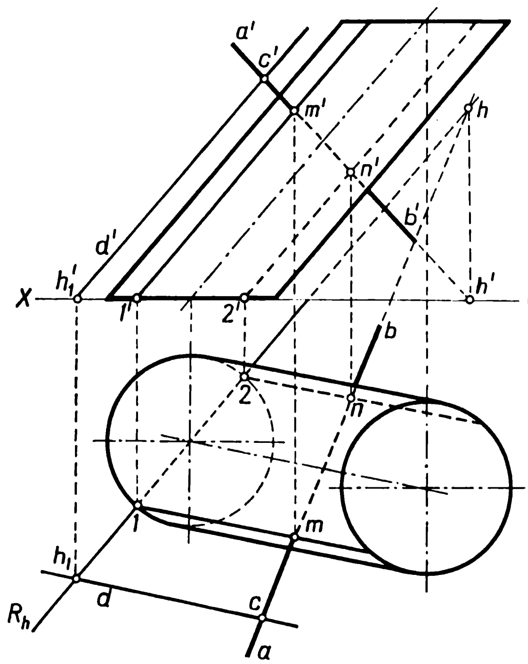


FIG. 1000.

yields the vertical projections ($m'$) and ($n'$) of the desired points. From $m'$ and $n'$ find $m$ and $n$ on $ab$. (Passing a horizontal or vertical projecting plane through $AB$ would complicate the solution of the problem—why?).

*Note.* In a particular case (when?) it is *more convenient* to specify the auxiliary plane passing through the line $AB$ by two straight lines parallel to the axis of the cylinder.

**Example 323**

Find the points of intersection of the straight line $AB$ with the surface of a cone (Fig. 1001).

*Solution.* The given line intersects the lateral surface of the cone in one point ($m$, $m'$). Through $AB$ pass a vertical projecting plane $R$ passing through the vertex $S$ of the cone. This plane intersects the surface of the cone along two straight lines which are generatrices (the drawing shows only one generatrix). The horizontal projection ($m$) of the desired point is found at the intersection of the horizontal projections of the given and obtained lines. From the point $m$ determine $m'$, which coincides with the vertical projection ($a'b'$) of the given line (why?).

**Example 324**

Determine the points of intersection of the straight line $AB$ with the surface of a cone (Fig. 1002).

*Solution.* Through $AB$ pass a plane $R$ parallel to the $H$ plane and intersecting the cone in a circle. The horizontal projections ($m$) and ($n$) of the desired points are found at the intersection of the horizontal projections of the circle obtained and the given line. From $m$ and $n$ find the points $m'$ and $n'$ on the line $a'b'$. We could place the line $AB$ in a horizontal projecting plane, but it would *greatly* complicate the solution of the problem (why?).

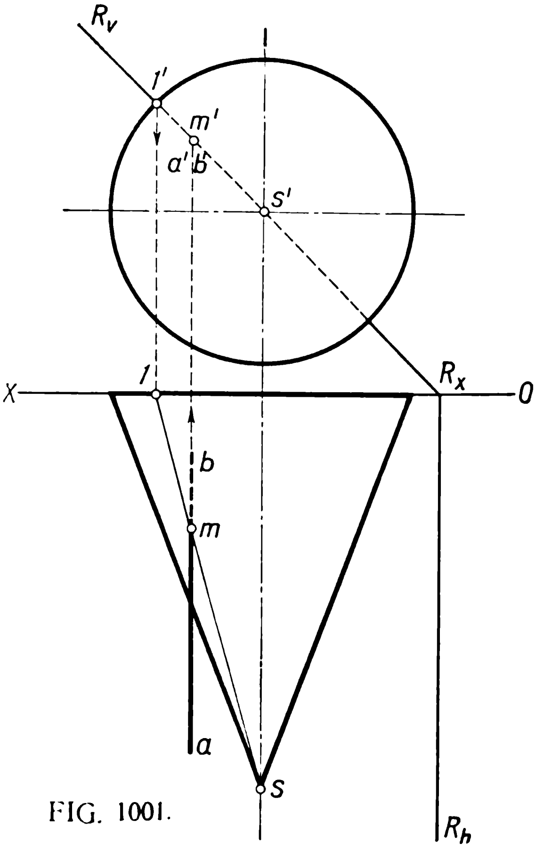



FIG. 1001.

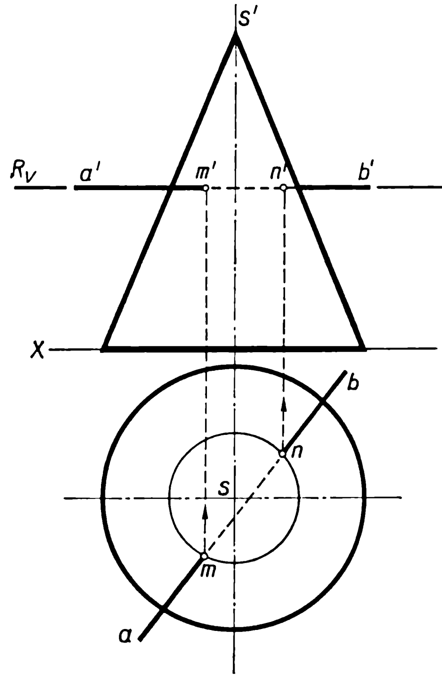



FIG. 1002.

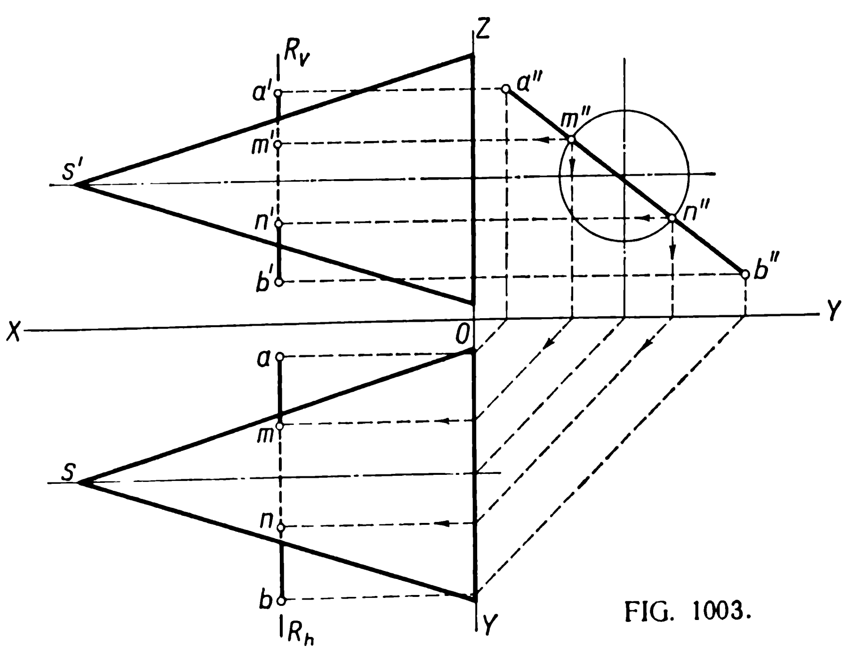



FIG. 1003.

**Example 325**

Determine the points of intersection of the straight line $AB$ with the surface of a cone (Fig. 1003).

*Solution.* Through $AB$ pass a profile plane $R$ which intersects the cone in a circle. The intersection of the profile projections of this circle and the given straight line yields the profile projections ($m''$) and ($n''$) of the desired points. Knowing them, find the horizontal and vertical projections of the desired points on the like projections of the line ($ab$, $a'b'$).

**Example 326**

Find the points of intersection of the straight line $AB$ with the surface of an oblique cone (Fig. 1004).

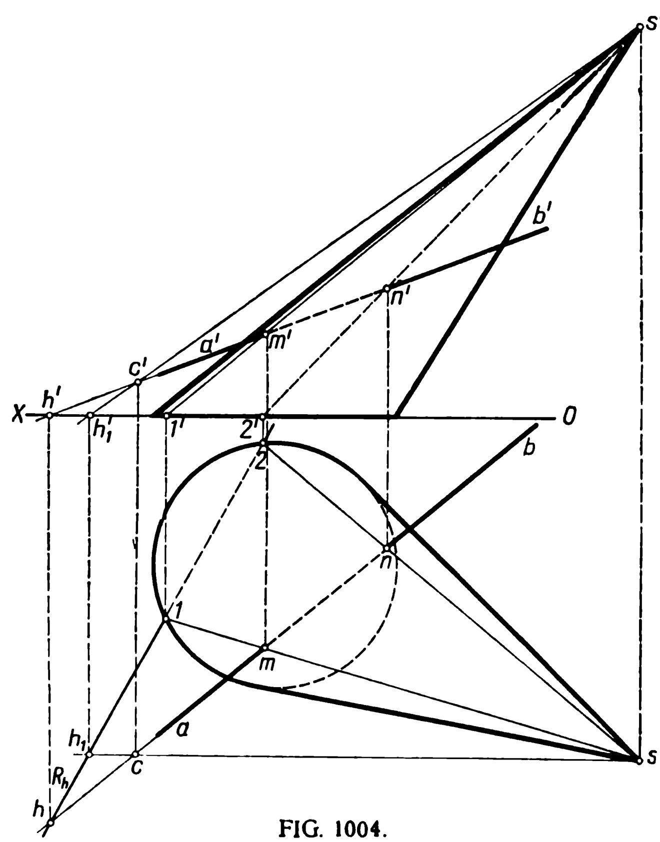



FIG. 1004.

*Solution.* Through $AB$ draw a plane $R$ passing through the vertex $S$ of the cone. This plane, specified by the line $AB$ and the point $S$, intersects the cone along two straight lines, which are generating lines.

To determine them, proceed as follows. Specify the auxiliary plane not by the line $AB$ and the point $S$ but by two intersecting lines $AB$ and $SC$ (the point $C$ on the line $AB$ is taken arbitrarily). Find the horizontal traces ($h$, $h'$) and ($h_1$, $h_1'$) of the straight lines ($ab$, $a'b'$) and ($sc$, $s'c'$) and through the points $h$ and $h_1$ draw the horizontal trace ($R_h$) of the plane (the vertical trace of the plane is not needed for the solution of the problem—why?). The plane $R$ intersects the base of the cone along the chord ($12$, $1'2'$), and its surface, along the generating lines ($s1$, $s'1'$) and ($s2$, $s'2'$). The intersection of the vertical projections of these generating lines with the vertical projection ($a'b'$) of the given line yields the vertical projections ($m'$) and ($n'$) of the desired points. From the points $m'$ and $n'$ find points $m$ and $n$ on the line $ab$. Passing a horizontal or vertical projecting plane through the line $AB$ would greatly complicate the solution of the problem (why?).

*Note.* In a particular case (when?) it is more convenient to specify the auxiliary plane passing through the line $AB$ and the point $S$ by two lines intersecting at point $S$.

**Example 327**

Determine the points of intersection of the straight line $AB$ with a cone (Fig. 1005).

*Solution.* Through the line $AB$ draw a plane $R$ passing through the vertex $S$ of the cone. Find the traces ($h$, $h'$) and ($v$, $v'$) of the line ($ab$, $a'b'$) and draw the traces of the plane: the horizontal one ($R_h$) through the points $h$, $s$, and the vertical one ($R_v$) through the points $R_x$, $v'$. Find the line of intersection of the plane $R$ with the plane of the base of the cone. The plane $R$ intersects the base of the cone along the chord ($12$, $1'2'$), and its lateral surface, along the generatrices ($s1$, $s'1'$) and ($s2$, $s'2'$). The intersection of the vertical projections of these generatrices with the vertical projection ($a'b'$) of the given straight line yields the vertical projections ($m'$) and ($n'$) of the desired points. From $m'$ and $n'$ find the points $m$ and $n$ on the line $ab$.

**Example 328**

Find the points of intersection of the straight line $AB$ with a sphere (Fig. 1006).

*Solution.* Through the line $AB$ pass a plane $R$ parallel to the $H$ plane and intersecting the sphere in a circle. The horizontal projections ($m$) and ($n$) of the desired points are found at the intersection of the horizontal projections of the obtained circle and the given line. From $m$ and $n$ find $m'$ and $n'$ on the line $a'b'$.

**Example 329**

Find the points of intersection of the straight line $AB$ with a sphere.

*Alternative solutions*:

1. *By the coincidence method* (Fig. 1007): through $AB$ pass a horizontal projecting plane $R$, which intersects the sphere in a circle of radius $r$ and with centre at point ($c$, $c'$).

To avoid the use of a French curve (for what kind of line?), bring the plane $R$ to coincidence with the $H$ plane. Find the circle and the given line in the coincident position; their intersection yields points $M_0$ and $N_0$, which help us to determine the desired points ($m$, $m'$) and ($n$, $n'$).

2. *By replacing the planes of projection* (Fig. 1008). Replace, say, the $V$ plane with a new one $V_1$, which is parallel to the given line. Through the line ($ab$, $a_1'b_1'$) pass a plane $R$ parallel to the plane $V_1$ and intersecting the sphere in a circle. The intersection of the vertical projections of the straight line and the circle yields the points $m_1'$ and $n_1'$. The latter help us to determine the desired points ($m$, $m'$) and ($n$, $n'$).

3. *By the displacement method* (Fig. 1009). Displace the given system, say, parallel to the $H$ plane so that the straight line becomes parallel to the $V$ plane. Through the line ($a_1b_1$, $a_1'b_1'$) pass a plane $R$ parallel to the $V$ plane and intersecting the sphere in a circle. The intersection of the vertical projections of the circle and the straight line yields the points $m_1'$ and $n_1'$. From them, by reverse construction, find the points $m'$ and $n'$ on $a'b'$, and then the points $m$ and $n$ on $ab$. The points ($m$, $m'$) and ($n$, $n'$) are the desired points.

**Example 330**

Find the points of intersection of the straight line $AB$ with a toroid (Fig. 1010).

*Solution.* Through the line $AB$ pass a plane $R$ parallel to the $V$ plane and intersecting the toroid in a circle. The vertical projections ($m'$) and ($n'$) of the desired

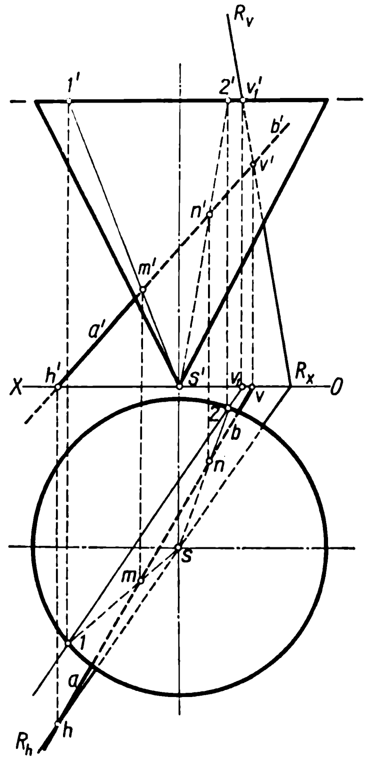



FIG. 1005.

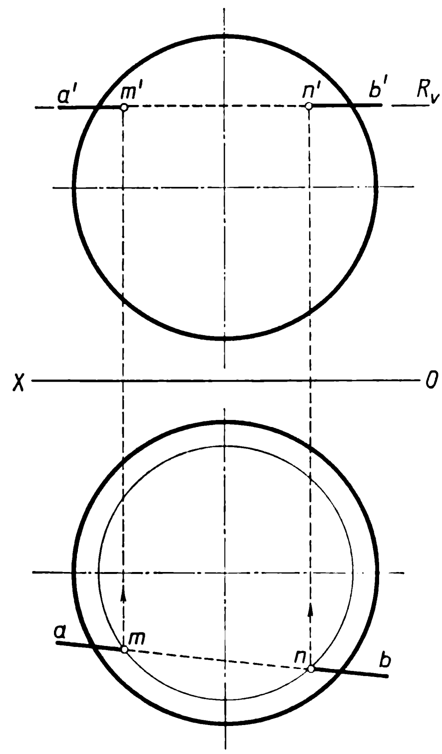



FIG. 1006.

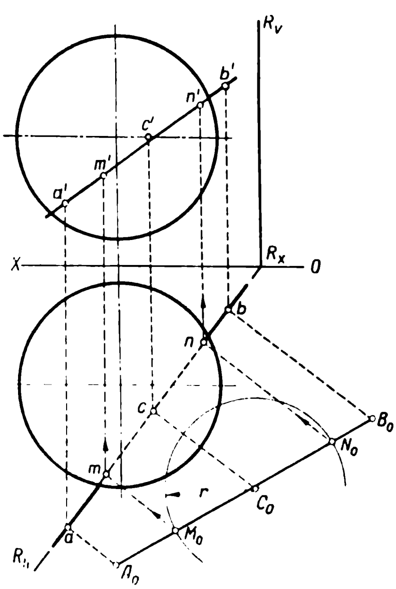



FIG. 1007.

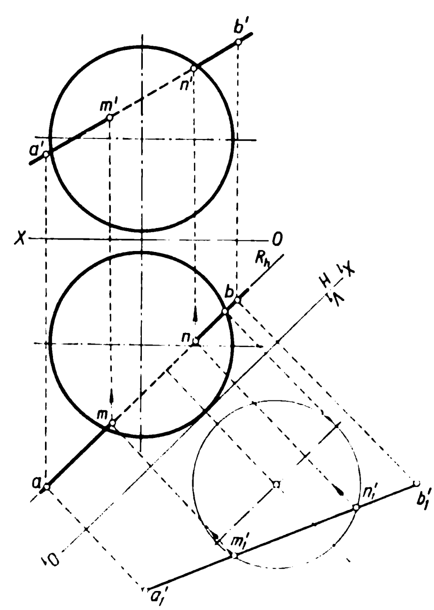



FIG. 1008.

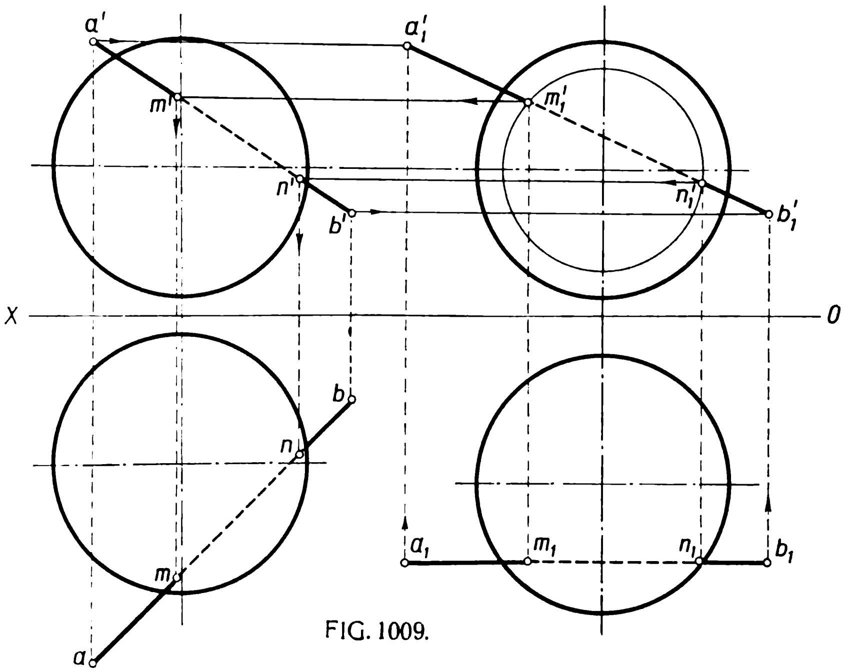



FIG. 1009.

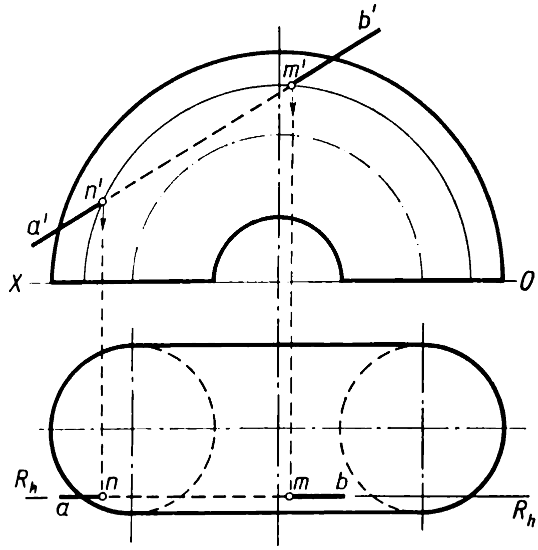



FIG. 1010.

points are found at the intersection of the vertical projections of the circle and the given line. From $m'$ and $n'$ find the points $m$ and $n$ on the line $ab$. (The straight line $AB$ could be placed in a vertical projecting plane, but this would greatly complicate the solution of the problem.)

**Example 331**

Find the points of intersection of the straight line $AB$ with a surface of revolution (Fig. 1011).

*Solution.* Through the line $AB$ pass a vertical projecting plane $R$ and find the projections of the line of intersection. The intersection of the horizontal projections of the obtained and given lines yields the points $m$ and $n$, which help us to find the points $m'$ and $n'$ on $a'b'$.

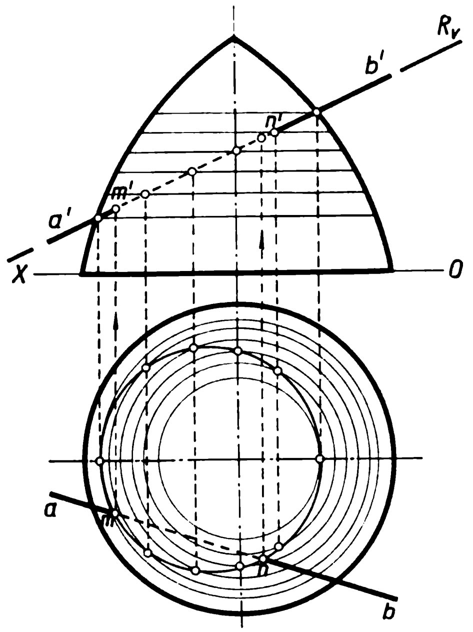


FIG. 1011.

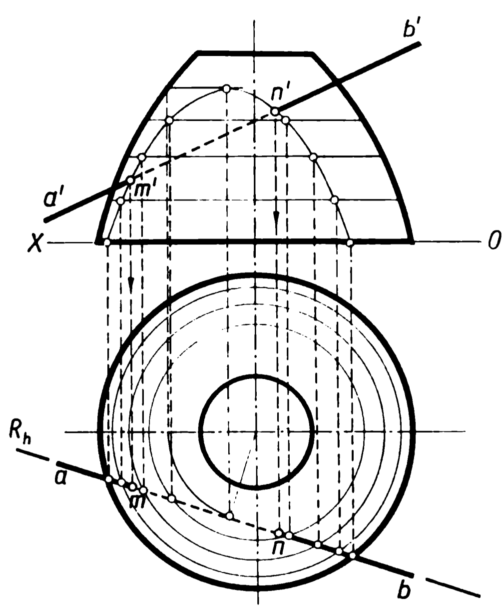


FIG. 1012.

Figure 1012 demonstrates how the problem can be solved by passing a horizontal projecting plane $R$ through $AB$.

**Example 332**

Given: a point $S$ and a straight line $AB$. Required: to draw through the point $S$ straight lines inclined at a given angle $\varphi$ to the $H$ plane and intersecting the line $AB$ (Fig. 1013).

*Solution.* The locus of straight lines in space passing through the point $S$ and inclined at the given angle $\varphi$ to the $H$ plane is the lateral surface of a right circular cone with vertex at $S$, whose generatrices make the same angle $\varphi$ with the $H$ plane. The desired lines are the generatrices passing through the points $(m, m')$ and $(n, n')$ of intersection of the surface of the cone with the given straight line (the construction is evident from the drawing).

**Example 333**

Given: a point $C$ and a straight line $AB$. Required: to find on $AB$ points distant 15 mm from the point $C$ (Fig. 1014).

*Solution.* The locus of points satisfying the given condition is a sphere of radius 15 mm with centre at $C$. The desired points are the points $(m, m')$ and $(n, n')$ of intersection of the sphere with the given straight line (the construction is obvious from the drawing).

PROBLEMS

489. Find the points of intersection of a straight line with the surface of a geometrical solid (prism, pyramid, cylinder, cone, sphere, etc. Figs. 1015 to 1034).

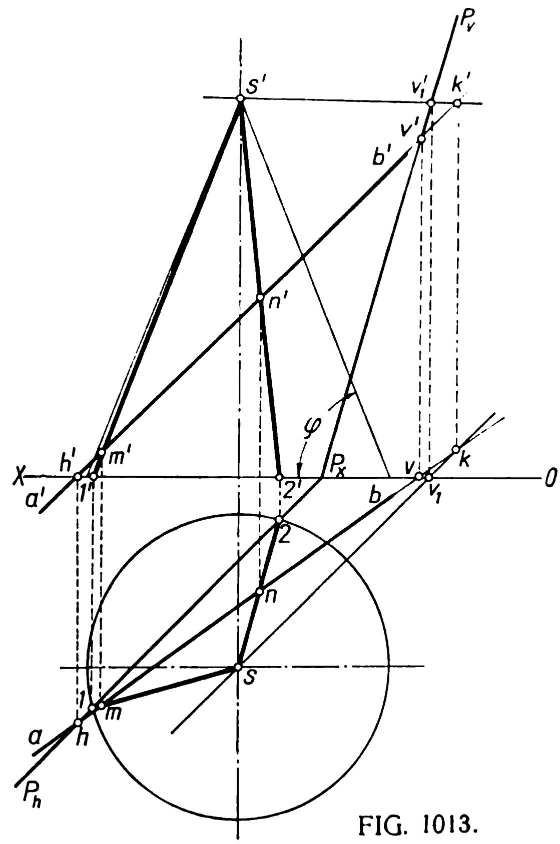



FIG. 1013.

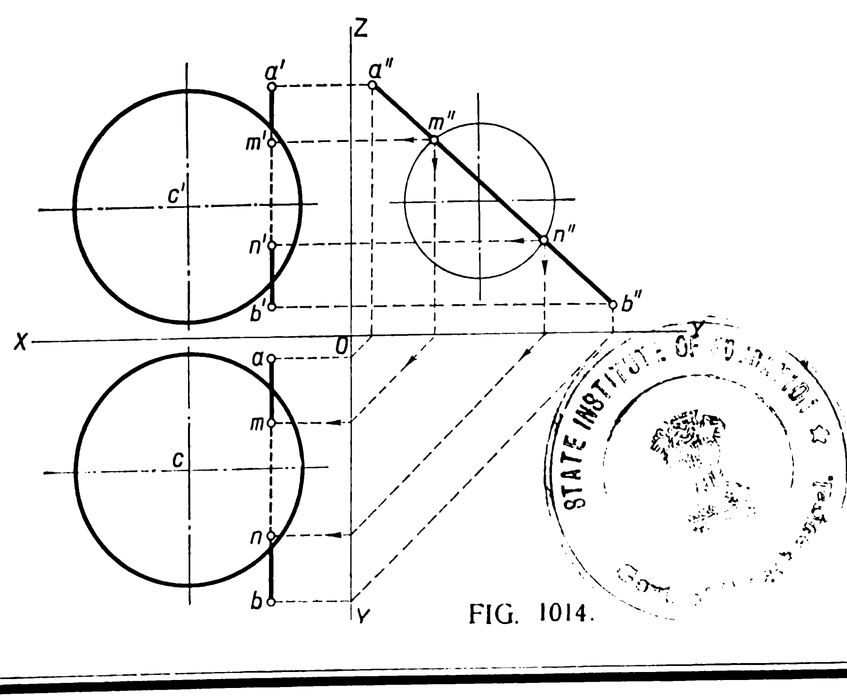



FIG. 1014.

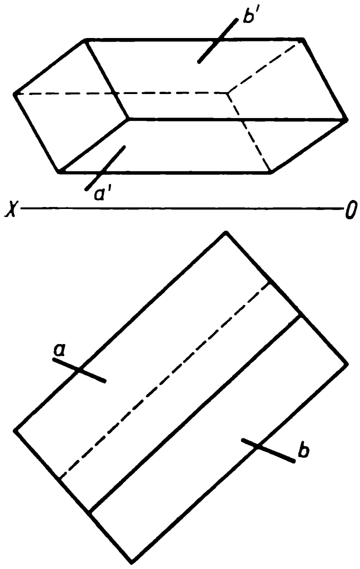



FIG. 1015.

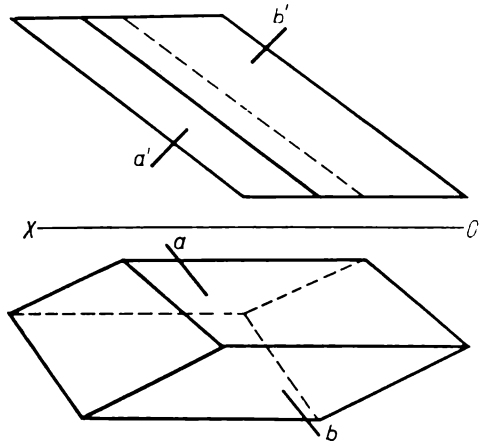



FIG. 1016.

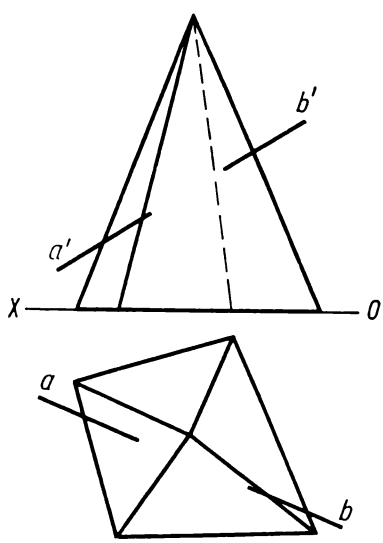



FIG. 1017.

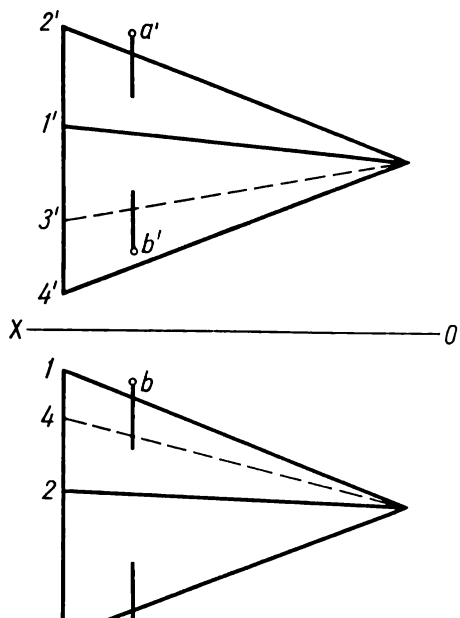



FIG. 1018.

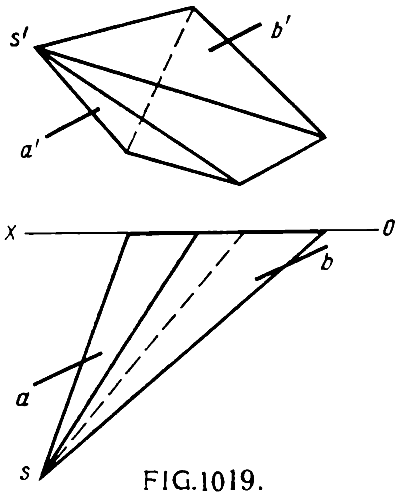


FIG. 1019.

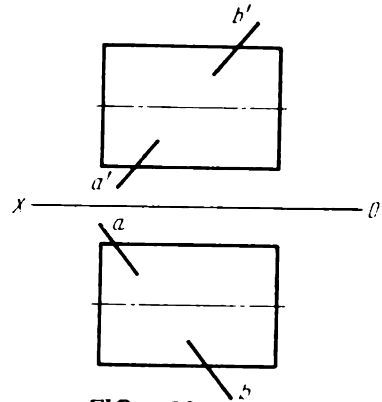


FIG. 1020.

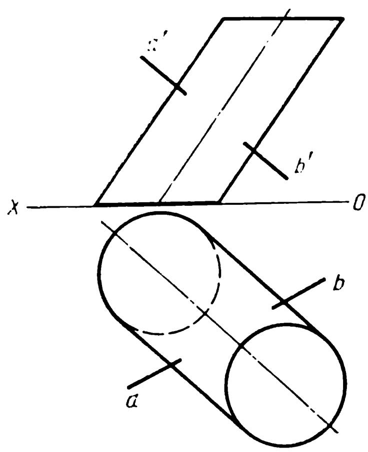


FIG. 1021.

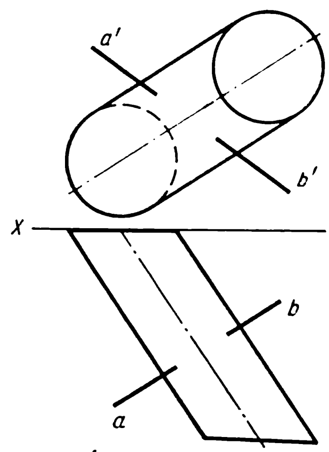


FIG. 1022.

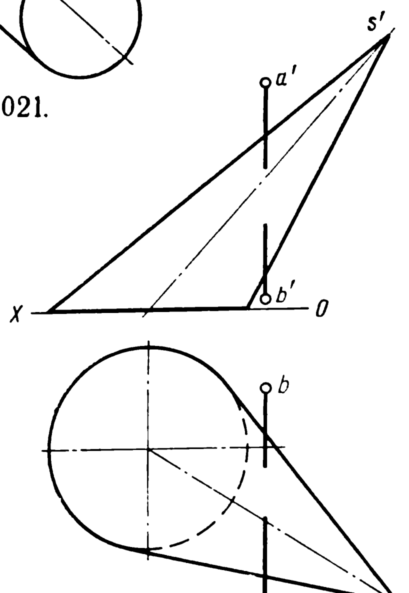


FIG. 1023.

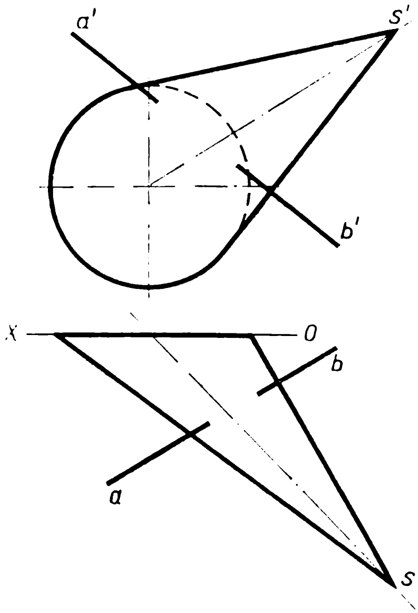



FIG. 1024.

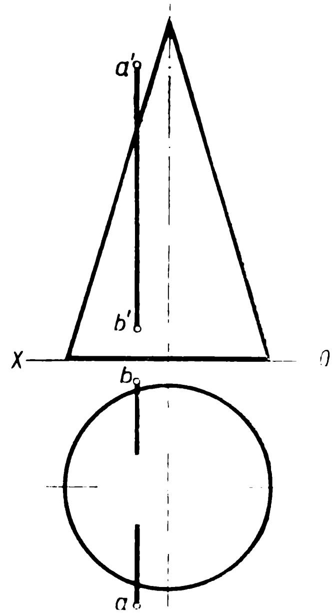



FIG. 1025.

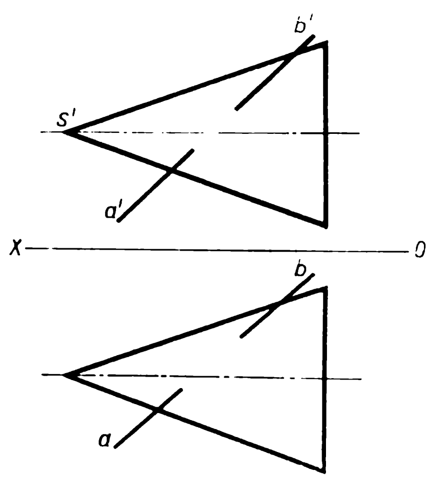



FIG. 1026.

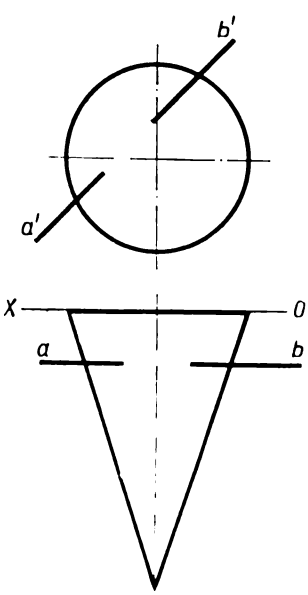



FIG. 1027.

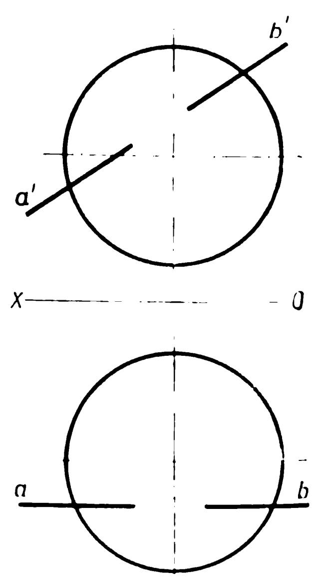



FIG. 1028.

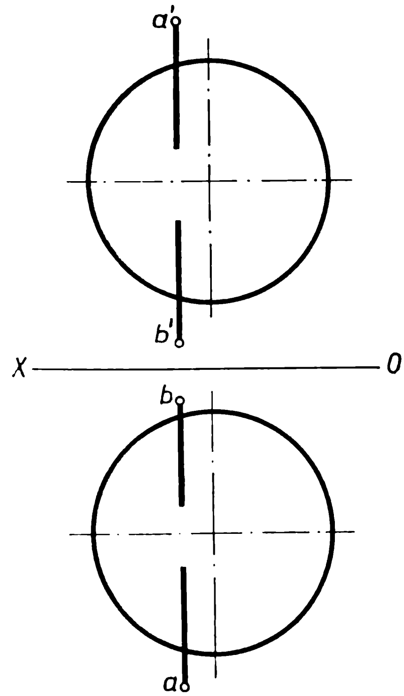



FIG. 1029.

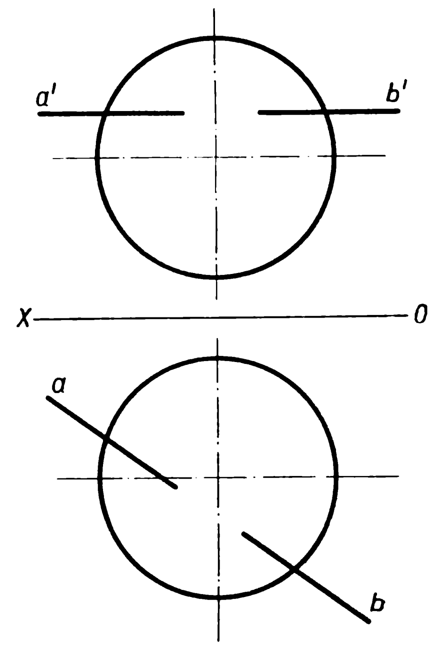



FIG. 1030.

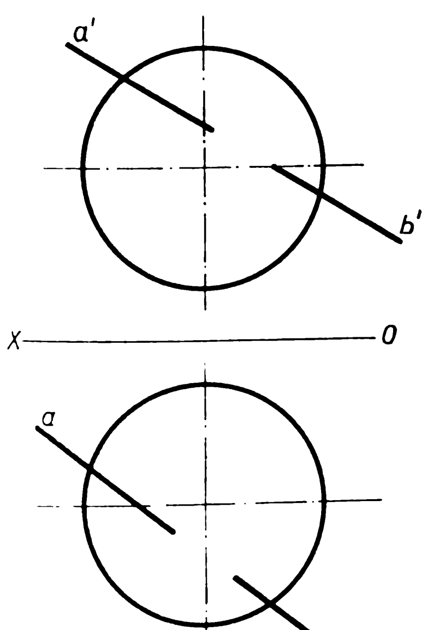



FIG. 1031.

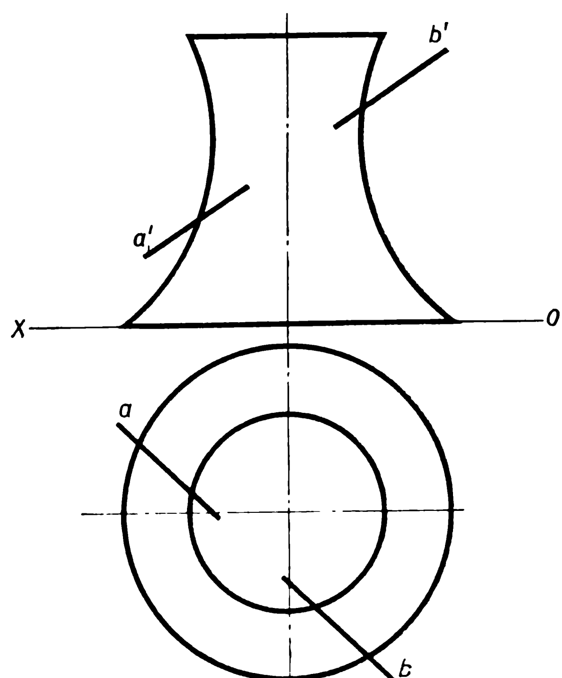



FIG. 1032.

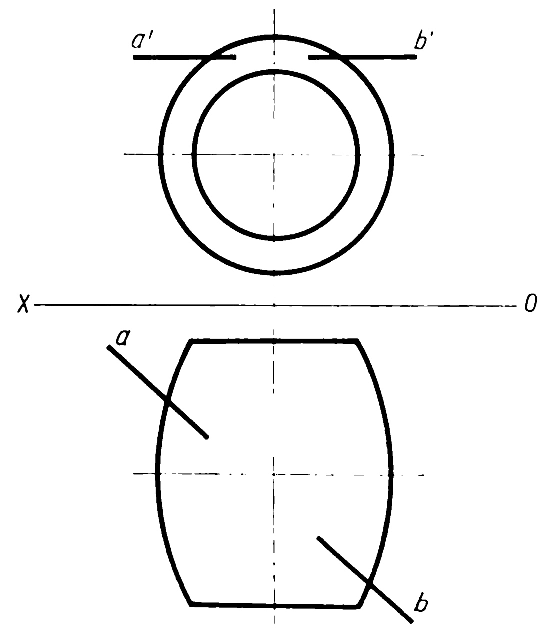



FIG. 1033.

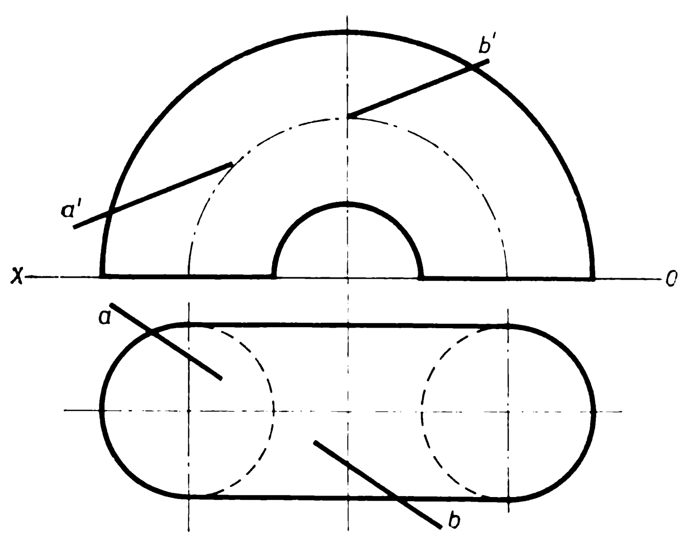



FIG.1034.

CHAPTER XXVI

# THE INTERSECTION OF SURFACES

To construct the line of intersection of two surfaces it is necessary to find a number of points common to both of them and then to join these points in a certain sequence.

Intersection lines are classified as:

(1) *space curves*, which are lines of intersection of two curved surfaces or a curved surface and a polyhedron;

(2) *polygonal lines in space*, which are lines of intersection of two polyhedrons.

Sometimes the line of intersection of two surfaces is a *plane* one: a straight line, a circle, an ellipse, etc. (see below).

To find an arbitrary point on an intersection line, proceed in the following sequence:

(1) introduce an auxiliary plane;

(2) determine the line of intersection of this plane with each of the two given surfaces;

(3) find the required points at the intersection of the lines obtained.

By introducing consecutively a number of auxiliary planes we can find the required number of points.

*Note.* An auxiliary plane should be chosen so that its line of intersection with each of the given surfaces is projected as a simple line—a straight line or a circle.

The line of intersection of two polyhedrons ($A$ and $B$) can alternatively be found in the following way:

(1) find the intersection points of the edges of *one* polyhedron ($A$) with the faces of the *other* ($B$);

(2) find the intersection points of the edges of the *other* polyhedron ($B$) with the faces of the first one ($A$);

(3) join these points with straight lines.

*Note.* Only those points lying on the same faces of *each* polyhedron must be joined.

If the generatrices of at least one of the surfaces are straight lines, the line of intersection can also be determined in the following manner: draw a number of generating lines on this surface and find their points of intersection with the other surface. Then join the points in a smooth curve.

Sometimes, in determining points of the line of intersection of two curved surfaces it is advisable to introduce a surface (not a plane)—cylindrical, conical or spherical.

Any surface of revolution intersects with a sphere in a *circle* provided the centre of the sphere *lies* on the axis of revolution.

## EXAMPLES

### Example 334

Find the line of intersection of two cylinders (Fig. 1035).

*Solution.* Introduce an auxiliary plane $R$ parallel to the $H$ plane and intersecting the vertical cylinder in a circle and the horizontal one along the generating lines; their intersection yields points ($1$, $1'$) and ($2$, $2'$). Likewise find some more points (see the figure). Join the points in a smooth curve to obtain the desired intersection line.

*Note.* In solving this problem we can make use of an auxiliary plane parallel to the $V$ plane.

### Example 335

Find the line of intersection of two cylinders (Fig. 1036).

*Solution.* Introduce an auxiliary plane $R$ parallel to the $V$ plane and intersecting the cylinders along the generating lines. Their intersection yields points ($1$, $1'$)

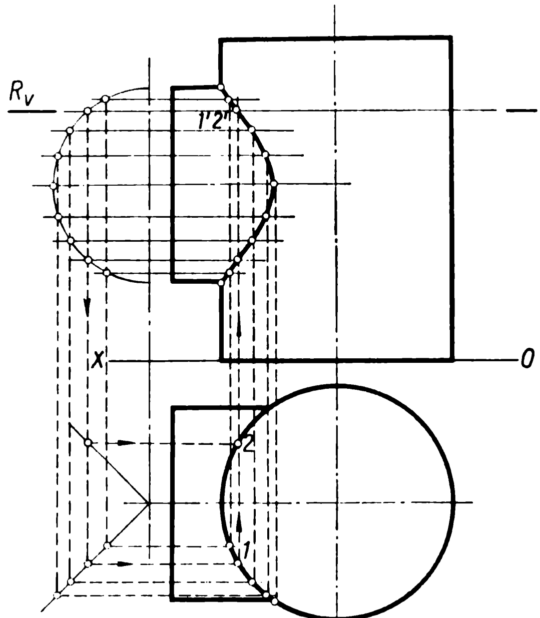



FIG. 1035.

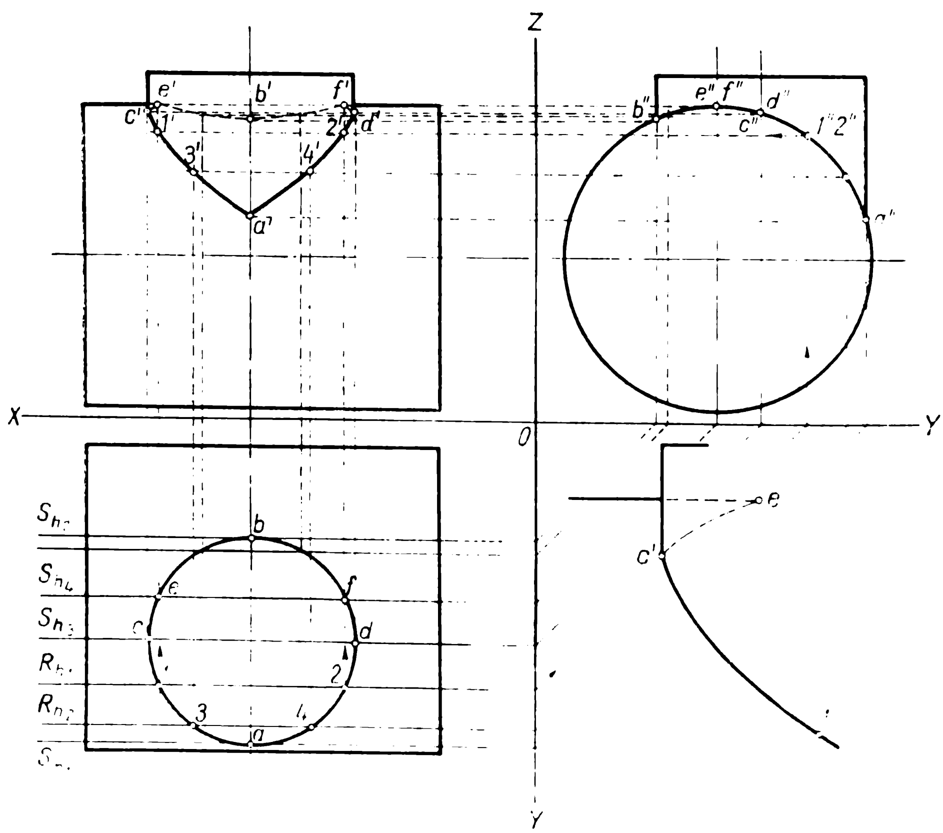



FIG. 1036.

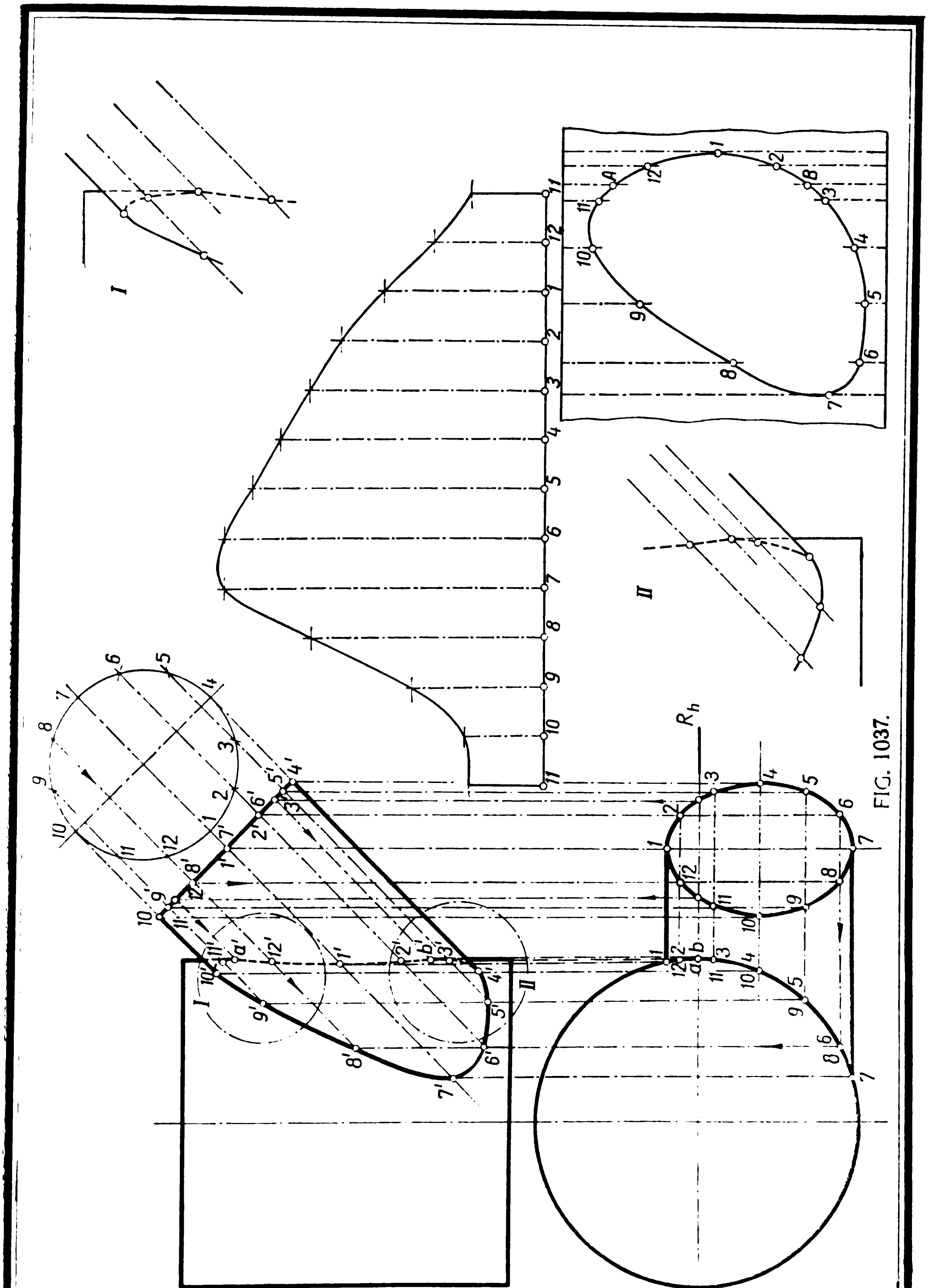



FIG. 1037.

and (*2*, *2'*). Likewise, find some more *arbitrary* points. Then determine the singular points $A$, $B$, $C$, $D$, $E$, $F$ with the aid of auxiliary planes $S_1$, $S_2$, $S_3$, and $S_4$ (see the accompanying drawing). Join the points in a smooth curve to obtain the desired intersection line.

*Note.* When constructing a line of intersection, try to determine those of its points which make it possible to give a fuller description of the desired curve.

**Example 336**

Find the line of intersection of two cylinders and construct developments of their lateral surfaces (Fig. 1037).

*Solution.* Divide the base of the oblique cylinder into twelve equal parts, draw the generating lines through the division points, and find their points of intersection with the surface of the vertical cylinder. Then introduce an auxiliary plane $R$ parallel to the $V$ plane and passing through the axis of the vertical cylinder. This plane intersects the cylinders along the generatrices whose intersection yields the singular points ($a$, $a'$) and ($b$, $b'$). Join all these points in a smooth curve to obtain the desired line of intersection.

Developments of the lateral surfaces of the cylinders are constructed according to the general rule (construction is obvious from the drawing).

**Example 337**

Find the line of intersection of a cylinder with a cone (Fig. 1038).

*Solution.* Introduce an auxiliary plane $R$ parallel to the $W$ plane and intersecting the cone in a circle and the cylinder along its generating lines. Their intersection yields points (*1*, *1'*) and (*2*, *2'*). Find other arbitrary points in a similar fashion. Then determine the singular points $A$, $B$; $C$, $D$; $E$, $F$ with the aid of auxiliary planes $P$, $Q$, $S$ (not shown in the drawing). Join all these points in a smooth curve to obtain the required line.

The use of an auxiliary plane parallel to the $H$ or $V$ plane greatly complicates the solution of the problem (why?).

**Example 338**

Find the line of intersection of a cylinder with a cone and construct developments of their lateral surfaces (Fig. 1039).

*Solution.* Divide the base of the cylinder into twelve equal parts, draw the generating lines through the division points and find their points of intersection with the cone. Join these points in a smooth curve to obtain the desired line.

Developments of the lateral surfaces of the cylinder and cone are constructed according to the general rule (construction is evident from the drawing).

**Example 339**

Find the line of intersection of a cylinder with a cone (Fig. 1040).

*Solution.* Draw twelve generating lines on the surface of the cone and find their points of intersection with the cylinder. Then find characteristic points $A$ and $B$. Join all these points in a smooth curve to obtain the required line (construction is evident from the drawing).

**Example 340**

Find the line of intersection of a cylinder and a cone by using spheres (Fig. 1041).

*Solution.* Take the point ($c$, $c'$) of intersection of the axes of the given solids as the centre of auxiliary spheres. Describe, from the point ($c$, $c'$) as centre, a sphere of *arbitrary* radius $R$ intersecting each of the given surfaces in a circle. Their vertical projections are straight lines, while the horizontal projections are ellipses (the latter are not needed for solving the problem and that is why they are not constructed). The intersection of the straight lines yields the vertical projections (*1'*) and (*2'*) of the points. Then find the horizontal projections (*1*) and (*2*) of the points with the aid of auxiliary generating lines (or circles). Likewise obtain a few more points, each time

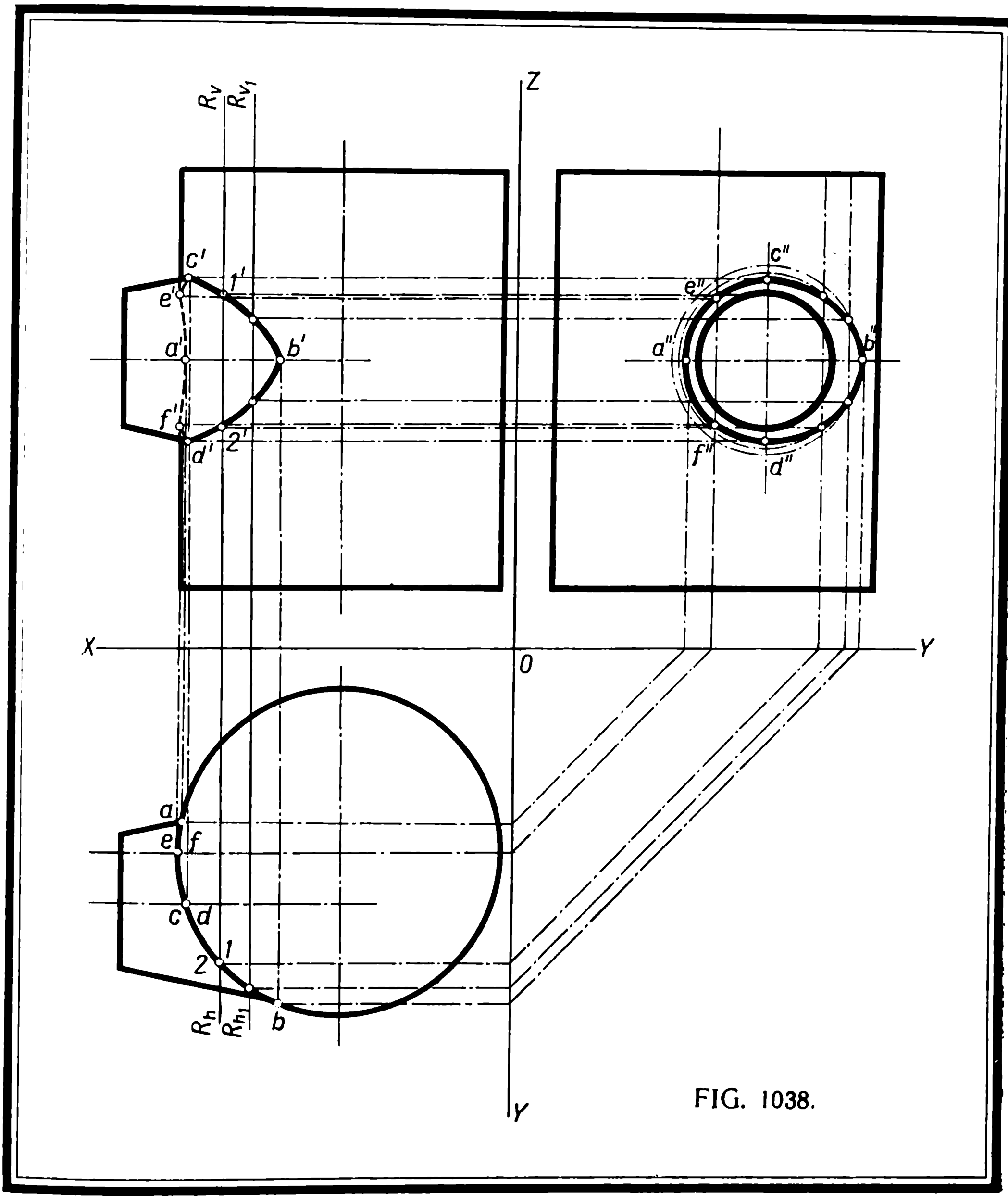



FIG. 1038.

changing only the radius of the sphere. Join these points in a smooth curve to obtain the desired line of intersection.

The intersection line can be determined with the aid of auxiliary planes as well. To avoid using a French curve (for what kind of curves?) pass the planes through the vertex of the cone and parallel to the axis of the cylinder. This method greatly complicates the solution of the problem.

*Notes*:

1. If the axes of intersecting surfaces are not parallel to a projection plane, it is expedient first to bring them to such a position (why?).

2. If the axes of cylinder and cone do not intersect, make use of auxiliary planes.

3. If the cylinder and cone are not surfaces of revolution, we still use auxiliary planes, irrespective of the relative positions of their axes.

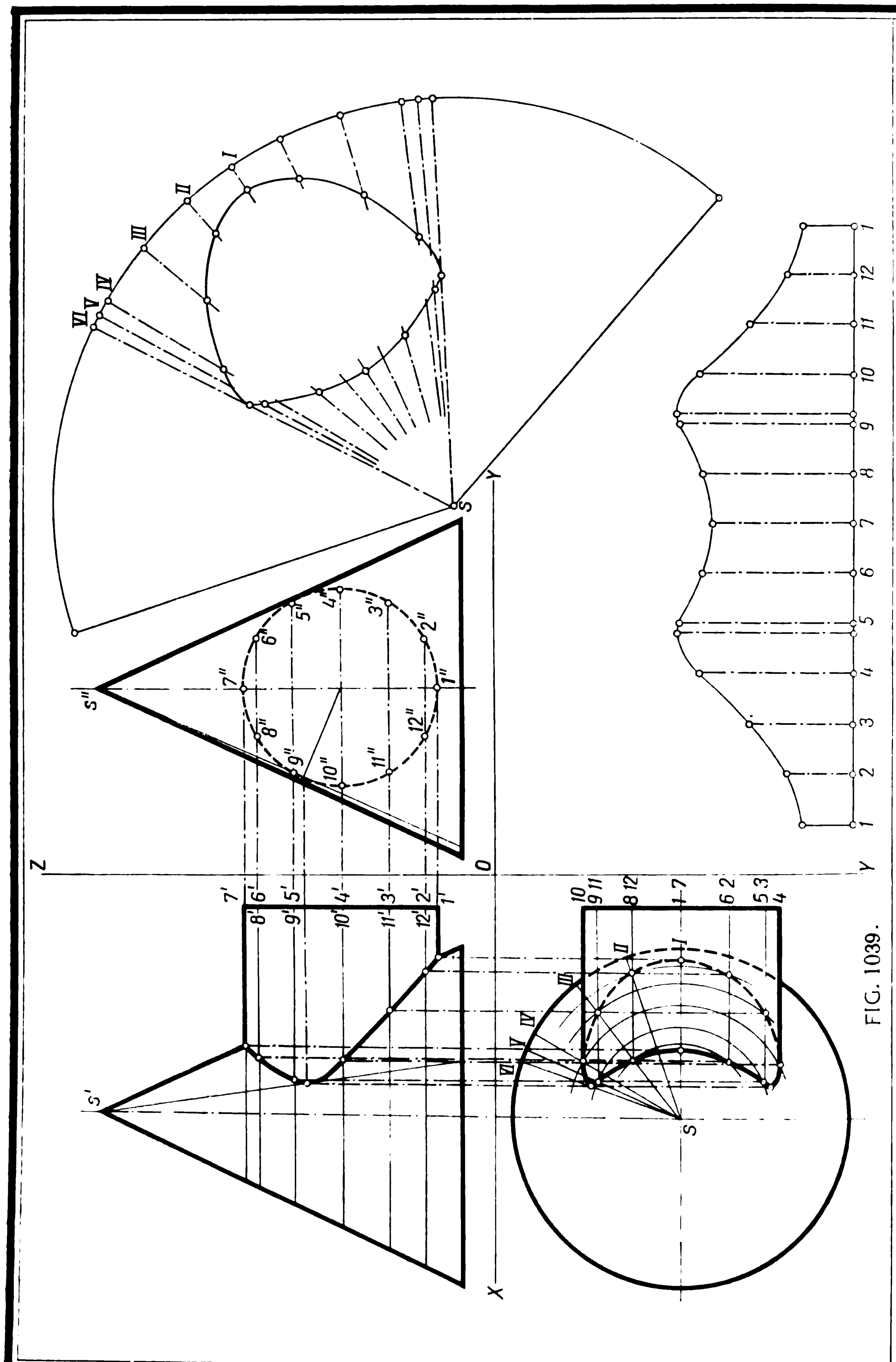



FIG. 1039.

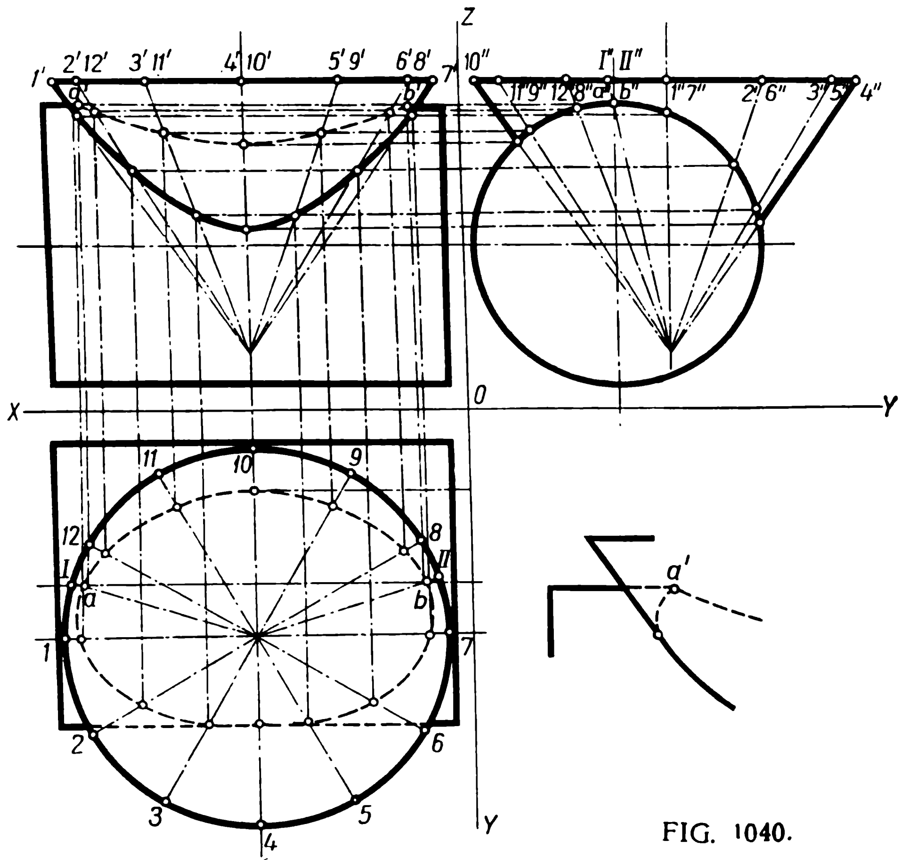



FIG. 1040.

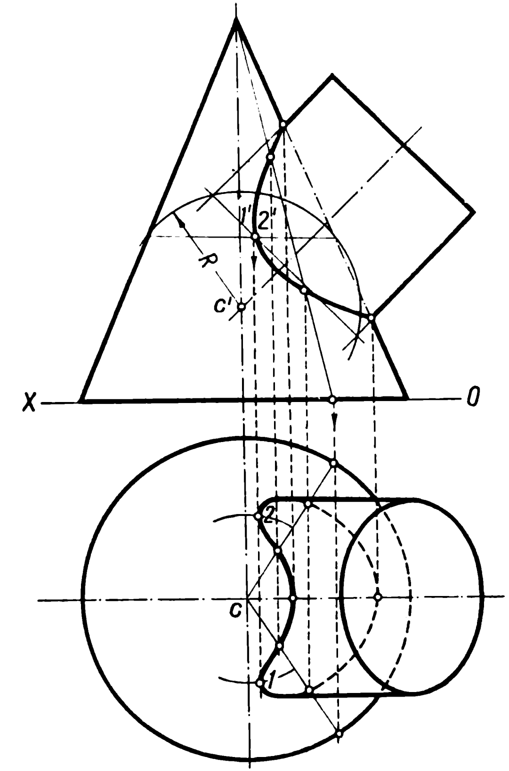



FIG. 1041.

**Example 341**

Find the line of intersection of two cones (Fig. 1042).

*Solution.* Since the axes of the given cones intersect at the point ($c$, $c'$), take this point for the centre of auxiliary spheres. From the point ($c$, $c'$) describe a sphere of arbitrary radius $R$ to intersect each given surface in a circle. The horizontal projections of these circles are straight lines, the vertical projections are ellipses (the latter are not needed for solving the problem and that is why they are not constructed).

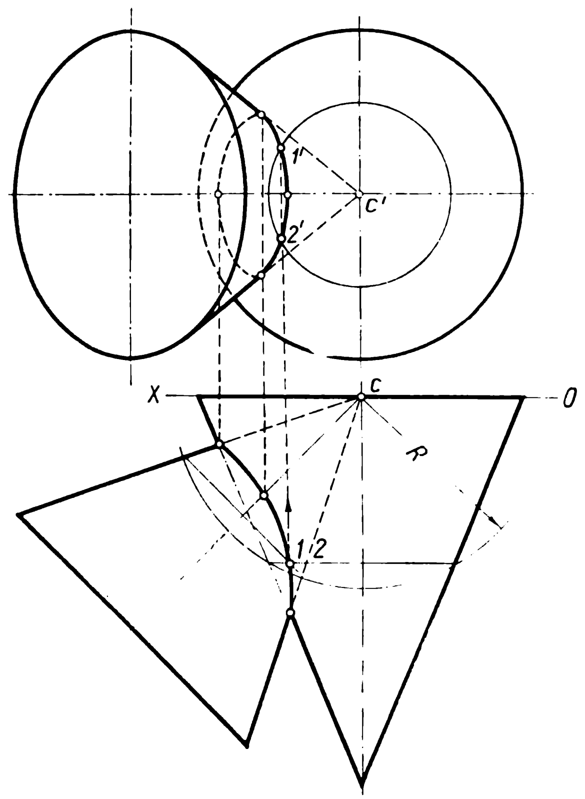



FIG. 1042.

The intersection of the lines yields the horizontal projections (*1*) and (*2*) of the points, from which we find the vertical projections (*1'*) and (*2'*) of the points by making use of auxiliary generating lines (or circles). In similar fashion obtain a few more points, each time changing only the radius of the sphere. Join the points in a smooth curve to obtain the desired line of intersection.

The intersection line can be determined with the aid of auxiliary planes as well. To avoid using a French curve (for what kind of curves?), pass the planes through the vertices of the cones. This method greatly complicates the solution of the problem.

*Notes*:

1. If the axes of two cones do not intersect, make use of auxiliary planes.
2. If the cones are not surfaces of revolution, also use auxiliary planes, irrespective of the relative positions of their axes.

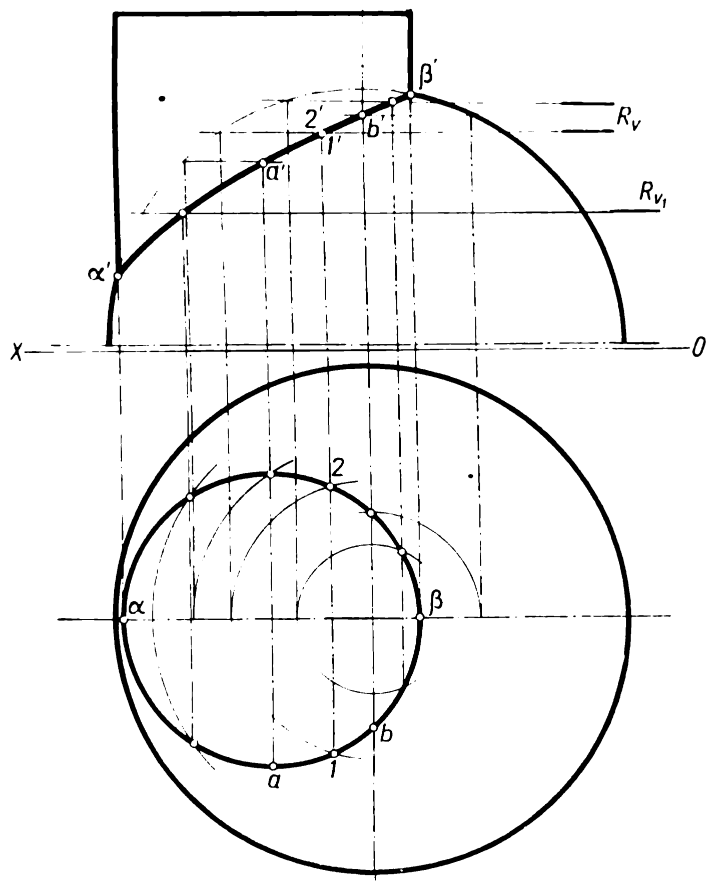



FIG. 1043.

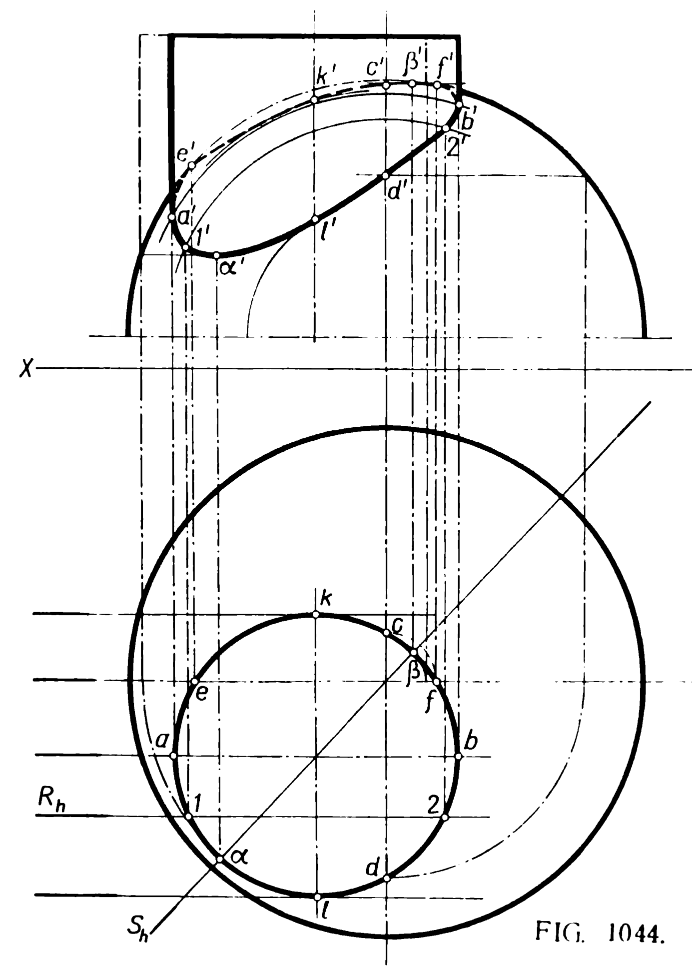



FIG. 1044.

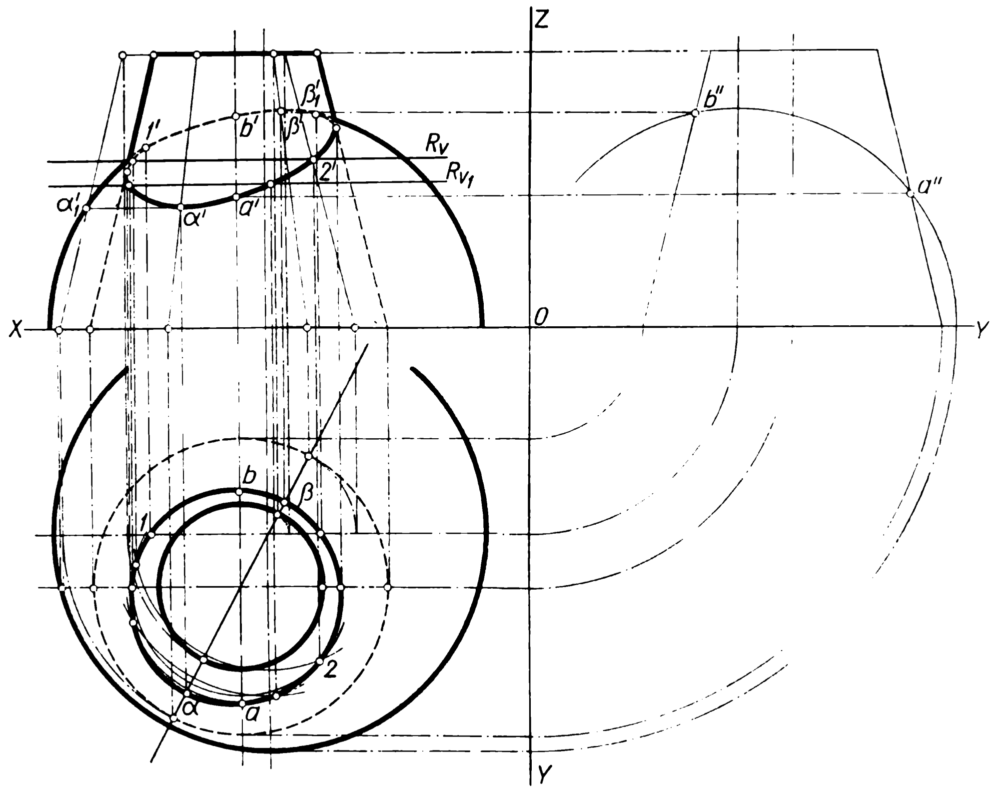



FIG. 1045.

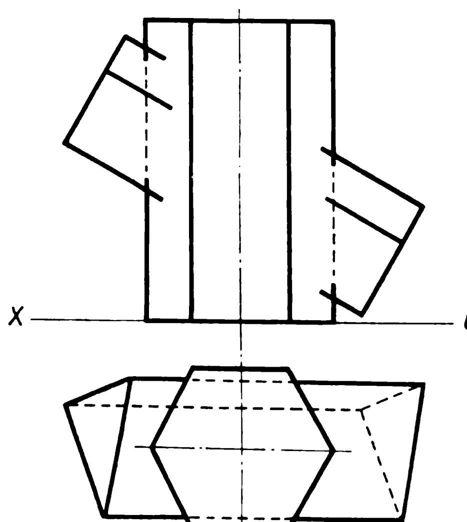



FIG. 1046.

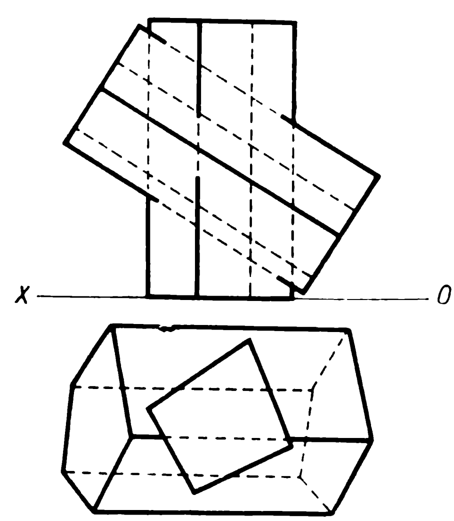



FIG. 1047.

**Example 342**

Find the line of intersection of a cylinder and a sphere (Fig. 1043).

*Solution.* Introduce an auxiliary plane $R$ parallel to the $H$ plane and intersecting both surfaces in circles. Their intersection yields points ($1$, $1'$) ($2$, $2'$). In similar fashion determine a few more auxiliary points. Then find the characteristic points: the lowest ($\alpha$, $\alpha'$) and highest ($\beta$, $\beta'$), then ($a$, $a'$) and ($b$, $b'$) (see the drawing). Join all the points in a smooth curve to obtain the desired line of intersection.

We can also use an auxiliary plane parallel to the $V$ plane.

**Example 343**

Find the line of intersection of a cylinder and a sphere (Fig. 1044).

*Solution.* Introduce an auxiliary plane $R$ parallel to the $V$ plane and intersecting the sphere in a circle and the cylinder along the generating lines. Their intersection yields points ($1$, $1'$) and ($2$, $2'$). In similar fashion find a few more arbitrary points. Then determine the characteristic points $A$, $B$; $C$, $D$; $E$, $F$ (see the drawing). The lowest ($\alpha$, $\alpha'$) and the highest ($\beta$, $\beta'$) points are found at the intersection with the sphere of the generating lines contained in the horizontal projecting plane $S$ passing through the axis of the cylinder and the centre of the sphere. Join all the points in a smooth curve to obtain the desired line of intersection.

**Example 344**

Find the line of intersection of a cone and a sphere (Fig. 1045).

*Solution.* Introduce an auxiliary plane $R$ parallel to the $H$ plane and intersecting the given surfaces in circles. Their intersection yields points ($1$, $1'$) and ($2$, $2'$). In similar fashion find some other arbitrary points (see the drawing). To determine points ($\alpha$, $\alpha'$) and ($\beta$, $\beta'$) revolve the given surfaces about the axis of the sphere, which is perpendicular to the $H$ plane, through the angle $\varphi$ (see the drawing) and construct the vertical projection of the cone in a new position (the vertical projection of the sphere remains unchanged). Then, by revolving the system in the opposite direction, obtain the projections of the desired point ($\alpha$, $\alpha'$) and ($\beta$, $\beta'$). Now find the points ($a$, $a'$) and ($b$, $b'$) of intersection of the profile generatrices of the cone with the sphere (see the drawing). Join the points in a smooth curve to obtain the desired line of intersection.

The points ($\alpha$, $\alpha'$) and ($\beta$, $\beta'$) can also be found by replacing the planes of projection (how?); too, the points ($a$, $a'$) and ($b$, $b'$) can be determined without using the profile plane of projection (how?).

PROBLEMS

490. Construct the line of intersection of the following pairs of surfaces:
prism and prism (Figs. 1046 to 1048);
prism and pyramid (Figs. 1049 to 1051);
pyramid and pyramid (Figs. 1052 to 1054);
prism and cylinder (Figs. 1055 to 1057);
prism and sphere (Figs. 1058, 1059);
prism and cone (Figs. 1060 to 1062);
pyramid and cylinder (Figs. 1063, 1064);
pyramid and sphere (Fig. 1065);
pyramid and cone (Fig. 1066);
cylinder and cylinder (Figs. 1067 to 1080);
cylinder and cone (Figs. 1081 to 1094);
cylinder and sphere (Figs. 1095 to 1100);
cone and sphere (Figs. 1101 to 1107);
cone and cone (Figs. 1108 to 1110).
Construct developments of the above surfaces with the lines of intersection shown on them.

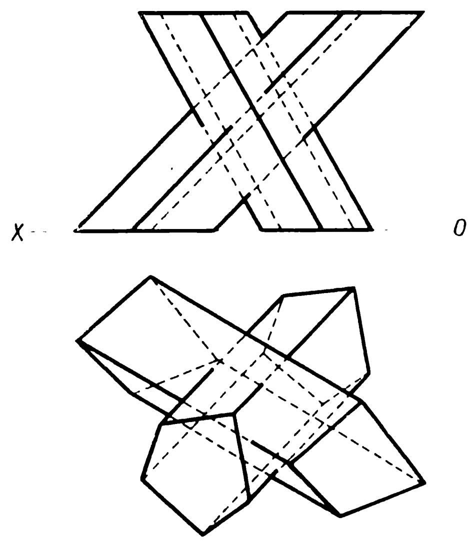


FIG. 1048.

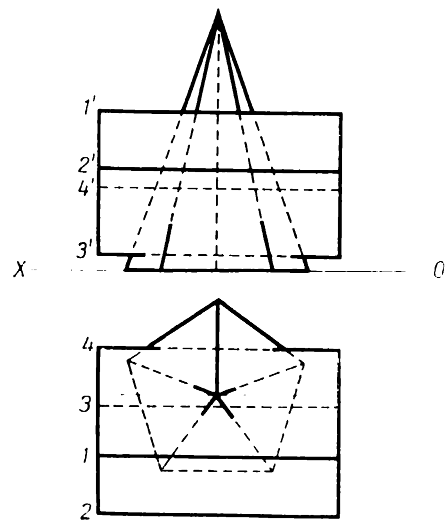


FIG. 1049.

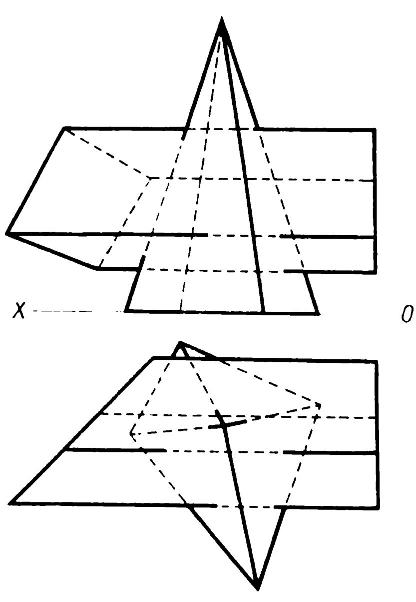


FIG. 1050.

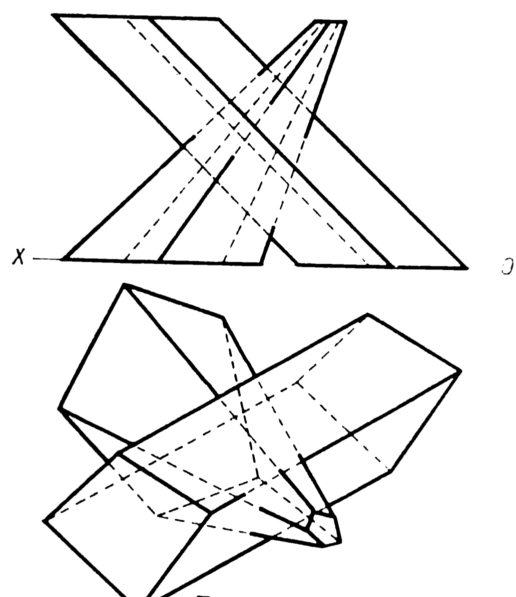


FIG. 1051.

491. Construct the projections of the line of intersection of a sphere with a right circular cylinder of radius $r = 20$ mm, whose axis coincides with a given straight line, and the centre of the upper base with the point ($c$, $c'$) (Fig. 1100a).

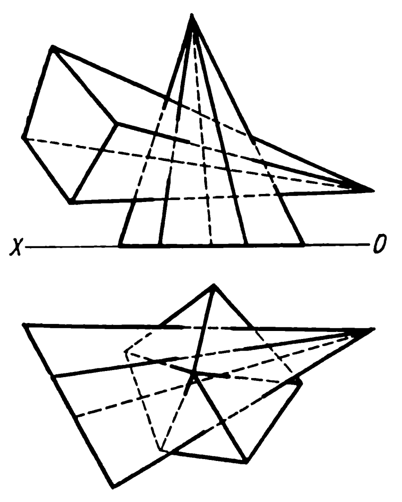



FIG. 1052.

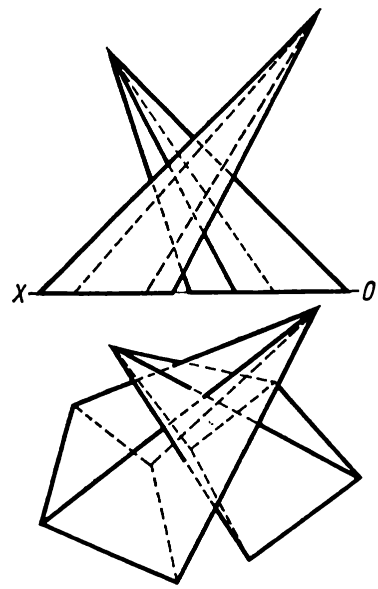



FIG. 1053.

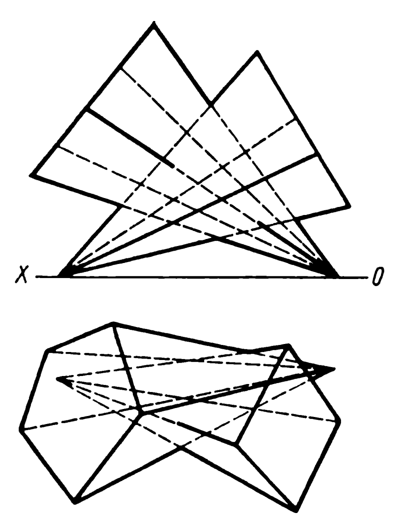



FIG. 1054.

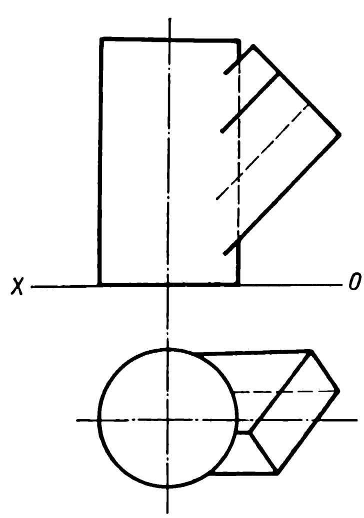



FIG. 1055.

492. Construct the projections of the line of intersection of a sphere with a right circular cone having the vertex at point $S$, given that the axis of the cone coincides with a given straight line, and the angle at its vertex is equal to 60°.

493. Using auxiliary spheres, construct the projections of the line of intersection of:

(1) two cylinders (Figs. 1067, 1069, 1072, 1074);

(2) a cylinder with a cone (Figs. 1079, 1081, 1083, 1086, 1087, 1090, 1091).

494. Construct the projections of the line of intersection of two cones using auxiliary spheres (Figs. 1108, 1109).

495. Construct the projections of the line of intersection of a toroid with a cylinder (Figs. 1111, 1112).

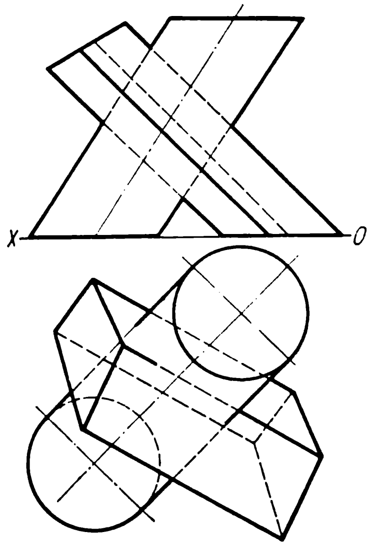



FIG. 1056.

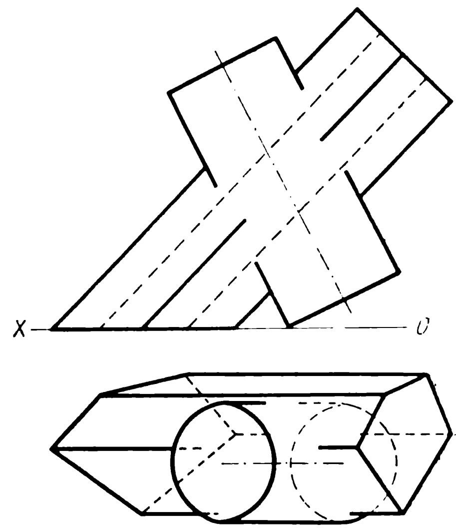



FIG. 1057.

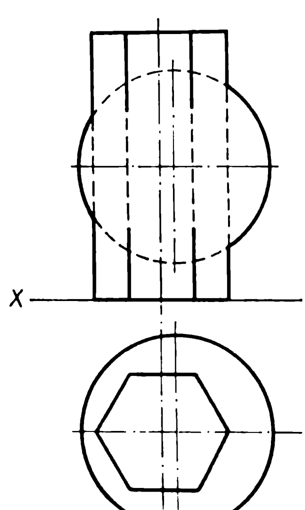



FIG. 1058.

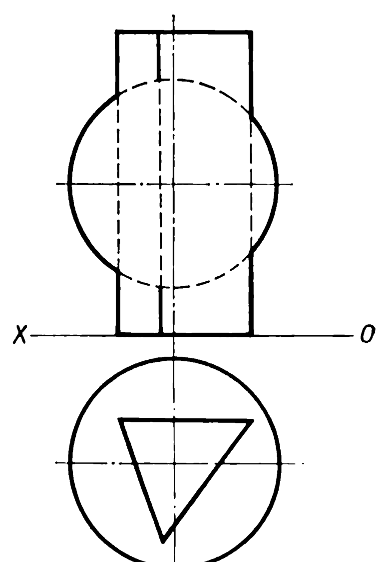



FIG. 1059.

496. Construct the lines of intersection of a toroid with a cone (Fig. 1113).

497. Construct the projections of the lines of intersection of a toroid with a cylinder ending in a hemisphere (Fig. 1114).

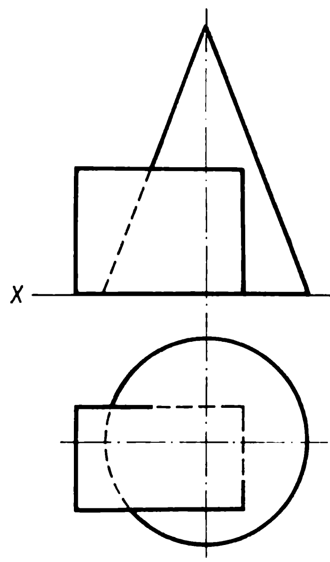


FIG. 1060.

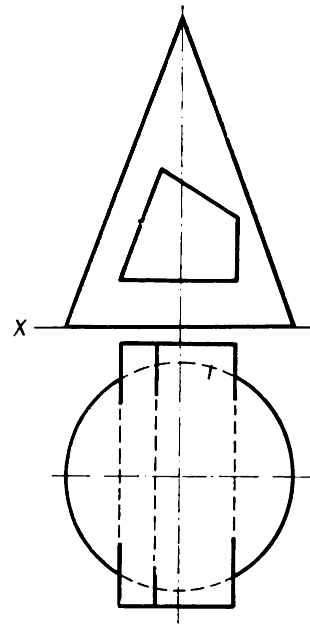


FIG. 1061.

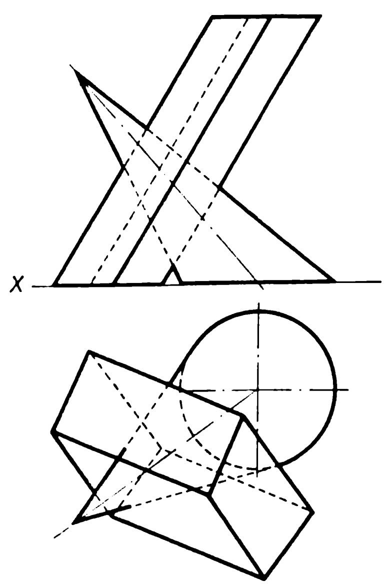


FIG. 1062.

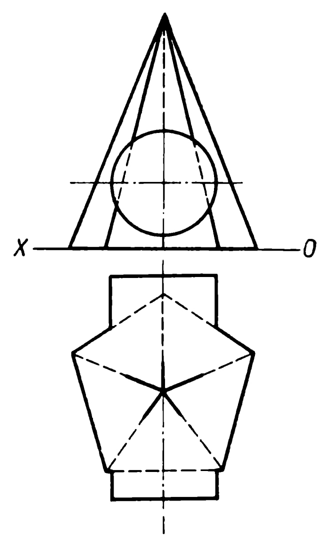


FIG. 1063.

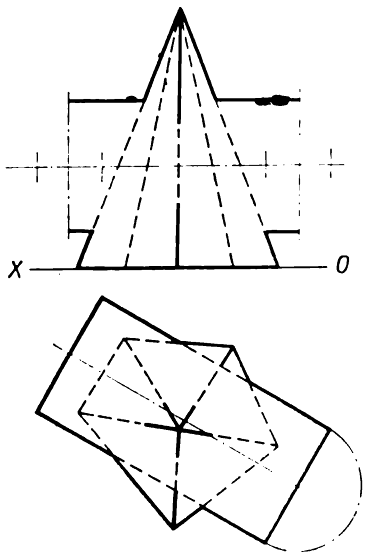



FIG. 1064.

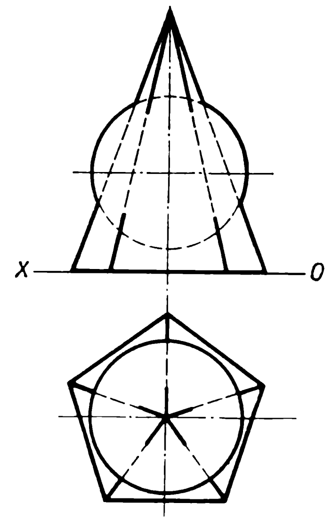



FIG. 1065.

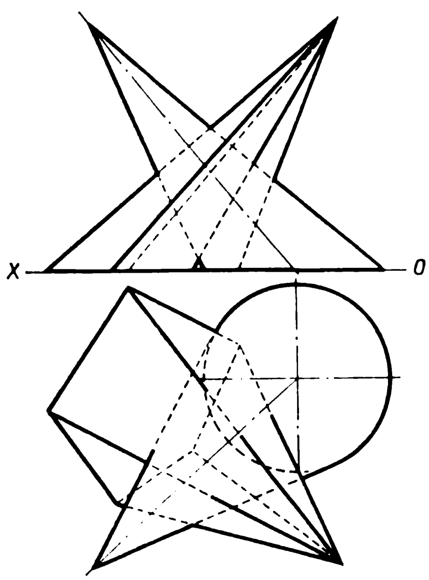



FIG. 1066.

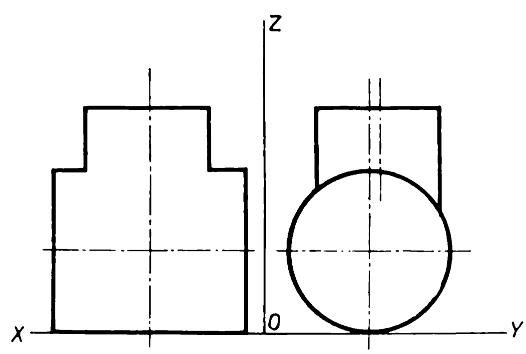



FIG. 1067.

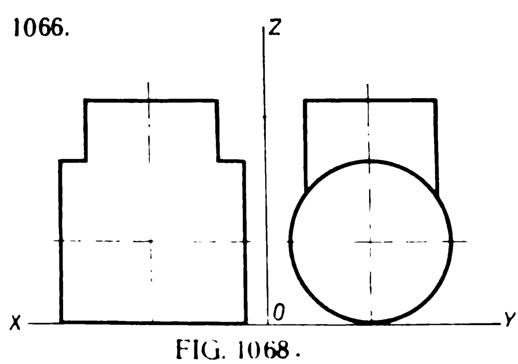



FIG. 1068.

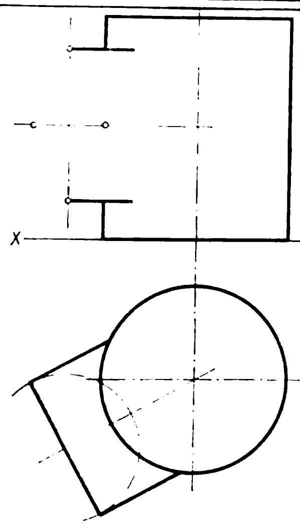



FIG. 1069.

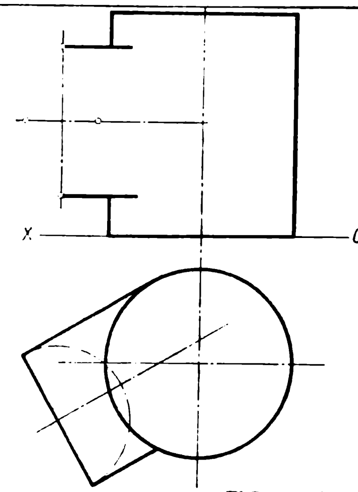



FIG. 1070.

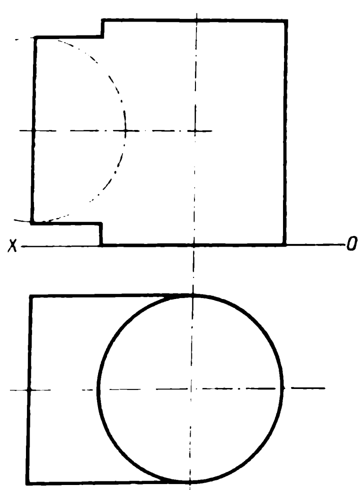



FIG 1071.

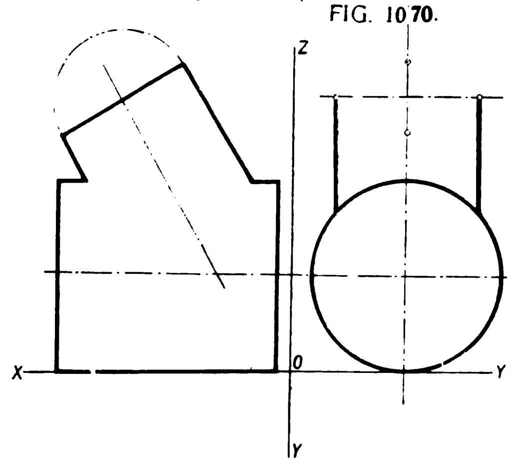



FIG. 1072.

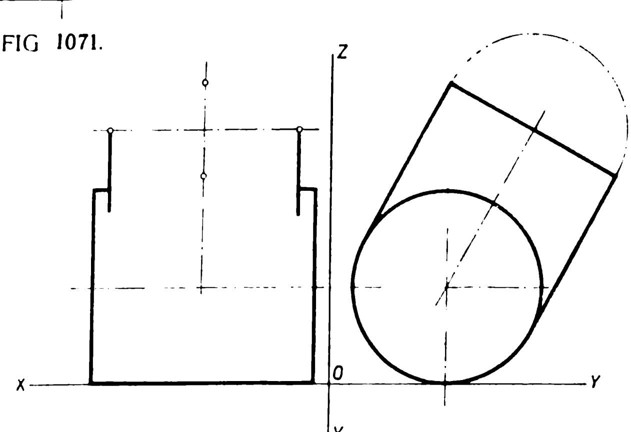



FIG. 1073.

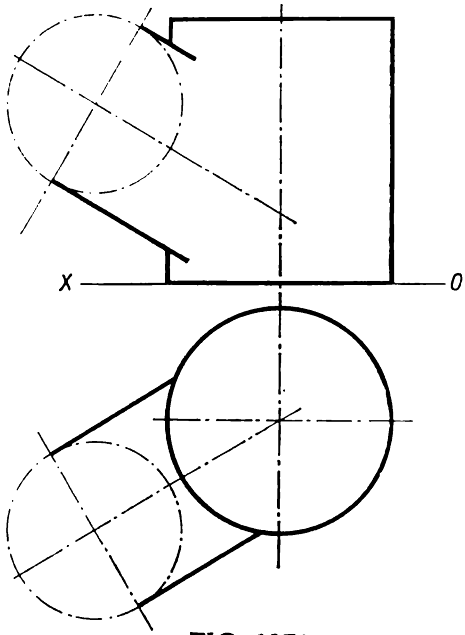



FIG. 1074.

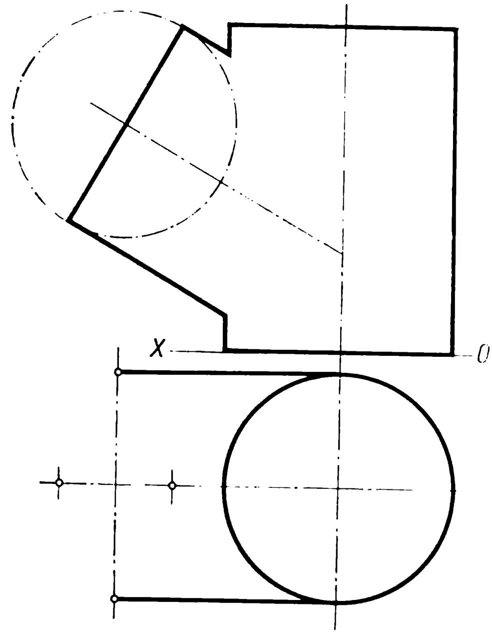



FIG. 1075.

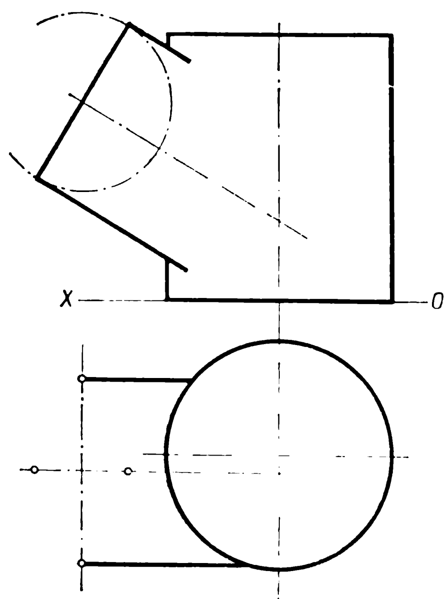



FIG. 1076.

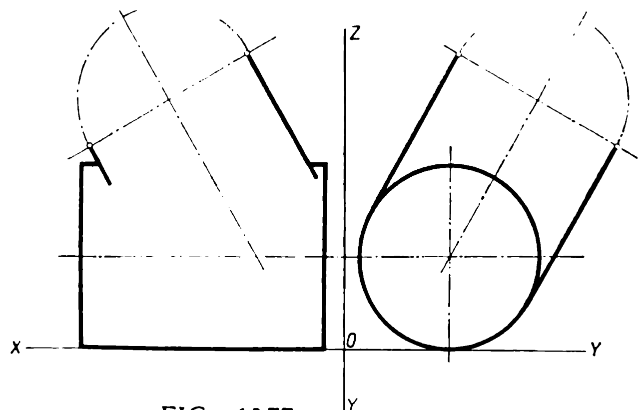



FIG. 1077.

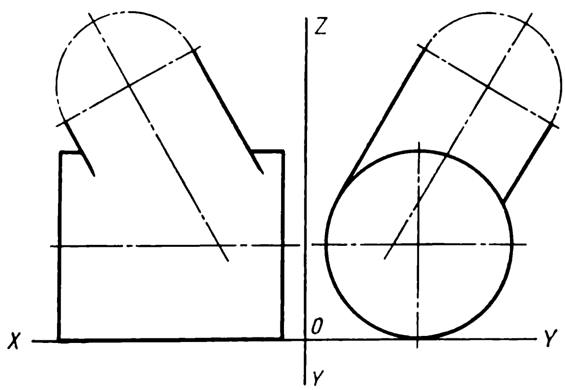



FIG. 1078.

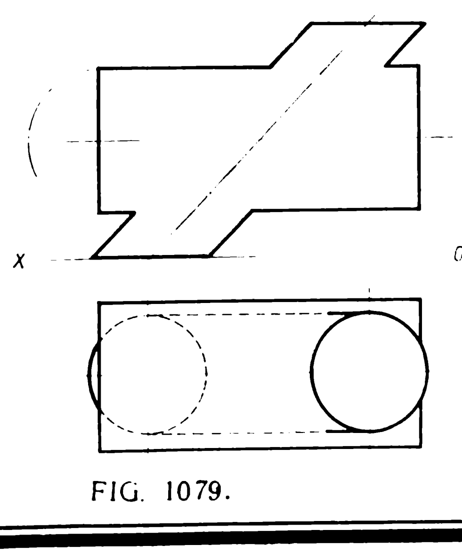



FIG. 1079.

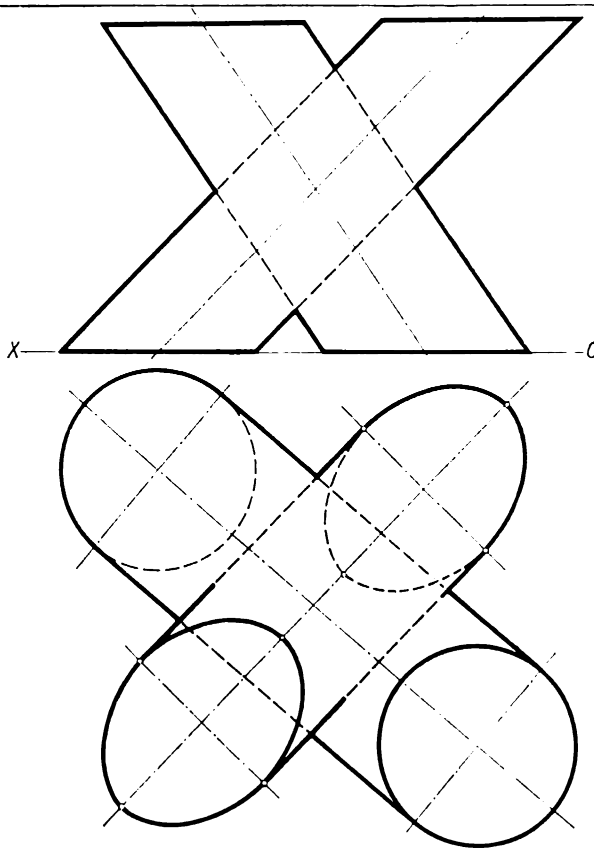



FIG. 1080.

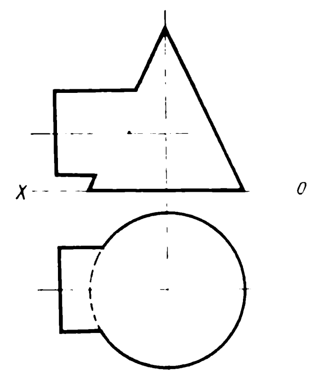



FIG. 1081.

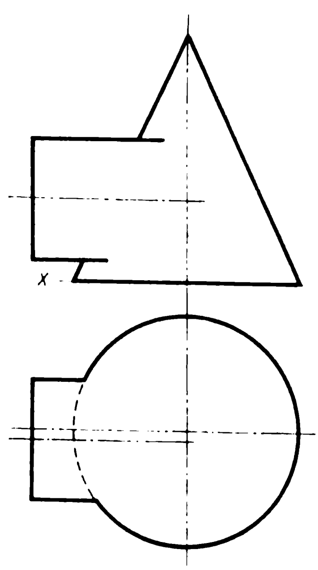



FIG 1082.

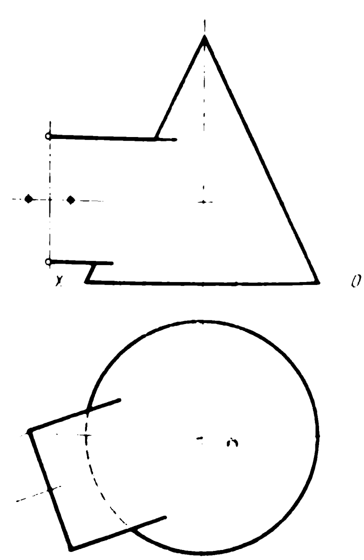



FIG. 1083.

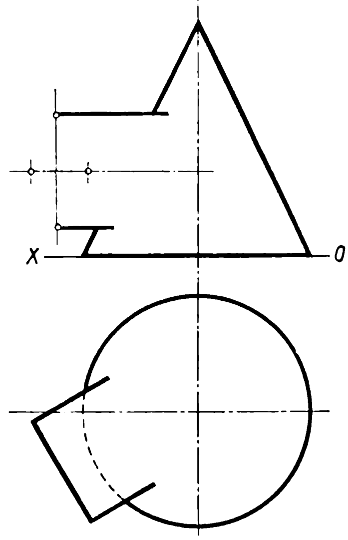



FIG. 1084.

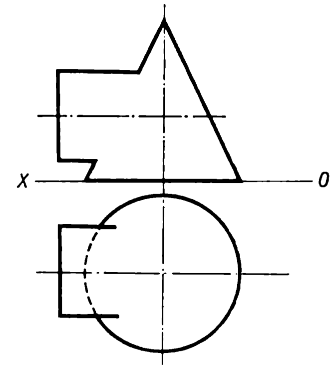



FIG. 1085

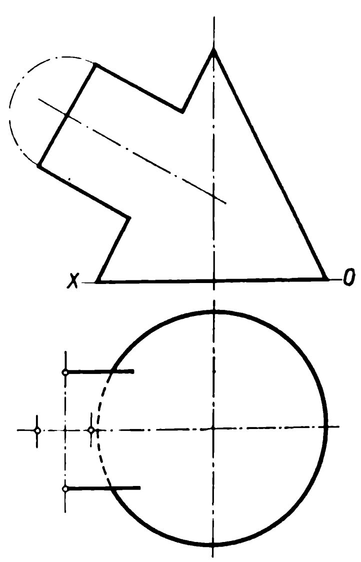



FIG. 1086.

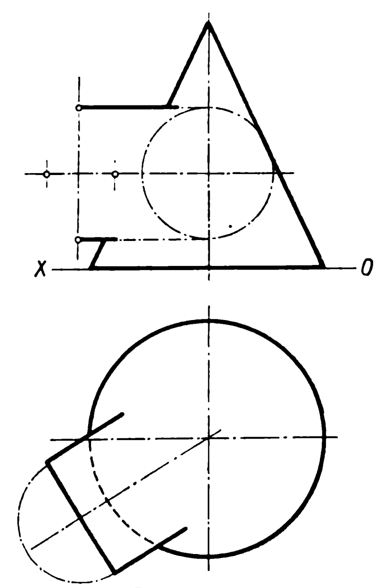



FIG. 1087.

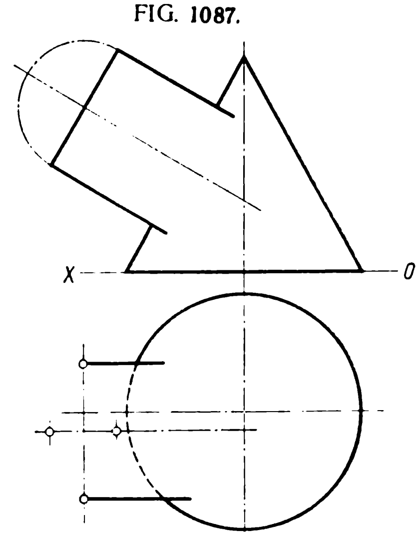



FIG. 1088.

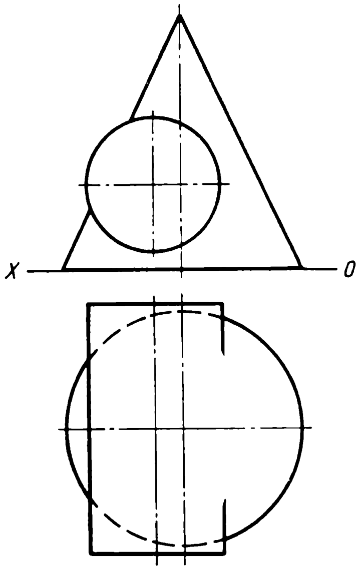



FIG. 1089.

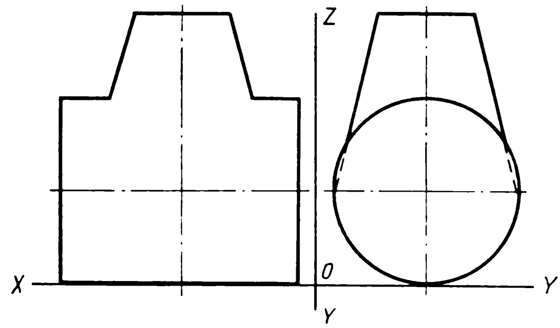



FIG. 1090.

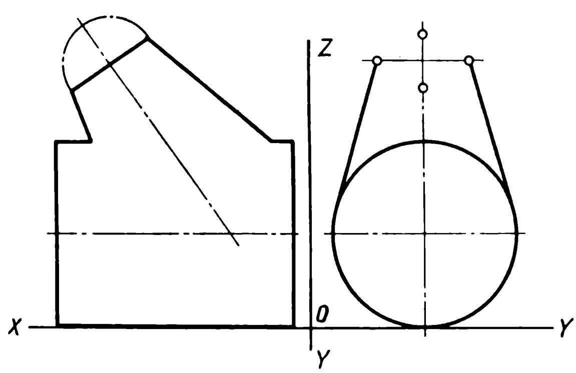



FIG. 1091.

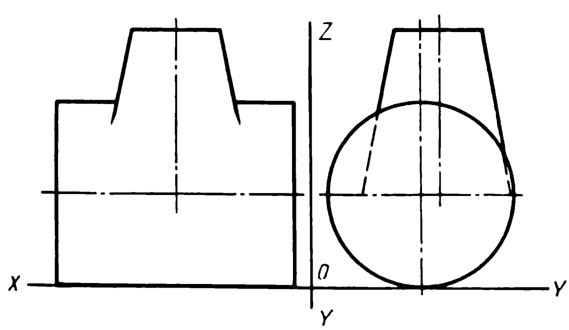



FIG. 1092.

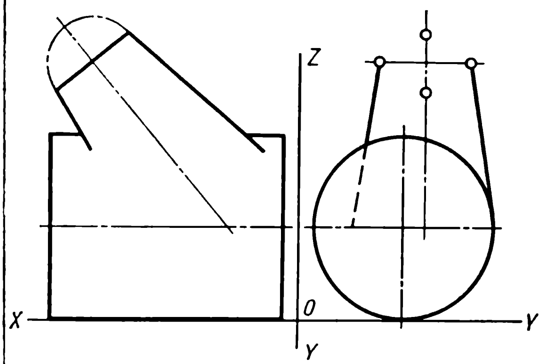


FIG.1093.

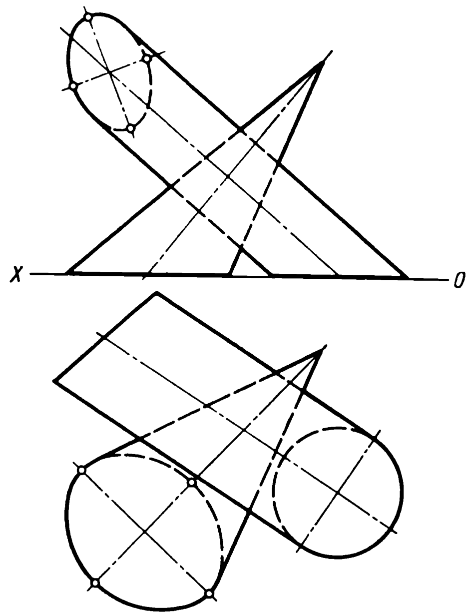


FIG.1094.

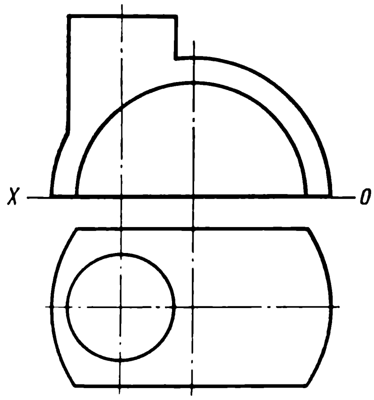


FIG. 1095.

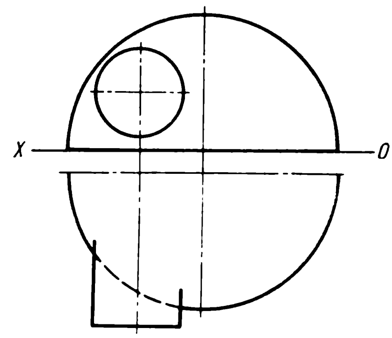


FIG. 1096.

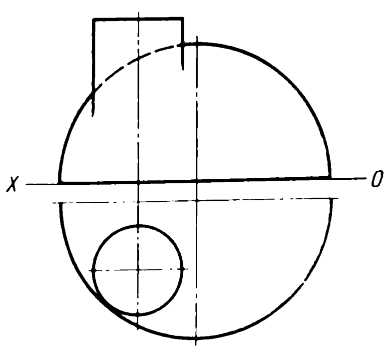


FIG.1097.

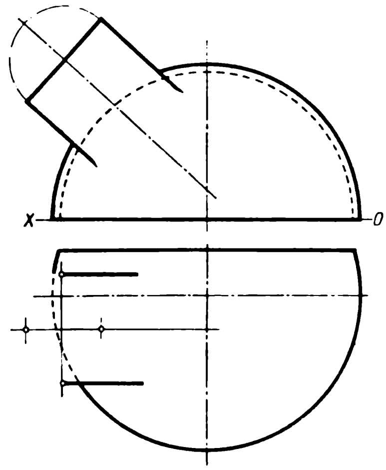



FIG. 1098.

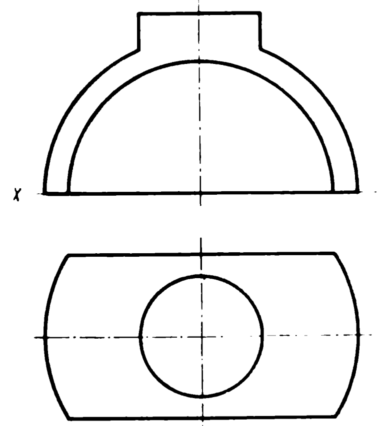



FIG. 1099.

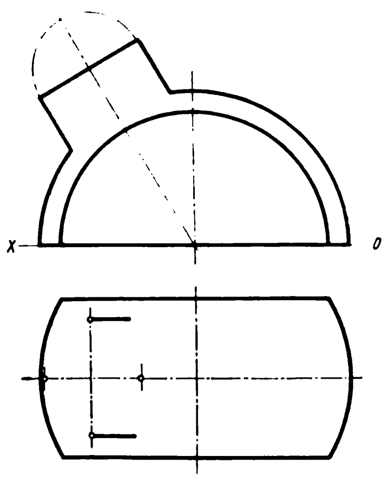



FIG. 1100.

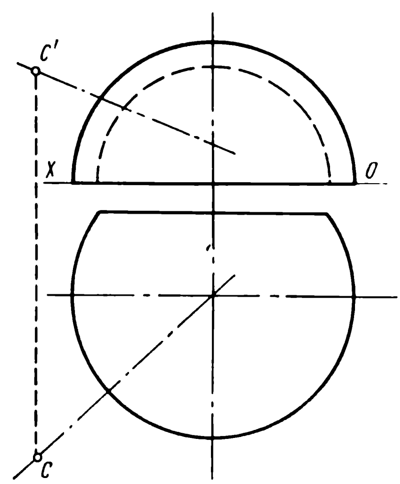



FIG. 1100a.

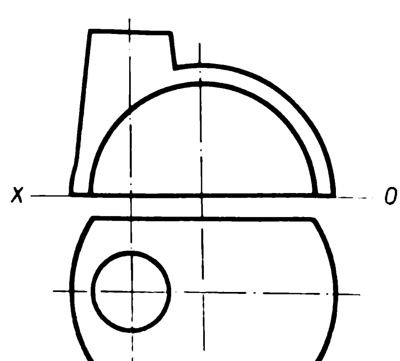



FIG. 1101.

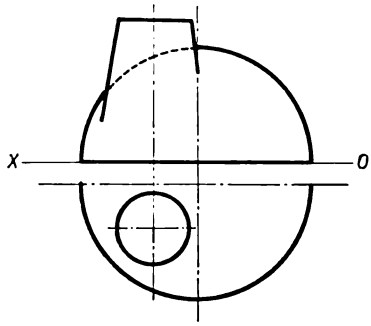



FIG. 1102.

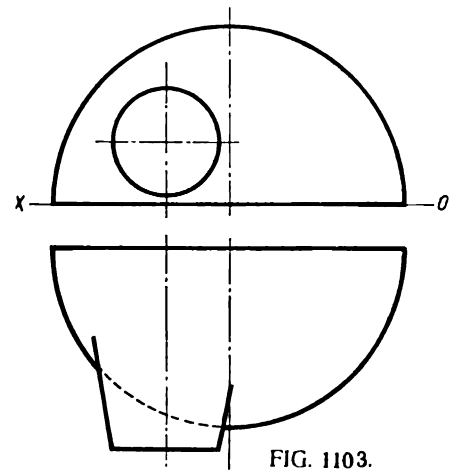



FIG. 1103.

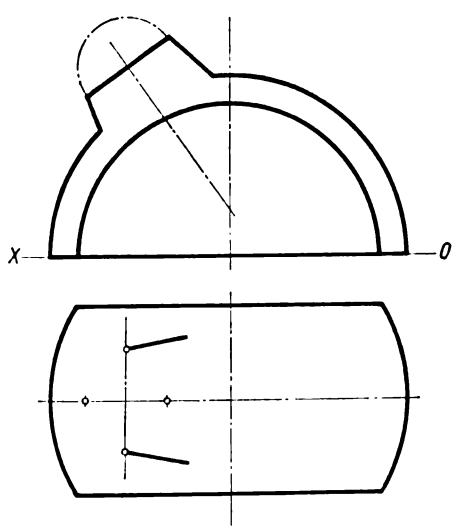



FIG. 1104.

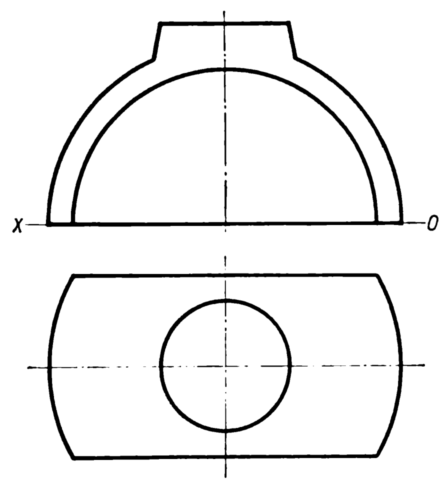



FIG. 1105.

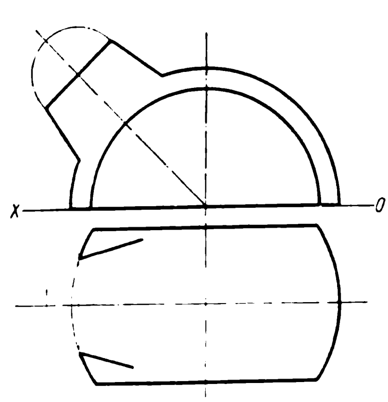



FIG. 1106.

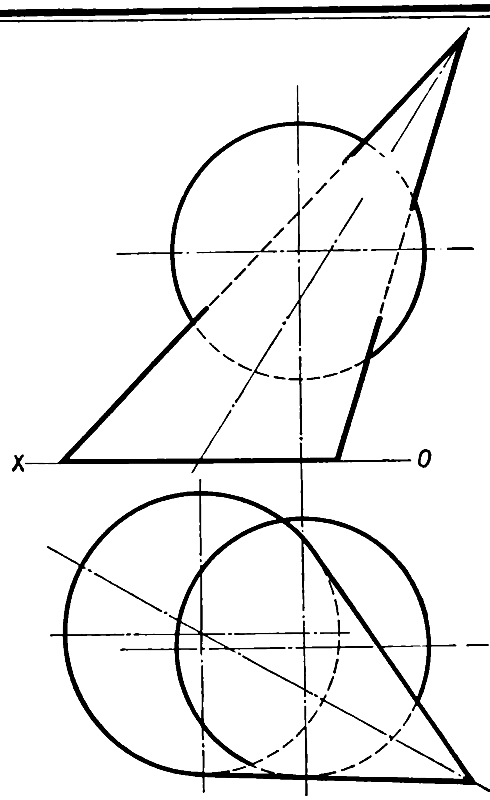



FIG. 1107.

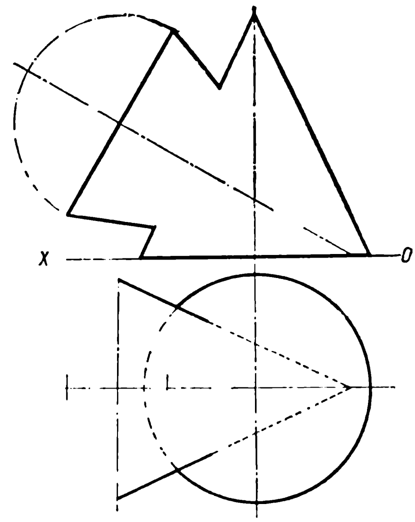



FIG. 1108.

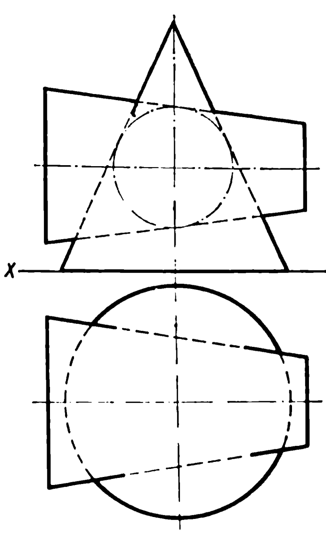



FIG. 1109.

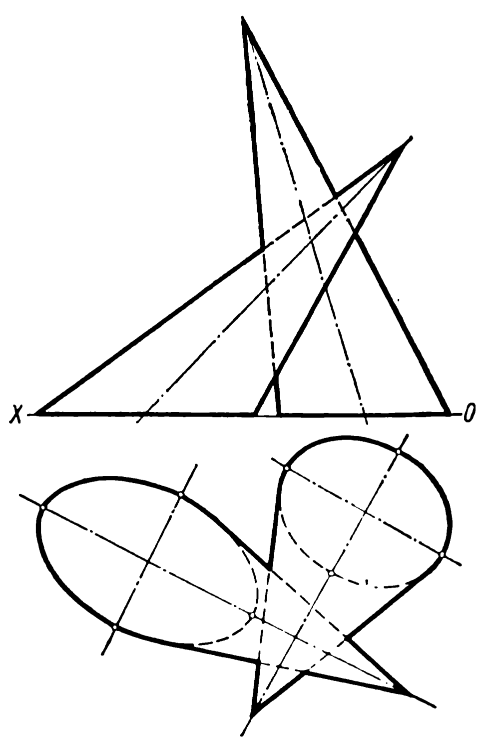



FIG. 1110.

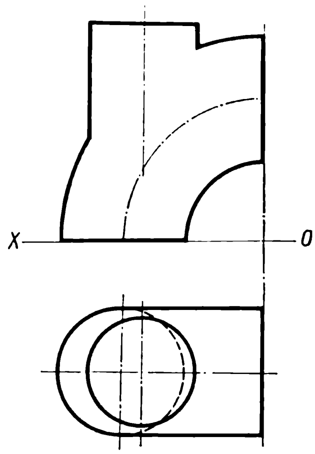



FIG. 1111.

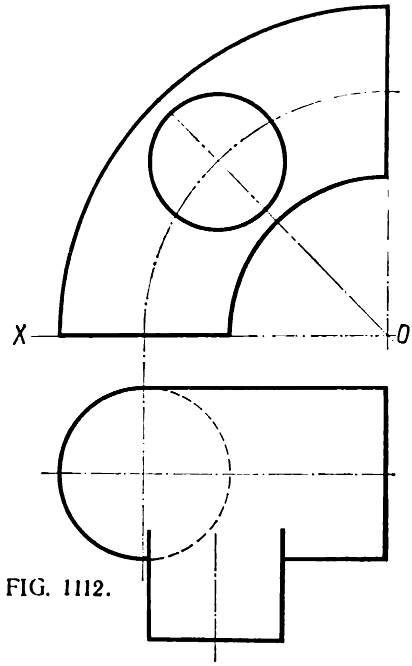



FIG. 1112.

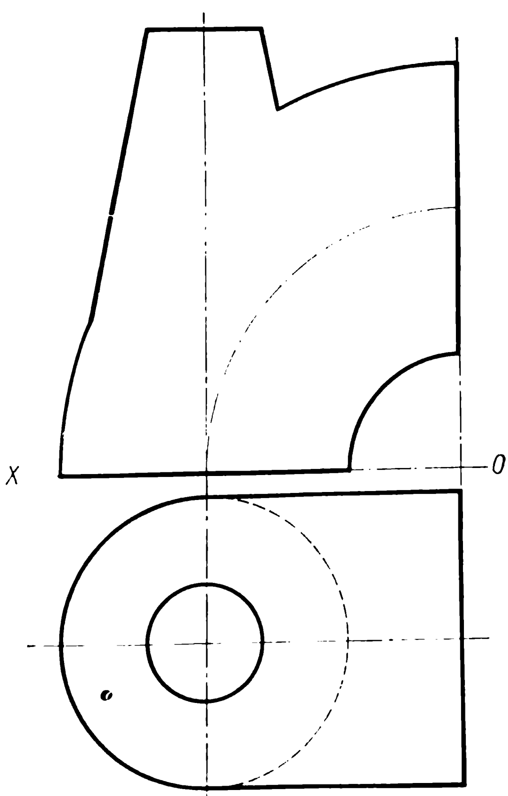



FIG. 1113.

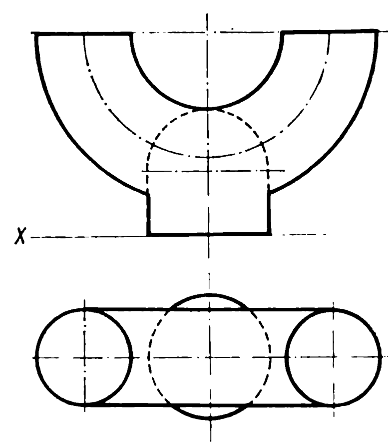



FIG. 1114.

# PART FIVE

## CHAPTER XXVII

## REVIEW PROBLEMS

Solve the following problems using a variety of possible methods.

498. Find the point of intersection of the planes $P$, $Q$, and $R$ (Figs. 1115, 1116).

499. Through the point $A$ draw a straight line parallel to the planes $P$ and $Q$ (Fig. 1117).

500. Through the point $K$ draw a straight line parallel to the horizontal projecting plane $R$ and to the plane specified by the straight line $AB$ and the point $C$ (Fig. 1118).

501. Through the point $K$ draw a straight line parallel to the vertical projecting plane $R$ and to the plane specified by the parallel lines $AB$ and $CD$ (Fig. 1119).

502. Through the point $K$ draw a straight line parallel to the profile projecting plane $P$ and to the plane specified by the intersecting lines $AB$ and $AC$ (Fig. 1120).

503. Through the point $M$ draw a straight line parallel to the planes specified by the intersecting lines $AB$ and $CD$ and by the parallel lines $EF$ and $KL$ (Fig. 1121).

504. In the plane $P$ draw a straight line through the point $A$ (lying in this plane) and parallel to the plane $Q$ (Fig. 1122).

505. Through the point $K$ draw a straight line $MN$ intersecting the given lines $AB$ and $CD$ (Fig. 1123).

506. Intersect the given lines $AB$, $CD$, and $EF$ with an *arbitrary* line $MN$ (Fig. 1124).

507. Intersect the given lines $AB$ and $CD$ with a line $MN$ parallel to the line $EF$ (Fig. 1124).

508. Intersect the given lines $AB$ and $CD$ with a line $MN$ parallel to the coordinate axis (Fig. 1125). (The profile plane of projection need not be used.)

509. Intersect the given lines $AB$ and $CD$ with an *arbitrary* line $MN$ inclined at equal angles to the projection planes (Fig. 1125).

510. Intersect the given lines $AB$ and $CD$ with a straight line inclined at 45° to the $H$ plane and at 30° to the $V$ plane (Fig. 1125).

511. Intersect the lines $AB$, $AC$, and $AD$ with a straight line $AM$ making equal angles with the given lines (Fig. 1126).

512. Through the point $M$ draw a straight line $KL$ inclined at equal angles to arbitrary lines $AB$, $CD$ and $EF$ (Fig. 1127).

513. In the plane $P$ find the locus of points equidistant from points $A$ and $B$ lying in this plane (Fig. 1128).

514. In the plane $P$ construct an isosceles triangle $ABC$ with vertex $A$ on the horizontal trace of the plane, given the vertical projection of the side $BC$ (Fig. 1129).

515. Through the point $M$ on the line $AB$ draw a straight line $MN$ perpendicular to $AB$ and intersecting a line $CD$ (Fig. 1130).

516. Construct the projections of a sphere of radius 25 mm covering up a circular hole of radius 20 mm cut in the plane $P$ (Fig. 1131).

517. Construct the projections of a sphere of radius 25 mm tangent to the plane $P$, given the vertical projection ($c'$) of the centre (Fig. 1132).

518. Construct the projections of a sphere of radius 25 mm tangent to the plane specified by the line $AB$ and the point $D$, given the horizontal projection ($c$) of the centre (Fig. 1133).

519. Construct the projections of a sphere of radius 30 mm tangent to the plane specified by the parallel lines $AB$ and $DE$, given the vertical projection ($c'$) of the centre (Fig. 1134).

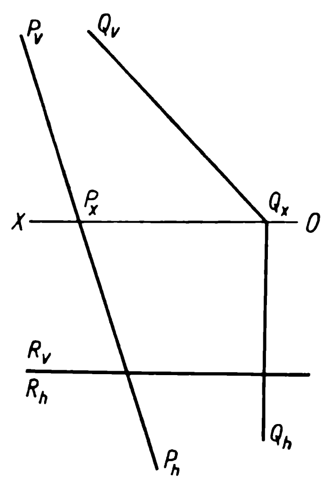



FIG. 1115.

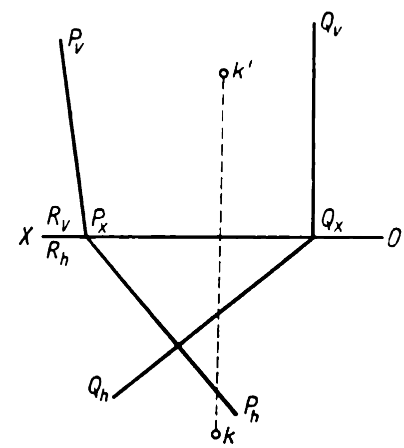



FIG. 1116.

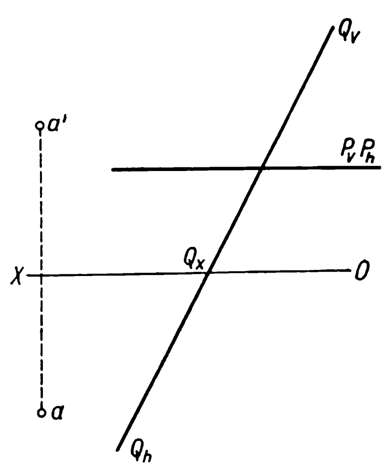



FIG. 1117.

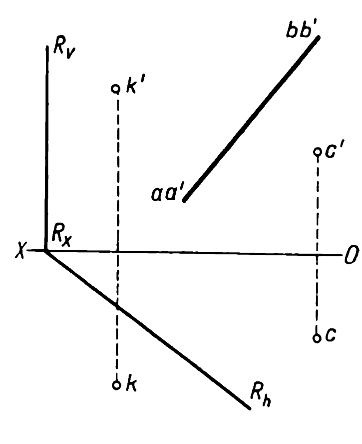



FIG. 1118.

520. Construct the projections of a sphere of radius 25 mm tangent to the plane of the triangle $KLM$, given the horizontal projection ($c$) of the centre (Fig. 1135).

521. Construct the projections of a sphere of radius 25 mm tangent to the plane $P$ and with centre on the line $AB$ (Fig. 466).

522. Construct the projections of a sphere of radius 25 mm tangent to the plane specified by the line $AB$ and the point $D$ and with centre on the line $MN$ (Fig. 1136).

523. Construct the projections of a sphere of radius 25 mm tangent to the plane specified by the parallel lines $AB$ and $DE$ and with centre on the line $MN$ (Fig. 1137).

524. Construct the projections of a sphere of radius 25 mm tangent to the plane of the triangle $DEF$ and with centre on the line $AB$ (Fig. 1138).

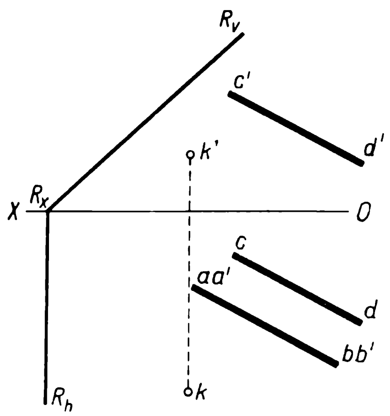


FIG. 1119.

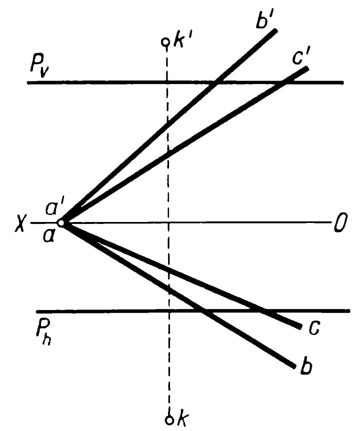


FIG. 1120.

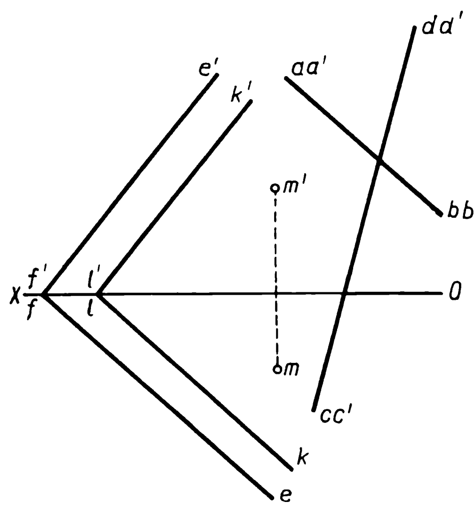


FIG. 1121.

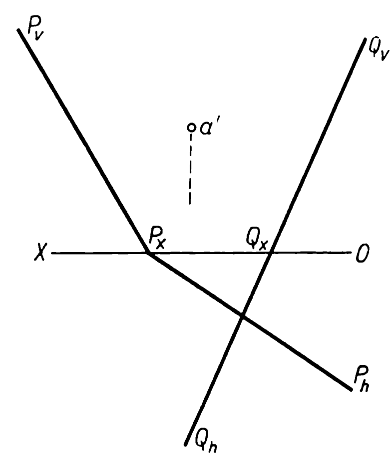


FIG. 1122.

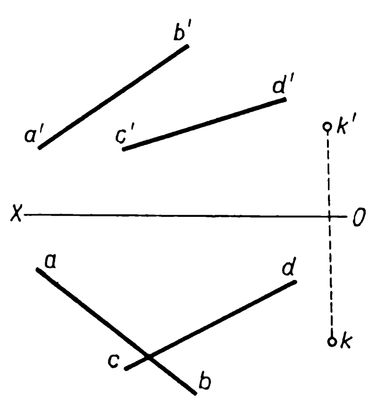


FIG. 1123.

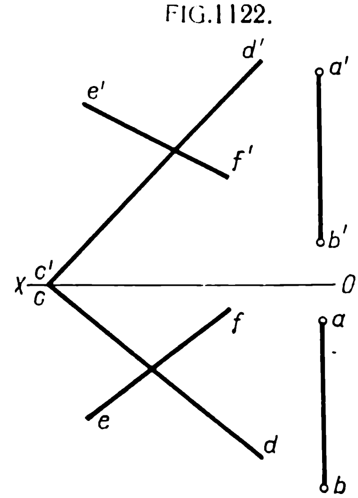


FIG. 1124.

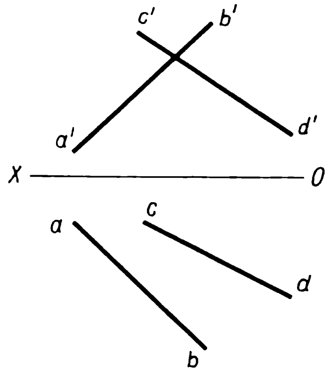



FIG. 1125.

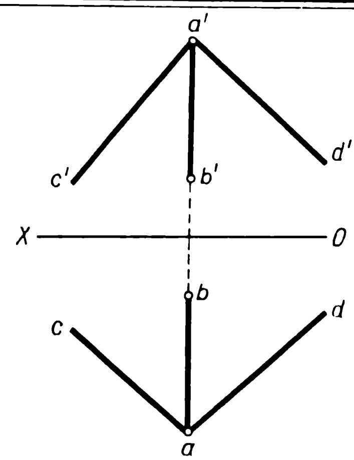



FIG. 1126.

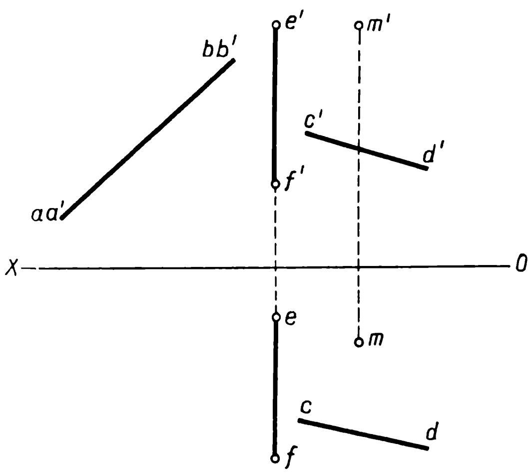



FIG. 1127.

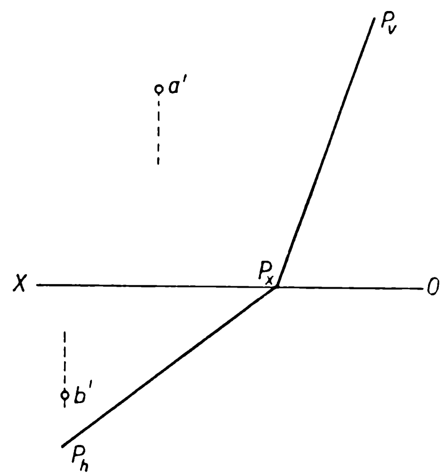



FIG. 1128.

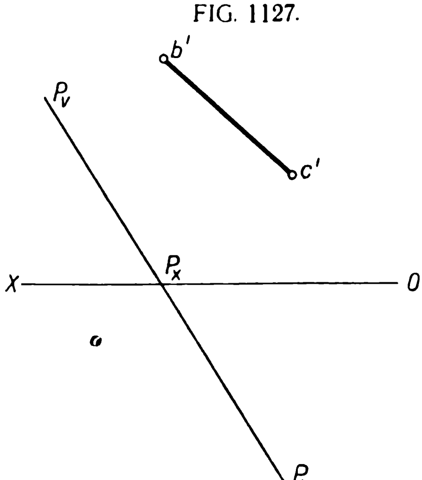



FIG. 1129

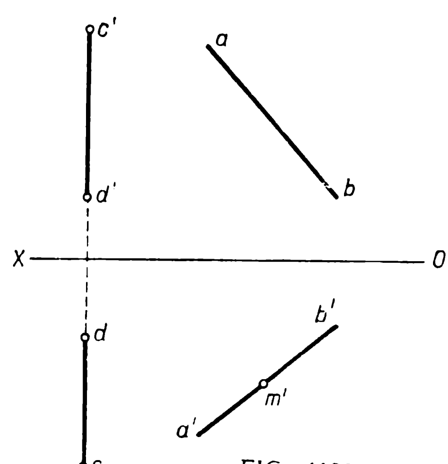



FIG. 1130.

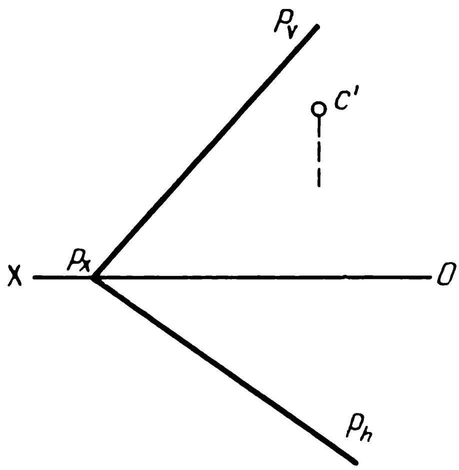



FIG. 1131.

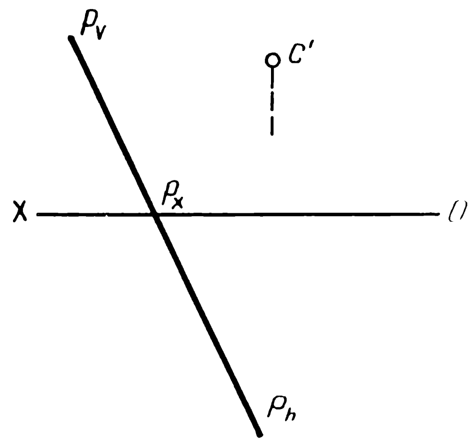



FIG. 1132.

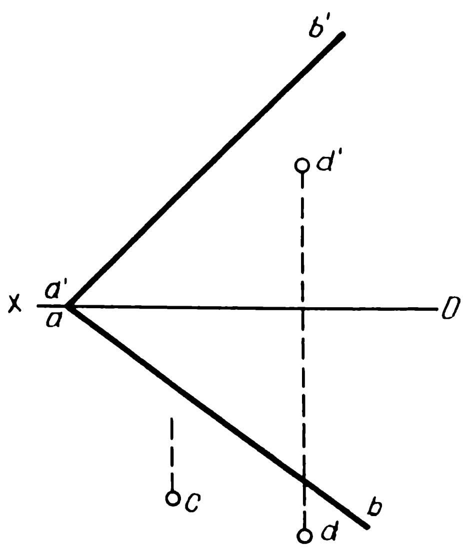



FIG. 1133.

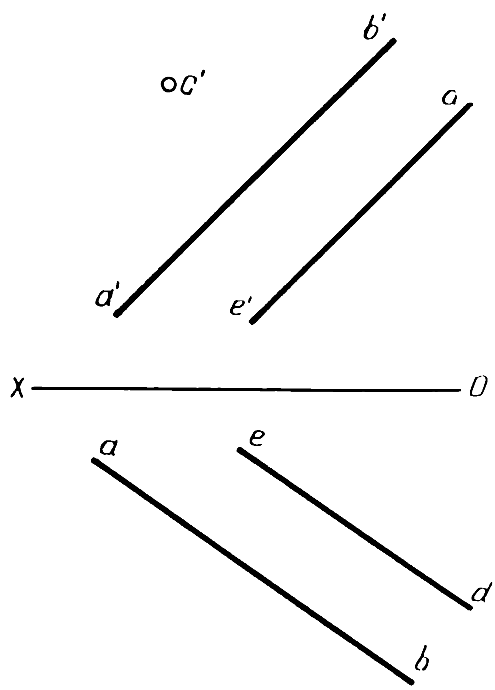



FIG. 1134.

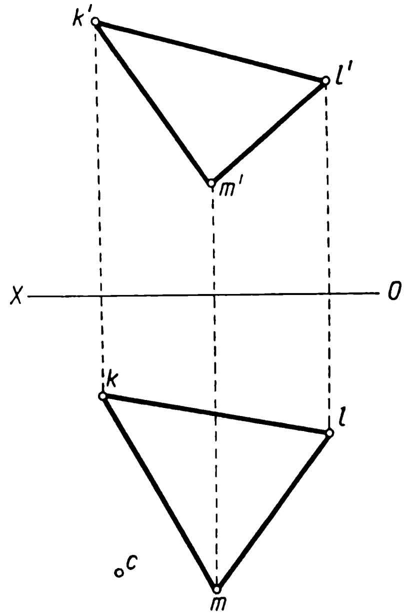



FIG. 1135.

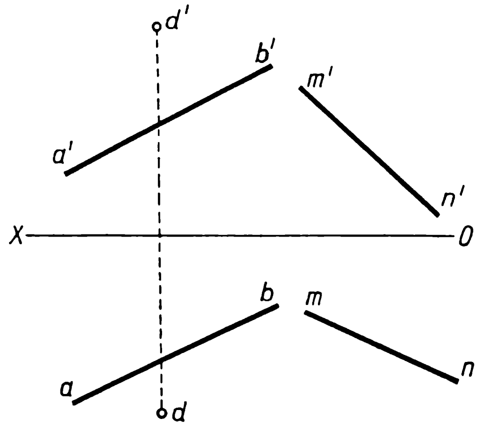



FIG. 1136.

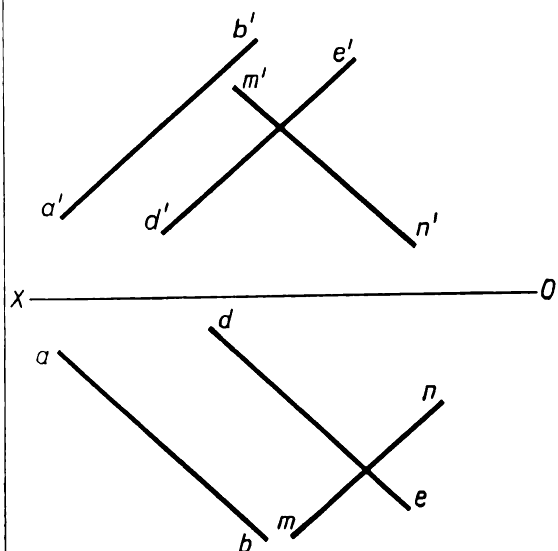



FIG. 1137.

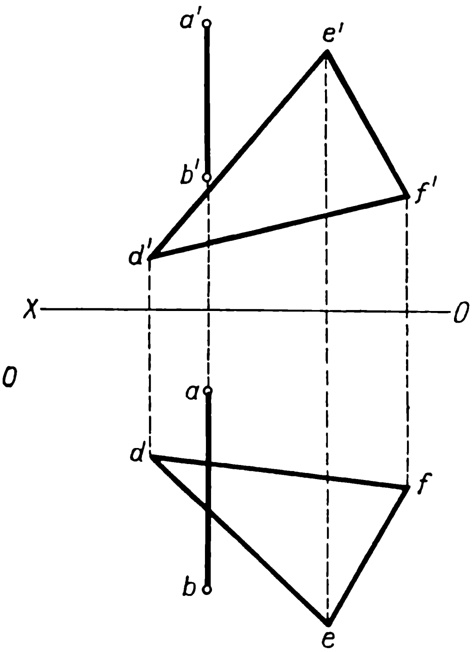



FIG. 1138.

525. Through the point $K$ pass a plane $Q$ perpendicular to the plane $P$ and equally inclined to the planes of projection (Figs. 640, 641).

526. Through the point $K$ pass a plane $P$ perpendicular to the plane specified by the line $AB$ and the point $C$ and equally inclined to the planes of projection (Fig. 642).

527. Through the point $K$ pass a plane $P$ perpendicular to the plane specified by the lines $AB \| CD$ and equally inclined to the planes of projection (Fig. 643).

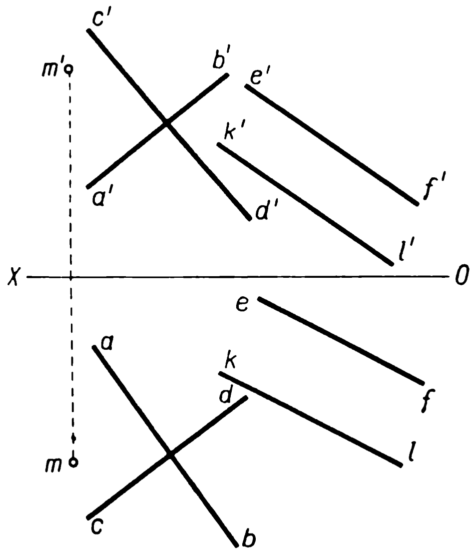



FIG. 1139.

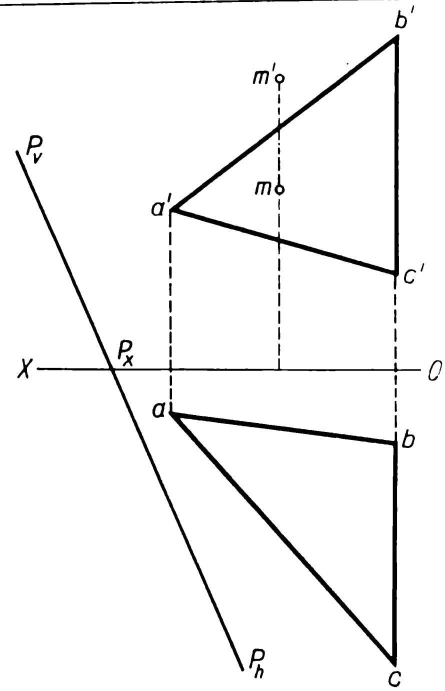



FIG. 1140.

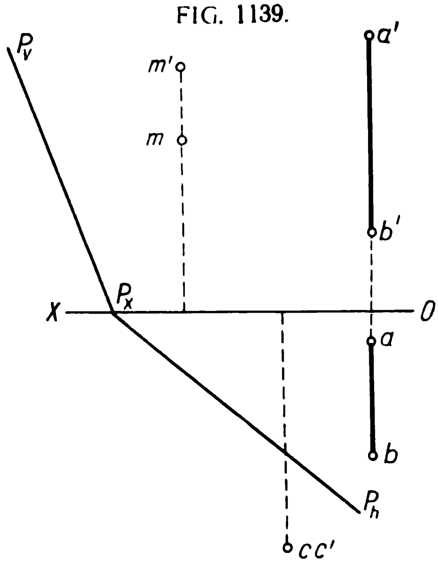



FIG 1141.

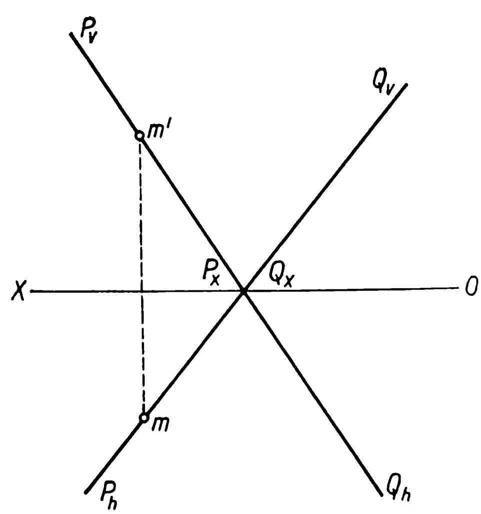



FIG. 1142.

528. Through the point $K$ pass a plane $P$ perpendicular to the plane of the triangle $ABC$ and equally inclined to the planes of projection (Fig. 644).

529. Through the point $M$ pass a plane $P$ perpendicular to the planes specified by the intersecting lines $AB$ and $CD$ and by the parallel lines $EF$ and $KL$ (Fig. 1139).

530. Through the point $M$ pass a plane perpendicular to the plane $P$ and to the plane of the triangle $ABC$ (Fig. 1140).

531. Through the point $M$ pass a plane perpendicular to the plane $P$ and to the plane specified by the line $AB$ and the point $C$ (Fig. 1141).

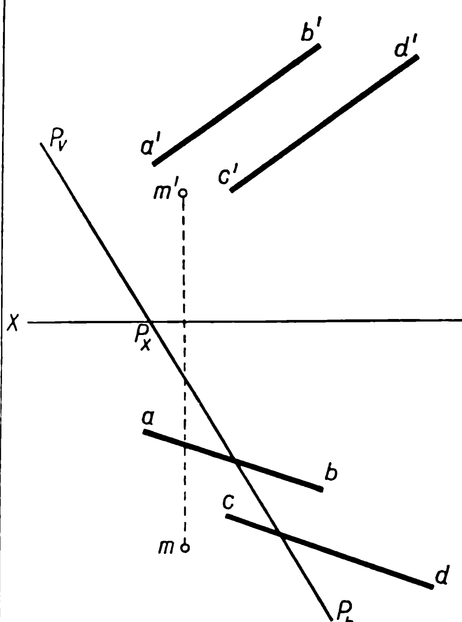



FIG. 1143.

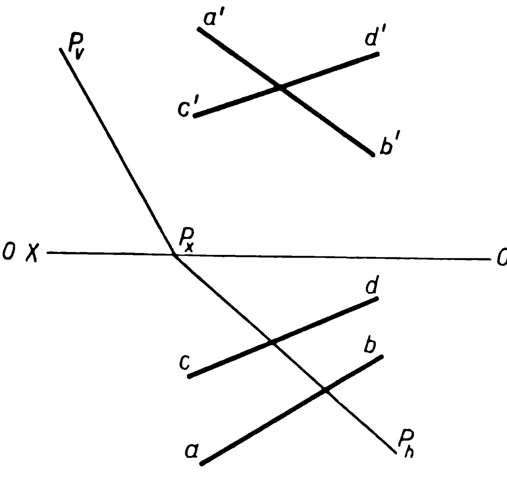



FIG. 1144.

532. Through the point $M$ pass a plane perpendicular to the planes $P$ and $Q$ (Fig. 1142).

533. Through the point $M$ pass a plane $Q$ perpendicular to the plane $P$ and to the plane specified by the parallel lines $AB$ and $CD$ (Fig. 1143).

534. Construct the locus of points in space equidistant from the parallel lines $AB$ and $CD$ (Fig. 599).

535. In the plane $P$ construct the locus of points equidistant from the parallel lines $AB$ and $CD$ (Fig. 602).

536. On the line $MN$ find a point $K$ equidistant from the lines $AB \| CD$ (Fig. 607).

537. Draw a straight line $MN$ perpendicular to the plane $P$ and intersecting the lines $AB$ and $CD$ (Fig. 1144).

538. Intersect arbitrary lines $AB$ and $CD$ with a line $MN$ perpendicular to them (Fig. 578).

539. Pass a plane $P$ equidistant from two arbitrary lines $AB$ and $CD$ (Fig. 578).

540. Construct the locus of points in space equidistant from the nearest points of two arbitrary lines $AB$ and $CD$ (Fig. 578).

541. In the plane $P$ find the locus of points equidistant from the nearest points of two arbitrary lines $AB$ and $CD$ (Fig. 1144).

542. On the line $EF$ find a point equidistant from the nearest points of arbitrary lines $AB$ and $CD$ (Fig. 1124).

543. In the plane $P$ draw a straight line perpendicular to the line $AB$ and passing through its point of intersection with the plane (Fig. 556).

544. In the plane of the triangle $ABC$ draw a straight line perpendicular to the line $MN$ and passing through its point of intersection with the plane (Fig. 698).

545. In the plane specified by the parallel lines $AB$ and $CD$ draw a straight line perpendicular to the line $MN$ and passing through its point of intersection with the plane (Fig. 607).

546. In the plane specified by the line $AB$ and the point $C$ draw a straight line perpendicular to the line $MN$ and passing through its point of intersection with the plane (Fig. 606).

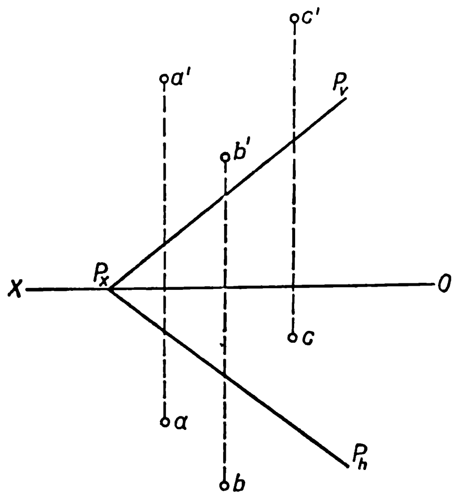


FIG. 1145.

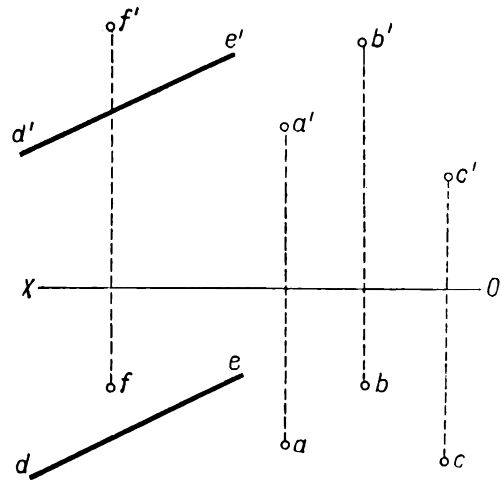


FIG. 1146.

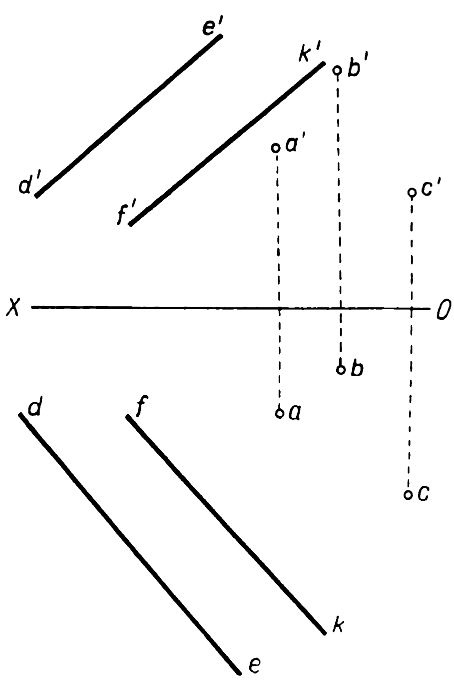


FIG. 1147.

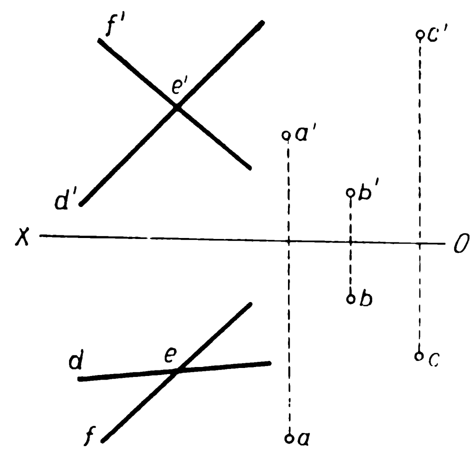


FIG. 1148

547. In the plane $P$ find a point equidistant from the outer points $A$, $B$, $C$ (Fig. 1145).

548. In the plane specified by the line $DE$ and the point $F$ find a point $K$ equidistant from the outer points $A$, $B$, $C$ (Fig. 1146).

549. In the plane specified by the parallel lines $DE$ and $FK$ find a point $M$ equidistant from the outer points $A$, $B$, $C$ (Fig. 1147).

550. In the plane specified by the intersecting lines $DE$ and $EF$ find a point $K$ equidistant from the outer points $A$, $B$, $C$ (Fig. 1148).

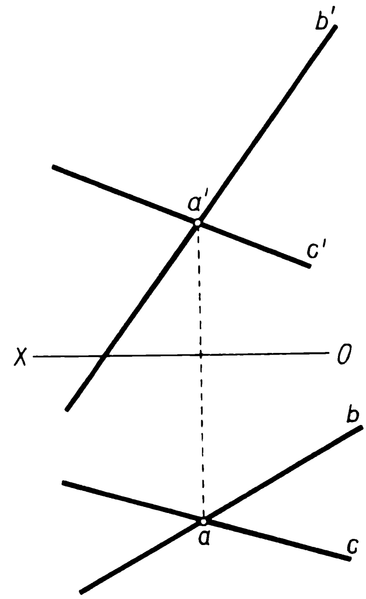


FIG. 1149.

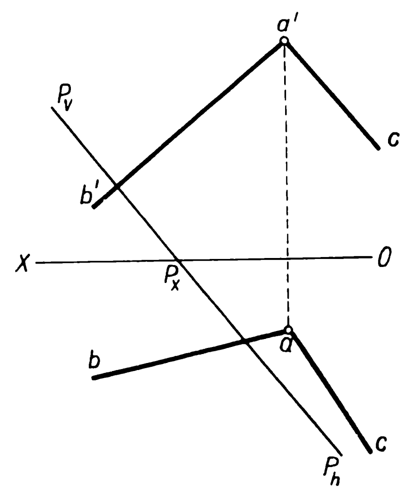


FIG. 1150.

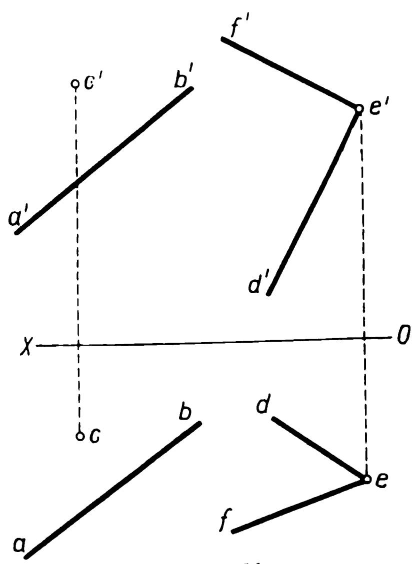


FIG. 1151.

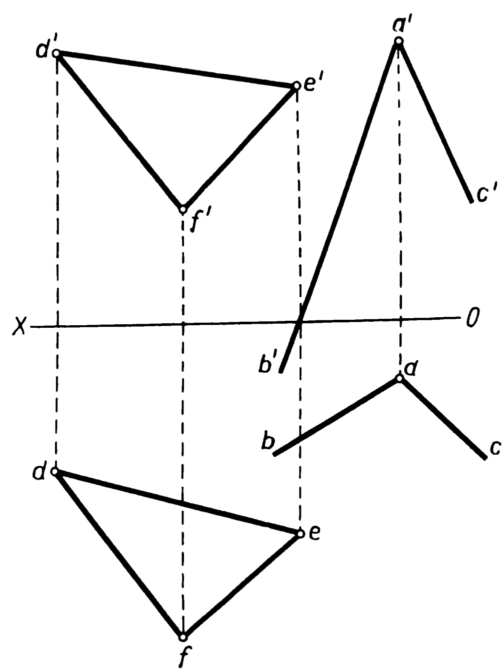


FIG. 1152.

551. Construct the locus of points in space equidistant from the intersecting lines $AB$ and $AC$ (Fig. 1149).

552. In the plane $P$ construct the locus of points equidistant from the intersecting lines $AB$ and $AC$ (Fig. 1150).

553. In the plane specified by the line $AB$ and the point $C$ construct the locus of points equidistant from the intersecting lines $DE$ and $EF$ (Fig. 1151).

554. In the plane of the triangle $DEF$ construct the locus of points equidistant from the intersecting lines $AB$ and $AC$ (Fig. 1152).

555. In the plane specified by the parallel lines $DE$ and $FK$ construct the locus of points equidistant from the intersecting lines $AB$ and $AC$ (Fig. 1153).



26—1063

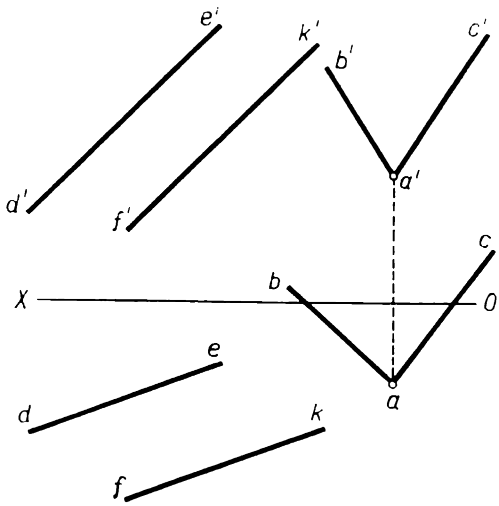


FIG. 1153.

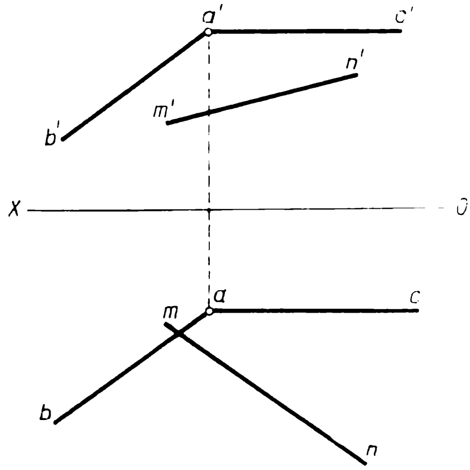


FIG. 1154.

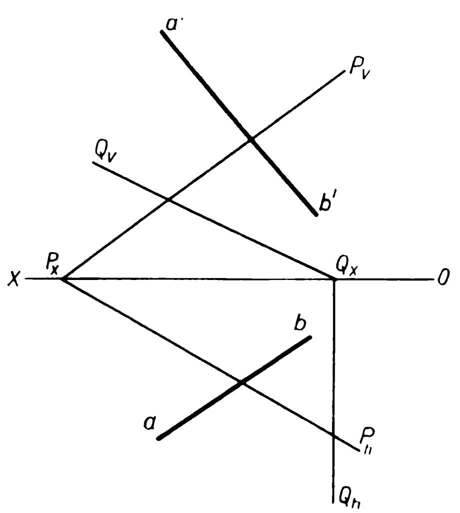


FIG. 1155.

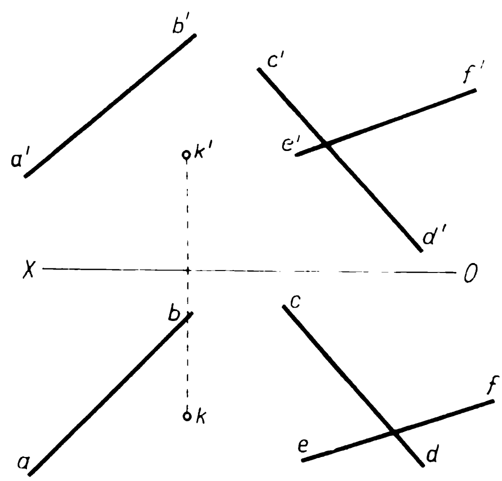


FIG. 1156.

556. In the outer line $MN$ find a point equidistant from the intersecting lines $AB$ and $AC$ (Fig. 1154).

557. Construct the locus of points in space equidistant from the traces of the plane $P$ (Figs. 596, 597).

558. Construct the locus of points in the plane $Q$ equidistant from the traces of the plane $P$ (Figs. 393, 394).

559. Construct the locus of points situated in the plane specified by the parallel lines $AB$ and $CD$ and equidistant from the traces of the plane $P$ (Fig. 602).

560. Construct the locus of points in the plane of the triangle $ABC$ equidistant from the traces of the plane $P$ (Figs. 474, 603).

561. Construct the locus of points in the plane specified by the line $AB$ and the point $C$ equidistant from the traces of the plane $P$ (Fig. 601).

562. On the line $AB$ contained in the plane $P$ find a point equidistant from the traces of this plane (Figs. 340, 341).

563. On the outer line $AB$ find a point equidistant from the traces of the plane $P$ (Fig. 499).

564. Construct the locus of points in space equidistant from the planes $P$ and $Q$ (Figs. 393, 394).

565. In the plane $P$ construct the locus of points equidistant from the planes $Q$ and $R$ (Figs. 1115, 1116).

566. On the line $AB$ find a point equidistant from the planes $P$ and $Q$ (Fig. 1155).

567. Through the point $K$ pass a plane equally inclined to the projection planes and on which the parallel lines $AB$ and $CD$ are projected in a single straight line (Fig. 643).

568. Through the point $K$ pass a plane parallel to the line $EF$ on which the parallel lines $AB$ and $CD$ are projected in a single straight line (Fig. 526).

569. Through the line $EF$ pass a plane on which the parallel lines $AB$ and $CD$ are projected in a single straight line (Fig. 531).

570. Through the point $K$ pass a plane equally inclined to the projection planes and on which arbitrary lines $AB$ and $CD$ are projected in parallel lines (Fig. 1123).

571. Through the line $AB$ pass a plane on which arbitrary lines $CD$ and $EF$ are projected in parallel lines (Fig. 1124).

572. Through the point $K$ pass a plane parallel to the line $AB$ on which arbitrary lines $CD$ and $EF$ are projected in parallel lines (Fig. 1156).

573. Find a point equidistant from four arbitrary points in space $A$, $B$, $C$, and $D$ (Fig. 251).

574. In the plane $P$ find a point equidistant from the point $P_x$ and from the traces of the line $AB$ contained in this plane (Figs. 340, 341).

575. In the plane $P$ construct a right-angled triangle $ABC$ with the vertex $B$ of the right angle on the horizontal trace of the plane, given the vertical projection of the side $AC$ (Fig. 1157).

576. In the plane $P$ and through its point $A$ draw a straight line making equal angles $\varphi$ with the traces of the plane (Figs. 352, 353).

577. In the plane $P$ find a point 20 mm and 30 mm distant from its points $A$ and $B$, respectively (Fig. 1158). What alternatives are possible?

578. In the plane $P$ find a point 20 mm distant from its point $C$ and 10 mm distant from its line $AB$ (Fig. 1159). What alternatives are possible?

579. In the plane $P$ and through its point $A$ draw a line 20 mm distant from another point $C$ (Fig. 1160).

580. In the plane $P$ draw a straight line 20 mm and 15 mm distant from its points $A$ and $B$, respectively (Fig. 1161).

581. In the plane $P$ construct the locus of points 60 mm distant from the outer point $A$ (Fig. 1162).

582. On the line $AB$ contained in the plane $P$ find a point 60 mm distant from the outer points $C$ (Fig. 1163).

583. In the plane $P$ find a point 15 mm distant from its point $C$ and 60 mm distant from the outer point $A$ (Fig. 1164).

584. In the plane $P$ find a point $l_1$ mm and $l_2$ mm distant from the outer points $A$ and $B$, respectively (Fig. 1165).

585. Through the point $M$ draw a straight line intersecting the line $AB$ and the plane $P$ at the points $K$ and $L$, respectively, given $MK = KL$ (Fig. 1166).

586. Construct an equilateral triangle $ABC$ with the vertex $C$ in the plane $P$, given that its altitude is equal to 40 mm (Fig. 1167).

587. Construct an equilateral triangle $ABC$, given that its vertex $C$ lies in plane $P$ (Fig. 1167).

588. Construct the trajectory of a point $C$ moving round the line $AB$ (Fig. 1168).

589. Construct the trajectory of the nearest point $M$ of the line-segment $AB$ revolving about an arbitrary line $CD$ (Fig. 578).

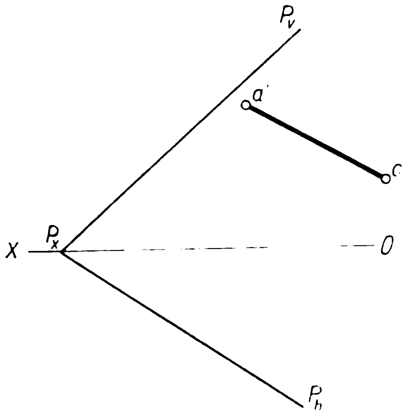



FIG. 1157.

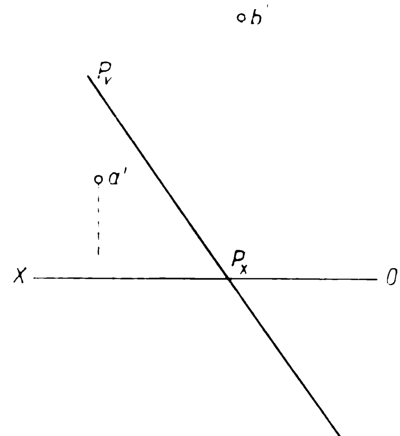



FIG. 1158.

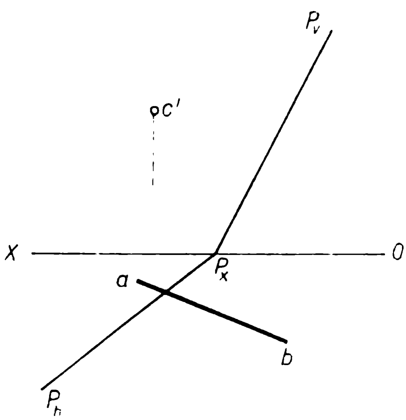



FIG. 1159

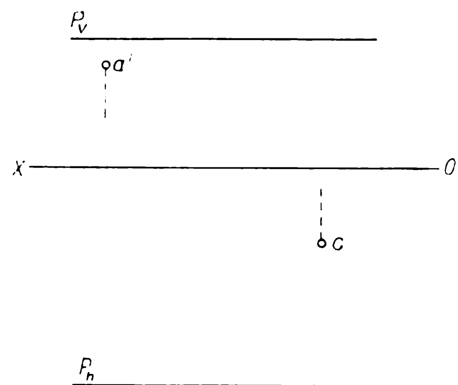



FIG. 1160.

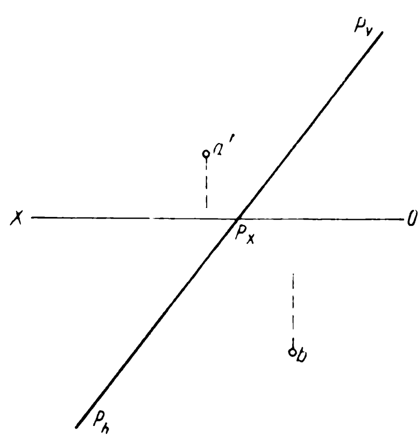



FIG. 1161.

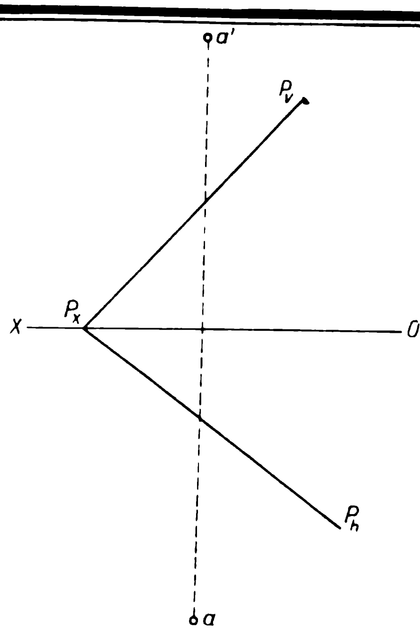


FIG. 1162.

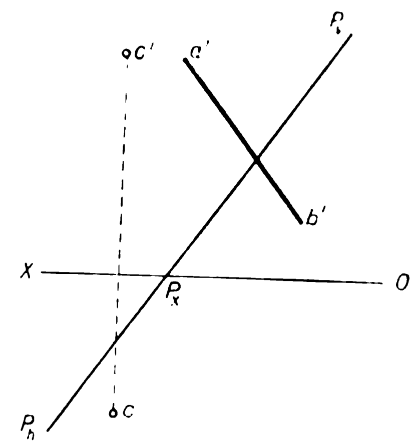


FIG. 1163.

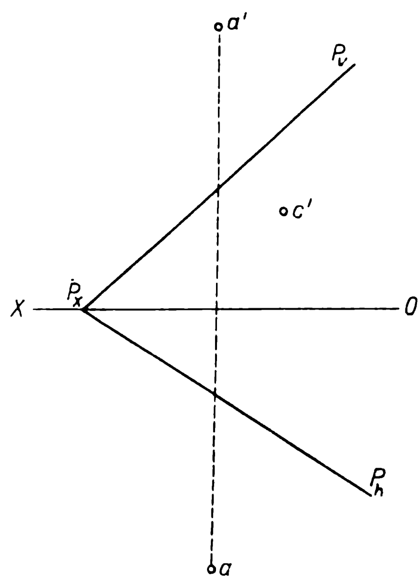


FIG. 1164.

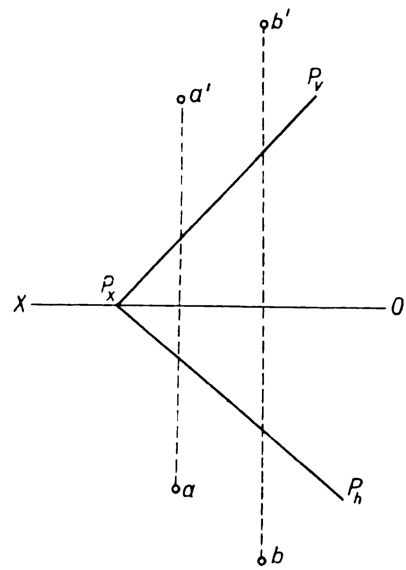


FIG. 1165.

590. Construct the projections of a right prism 60 mm high, given that its base is a square $ABCD$ with the diagonal $BD$ on the line $MN$ (Fig. 658).

591. Construct the projections of a right prism 60 mm high, given that its base is an isosceles triangle $ABC$ with the vertex $A$ on the line $EF$ (Fig. 661).

592. Construct the projections of a cube with the base $ABCD$ and its side $BC$ on the line $BM$, given the line-segment $AB$ and the horizontal projection of a line $BM$ perpendicular to it (Fig. 1169).

593. Construct a pyramid $SABC$ 60 mm high, given that its base is an isosceles triangle $ABC$ with the vertex $A$ on the line $EF$ and the projection of the vertex $S$

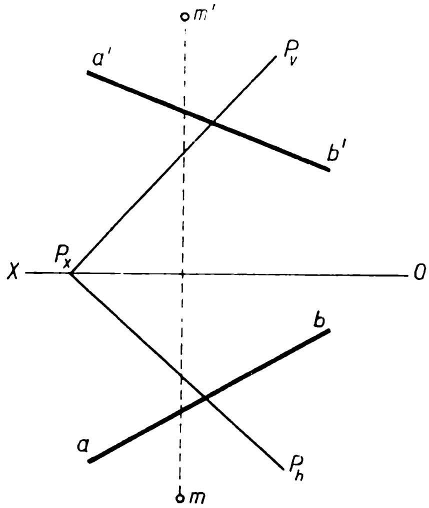



FIG. 1166.

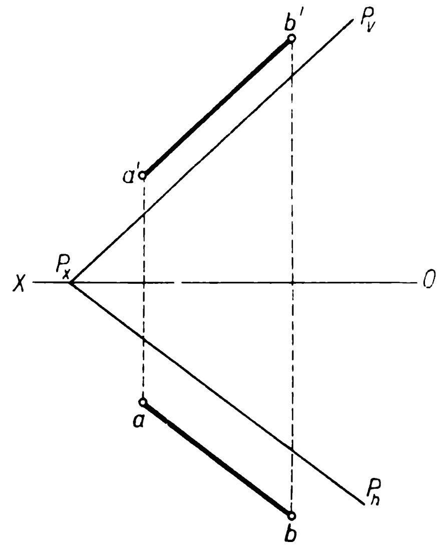



FIG. 1167.

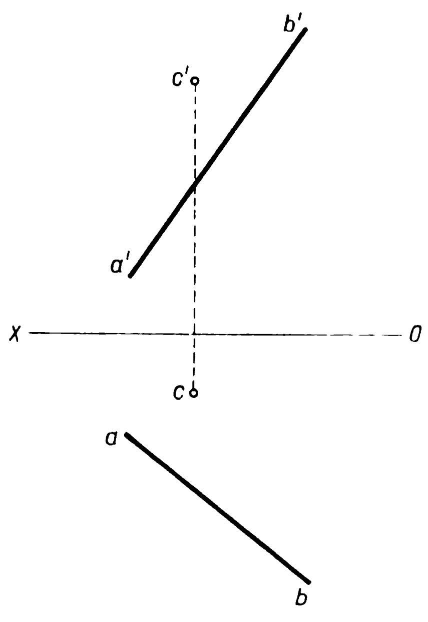



FIG. 1168.

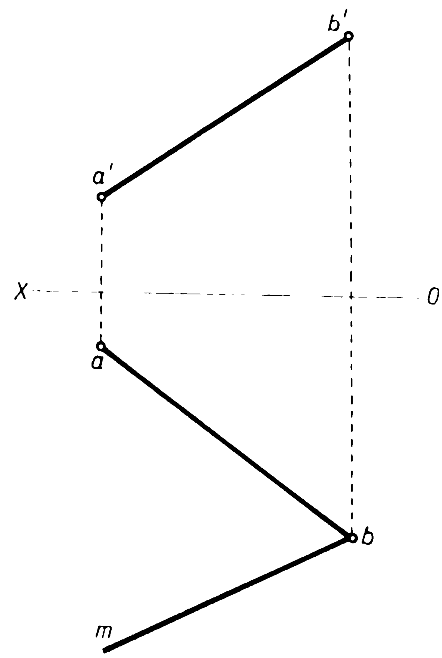



FIG. 1169.

on the base coincides with the centre of a circle circumscribed about the triangle *ABC* (Fig 661).

594. Construct the projections of a pyramid *SABC*, given that the length of the lateral edge *SA* is equal to 65 mm and the projection of the vertex *S* on the base coincides with the centre of a circle inscribed in the triangle *ABC* (Fig. 163).

595. Construct the projections of a pyramid *SABC* 60 mm high, given that the projection of the vertex *S* on the base coincides with the centre of a circle circumscribed about the triangle *ABC* (Fig. 163).

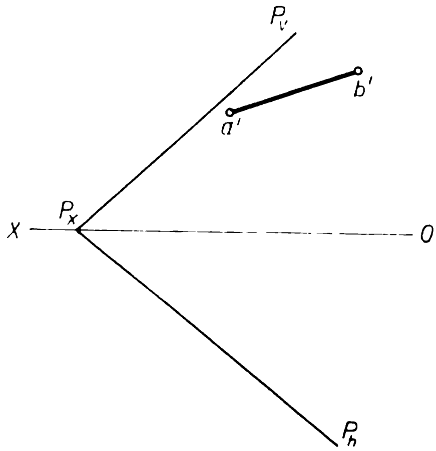



FIG. 1170.

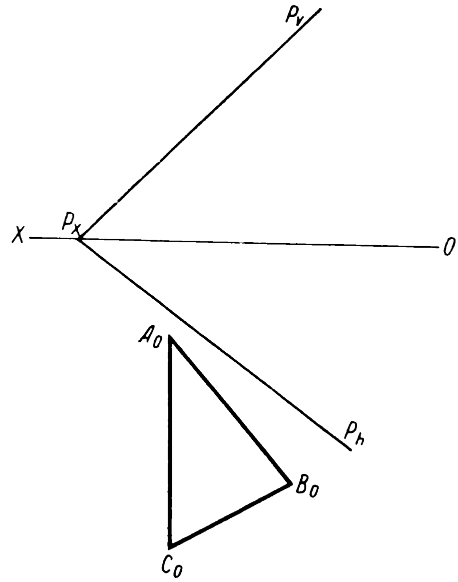



FIG. 1171.

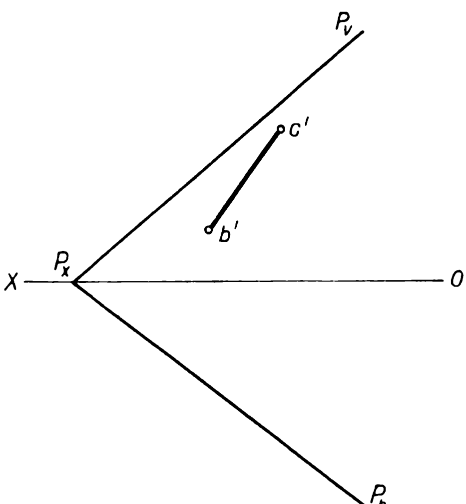



FIG. 1172.

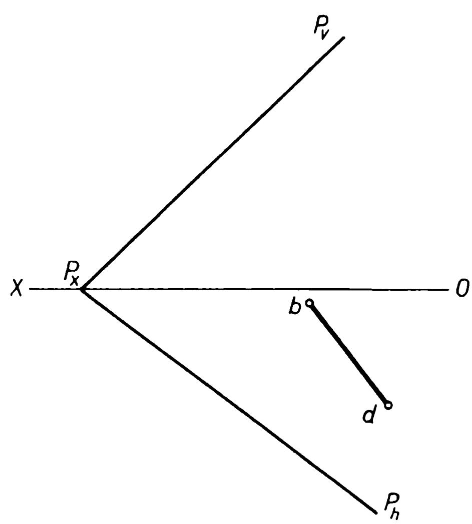



FIG. 1173.

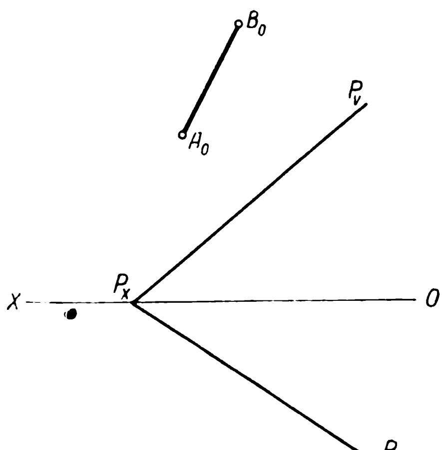



FIG. 1174.

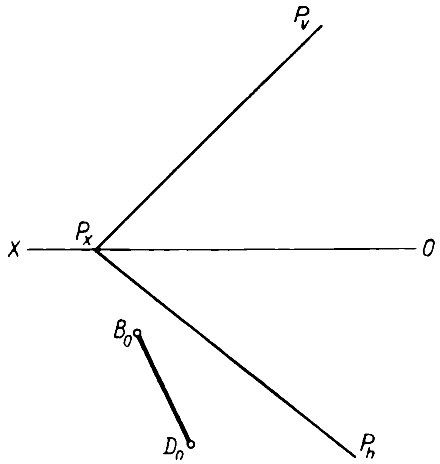



FIG. 1175.

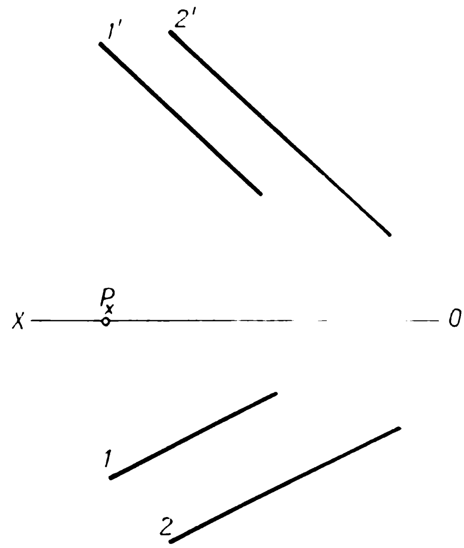



FIG. 1176.

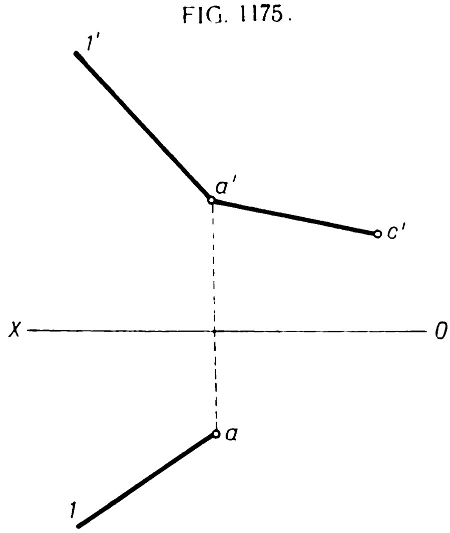



FIG. 1177.

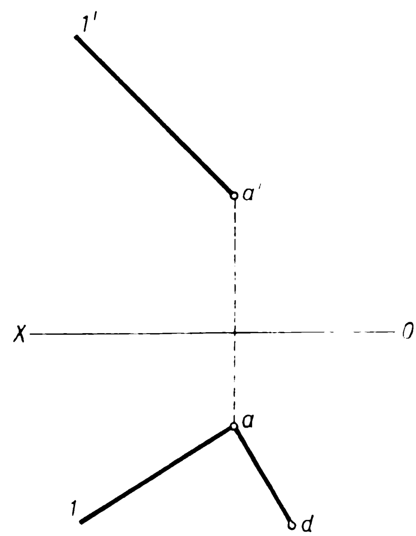



FIG. 1178.

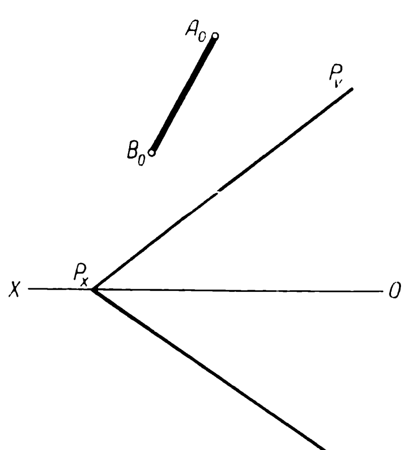



FIG. 1179.

596. Construct the projections of the pyramid $SABC$, given that the projection of its vertex $S$ on the base coincides with the centre of gravity of its area, and the lateral edge $SC$ is inclined at an angle $\varphi$ to the base (Fig. 163).

597. Construct the projections of a regular right prism of altitude 60 mm and with the base $ABC$ in the plane $P$, given the vertical projection of the side $AB$ of its base (Fig. 1170).

598. Construct the projections of a right prism of altitude 60 mm and with the base $ABC$ in the plane $R$, given the coincident position of the base in the $H$ plane (Fig. 1171).

599. Construct the projections of a cube with the base $ABCD$ in the plane $P$, given the vertical projection of the side $BC$ of its base (Fig. 1172).

600. Construct the projections of a cube with the base $ABCD$ in the plane $P$, given the horizontal projection of the diagonal $BD$ of its base (Fig. 1173).

601. Construct the projections of a cube with the base $ABCD$ in the plane $P$, given the side $AB$ of its base in the coincident position in the $V$ plane (Fig. 1174).

602. Construct the projections of a cube with the base $ABCD$ in the plane $P$, given the coincident position of the diagonal $BD$ of its base in the $H$ plane (Fig. 1175).

603. Construct the projections of a regular right triangular prism 60 mm high, given that two of its lateral edges coincide with the given straight lines ($1$, $1'$) and ($2$, $2'$), and $P_x$ is the vanishing point of the plane of the base (Fig. 1176).

604. Construct the projections of a cube with base $ABCD$, given that one of its lateral edges coincides with the line ($1$, $1'$) and given the vertical projection of the line-segment $AC$ perpendicular to it (Fig. 1177).

605. Construct the projections of a right prism of altitude 60 mm, given that its base is a square $ABCD$ and one of its lateral edges coincides with the line ($1$, $1'$) and given the horizontal projection of the line-segment $AD$ perpendicular to the line ($1$, $1'$) (Fig. 1178).

606. Construct the projections of a tetrahedron with the base $ABC$ in the plane $P$, given the side $AB$ of its base in a coincident position in the $V$ plane (Fig. 1179).

607. Construct the projections of a pyramid of altitude 60 mm and with the base $ABC$ on the plane $P$, given the vertical projections of its base and vertex $S$ (Fig. 1180).

608. Construct the projections of a right regular triangular pyramid of altitude 60 mm and with the base $ABC$ in the plane $P$, given the horizontal projection of the side $AB$ of its base (Fig. 1181).

609. Construct the projections of a right regular pyramid of altitude 60 mm and with the base $ABC$ in the plane $P$, given the side $AB$ of its base in a coincident position in the $H$ plane (Fig. 1182).

610. Construct the projections of a right circular cone whose base has a radius of 20 mm and lies in the plane $P$, given the vertex of the cone (point $S$) (Fig. 1183).

611. Construct the locus of straight lines in space passing through a point $S$ and inclined at a given angle $\varphi$ to the plane $P$ (Fig. 1183).

612. Construct the projections of a right circular cone of altitude 60 mm whose base has a radius of 20 mm and lies in the plane $P$, given the vertical projection of the centre of the base (Fig. 1184).

613. Construct the projections of a right circular cone of altitude 60 mm whose base has a radius of 20 mm and lies in the plane $P$, given the coincident position of the point $C$ (the centre of the base) in the $H$ plane (Fig. 1185).

614. Construct the projections of a right circular cone of altitude 60 mm whose axis is situated on the line ($1$, $1'$), given the point $A$ on the base circle in a coincident position in the $H$ plane, and the point $P_x$, which is the vanishing point of the plane of the base (Fig. 1186).

615. Construct the projections of a right circular cone of altitude 60 mm whose base has a radius of 20 mm and lies in the plane $P$ and whose axis lies on the line ($1$, $1'$), given the point $P_x$, which is the vanishing point of the plane of the base (Fig. 1187).

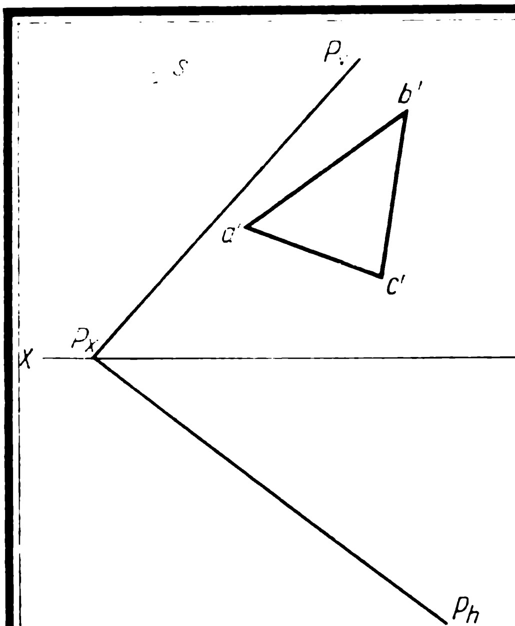


FIG. 1180.

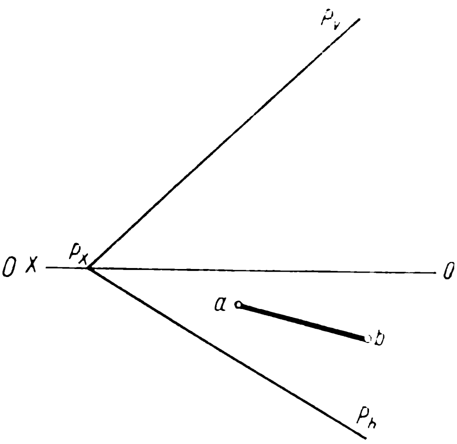


FIG. 1181.

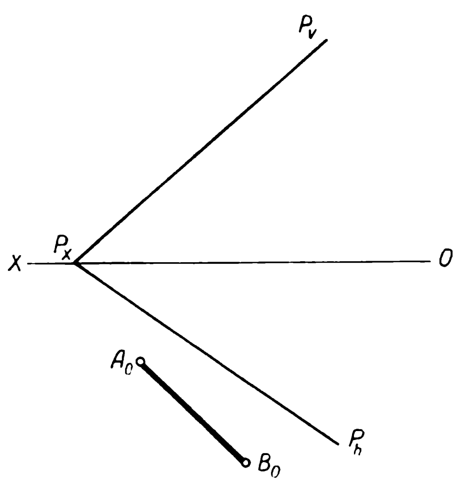


FIG. 1182.

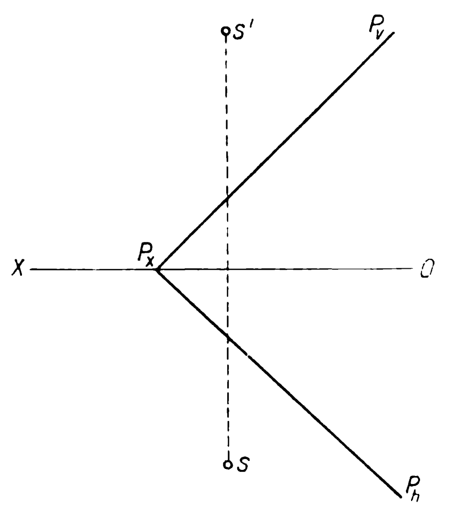


FIG. 1183.

616. Construct the projections of a right circular cone of altitude 60 mm whose axis lies on the line (*1*, *1'*), given the point $C$, which is the centre of the base circle of radius 20 mm (Fig. 1188).

617. Construct the projections of a right circular cone of altitude 60 mm whose axis lies on the line (*1*, *1'*), given the point $A$ lying on the base circle (Fig. 1189).

618. Construct a pyramid $SABC$, given its base $ABC$ and the lengths of the lateral edges: $SA = 55$ mm; $SB = 60$ mm; $SC = 60$ mm (Fig. 1190).

(The problem should be solved using the development of the surface of the pyramid.)

619. In the plane $P$ draw a straight line intersecting the coordinate axis at an angle of 30° (Fig. 1191).

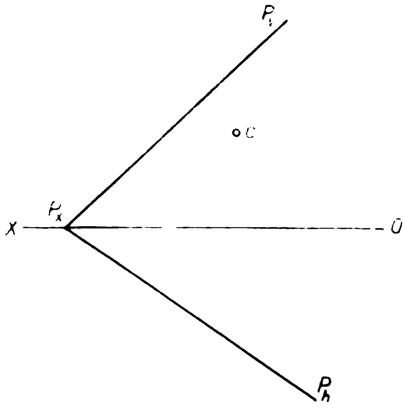



FIG. 1184.

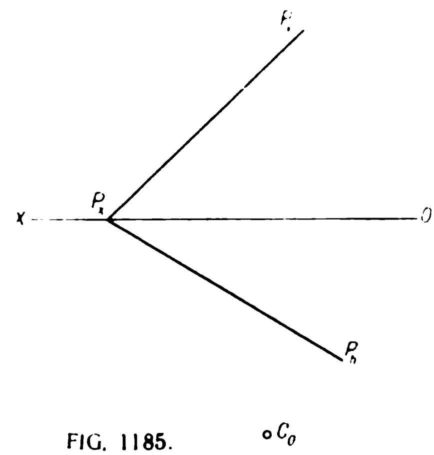



FIG. 1185.

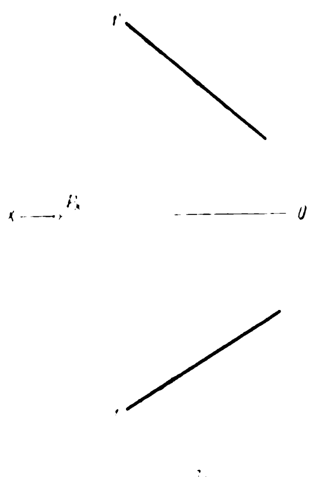


FIG. 1186.

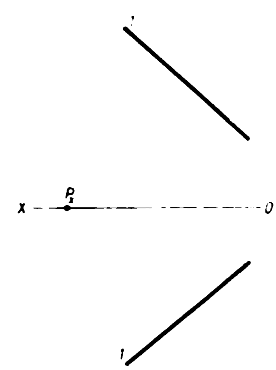


FIG. 1187.

620. Through the point $S$ in plane $P$ draw a straight line making an angle of 60° with the $H$ plane (Fig. 1192).

621. Through the point $S$ draw a straight line parallel to the plane $P$ and making an angle of 60° with the $V$ plane (Fig. 1193).

622. Through the point $S$ draw a straight line parallel to the plane $P$ and making an angle of 30° with the line $AB$ (Fig. 1194).

623. Construct a plane $Q$ parallel to the plane $P$ and intersecting a sphere along a circle of radius $r = 20$ mm (Fig. 1195).

624. On the line $AB$ find points 25 mm distant from the point $C$ (Fig. 1196). (Use the locus of the points.)

625. Through the point $S$ draw a straight line intersecting the line $AB$ and making an angle of 60° with the plane $P$ (Fig. 1197).

626. On the line $CD$ find a point 20 mm distant from the line $AB$ (Fig. 1198).

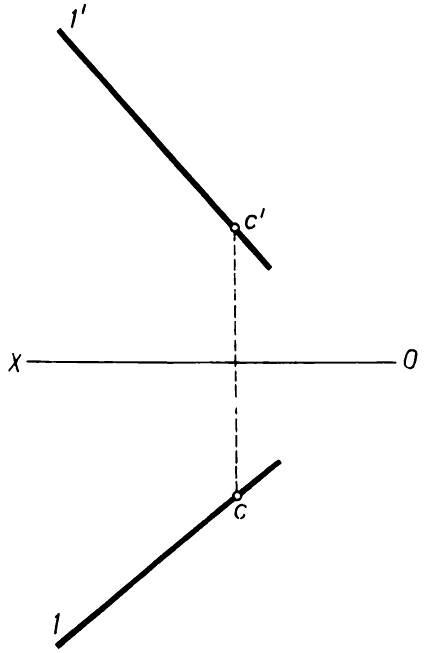



FIG. 1188.

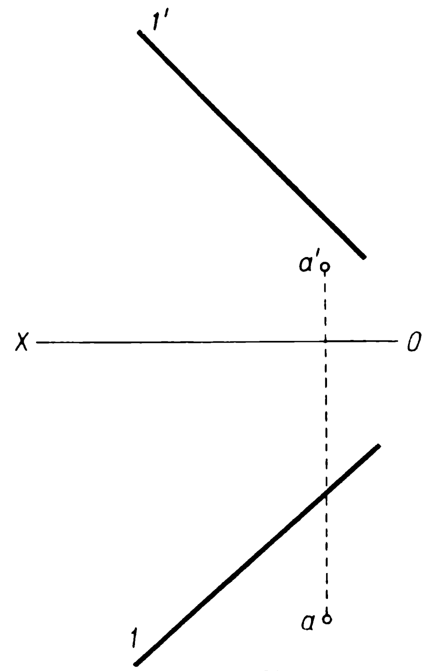



FIG. 1189.

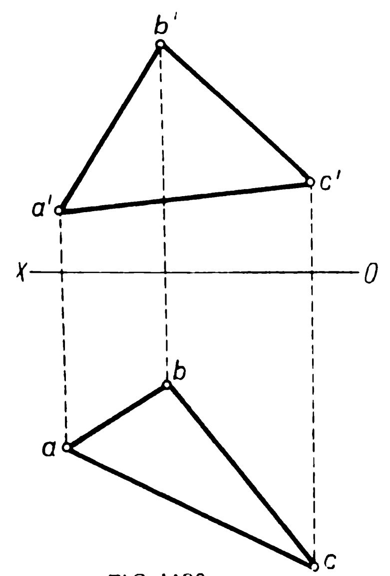



FIG. 1190.

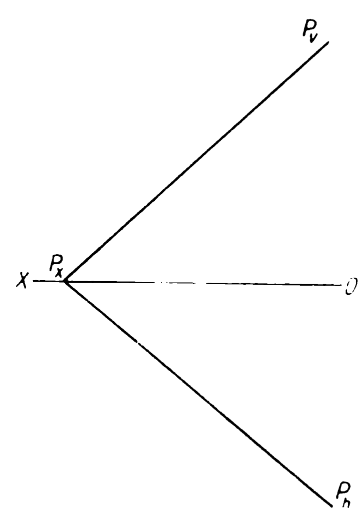



FIG. 1191.

627. Through the point $S$ draw a straight line intersecting the line $CD$ and making an angle of 30° with the line $AB$ (Fig. 1199).

628. Construct a right-angled triangle $CDK$ with the vertex $K$ of the right angle on the line $AB$ (Fig. 1200).

629. Construct the projections of a sphere of radius $r = 20$ mm tangent to a given sphere, given that the centre of the desired sphere lies on the line $AB$ (Fig. 1201)

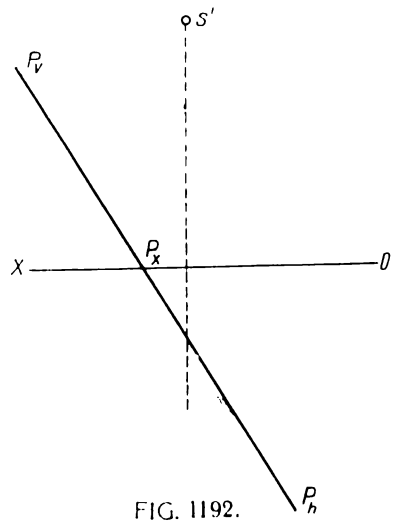



FIG. 1192.

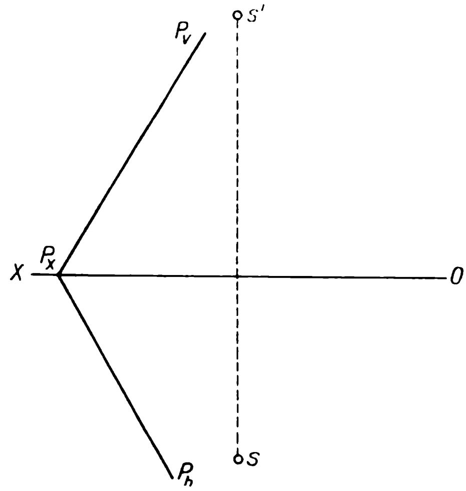



FIG. 1193.

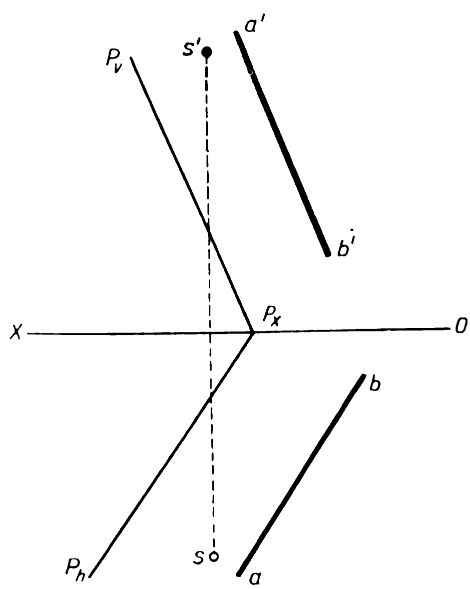



FIG. 1194.

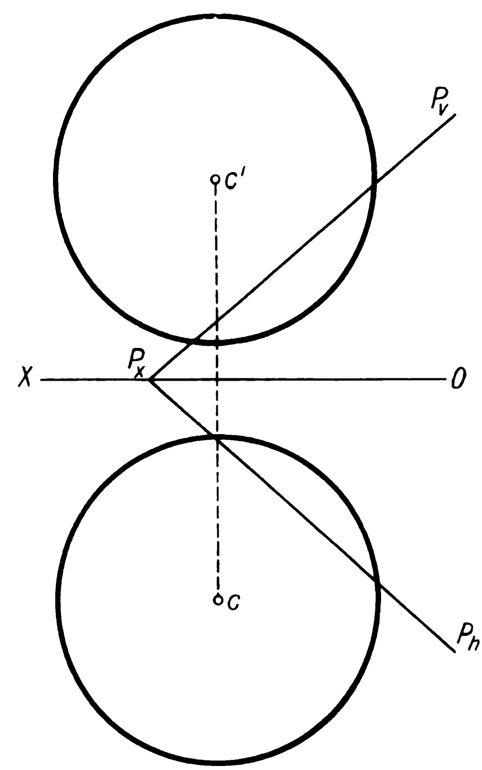



FIG. 1195.

630. Through the line $AB$ pass a plane making an angle of 60° with the $H$ plane (Fig. 1202).

631. Construct the traces of a plane $Q$ tangent to the surface of a cone and perpendicular to the plane $P$ (Fig. 1203).

632. Through the point $S$ draw a plane $P$ inclined to the $H$ plane at a given angle $\varphi$ and perpendicular to the plane $Q$ (Fig. 1204).

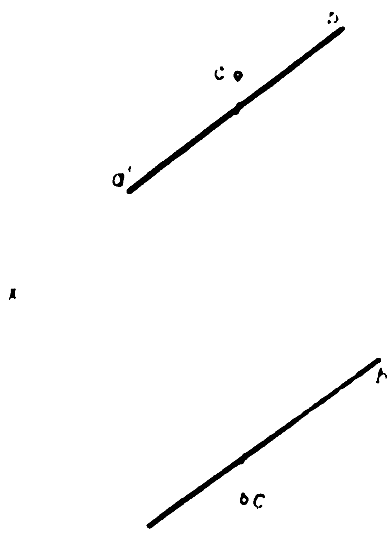



FIG. 1196.

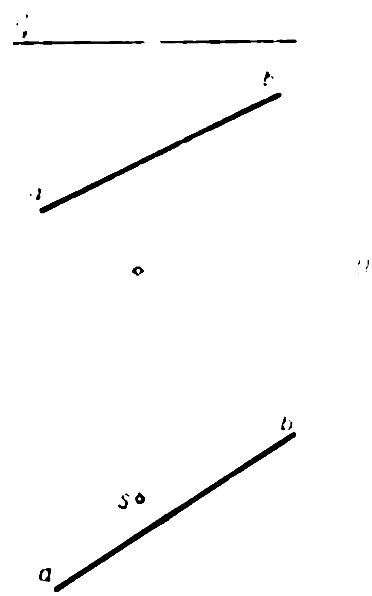



FIG. 1197.

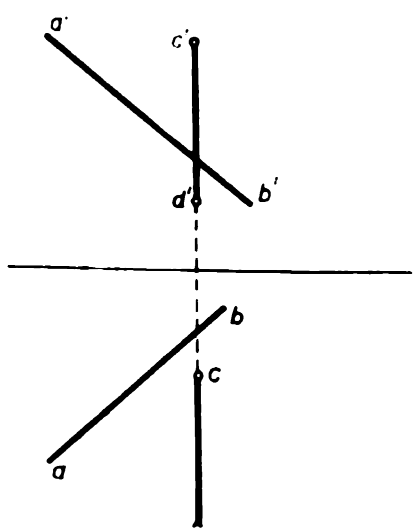



FIG. 1198.

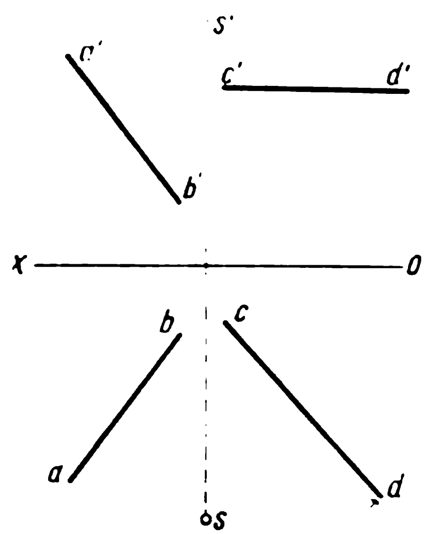



FIG. 1199.

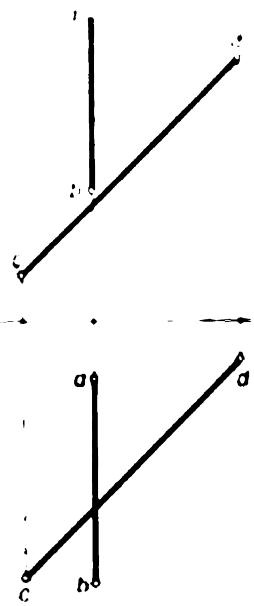



FIG. 1200

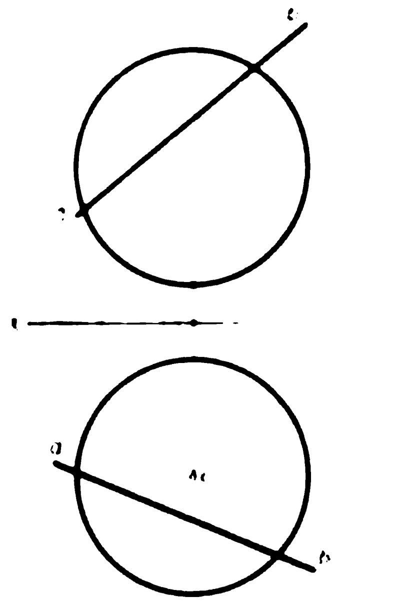



FIG. 1201

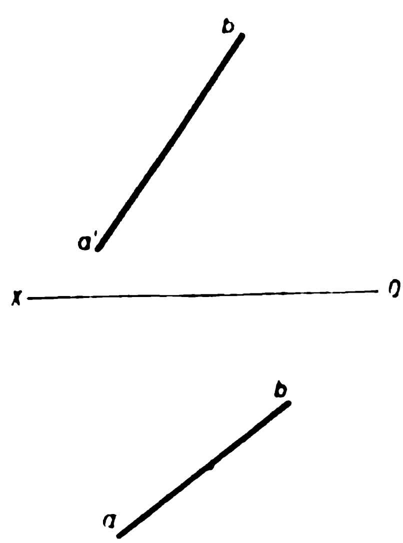



FIG. 1202

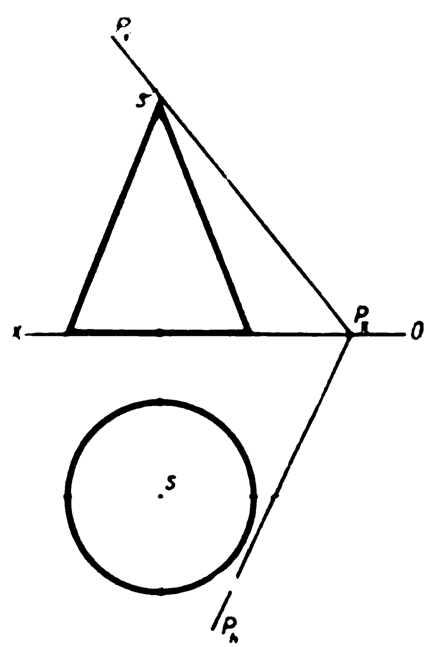



FIG. 1203

633. Through the point $S$ pass a plane $P$ inclined at a given angle $\varphi$ to the $H$ plane and parallel to the line $AB$ (Fig. 1205).

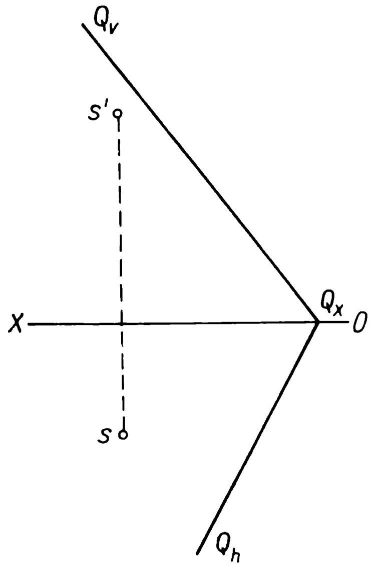



FIG. 1204.

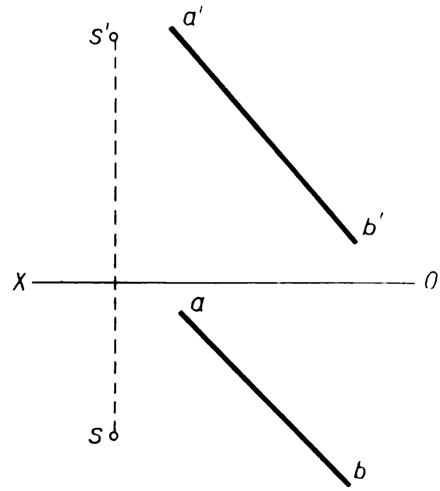



FIG. 1205.

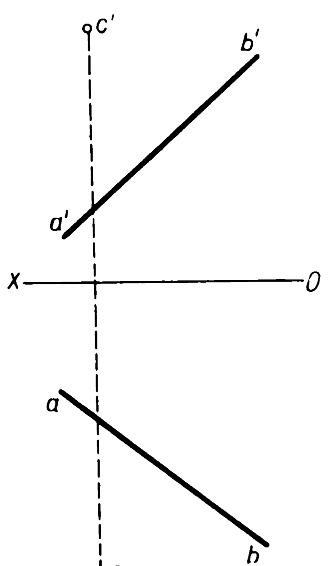



FIG. 1206.

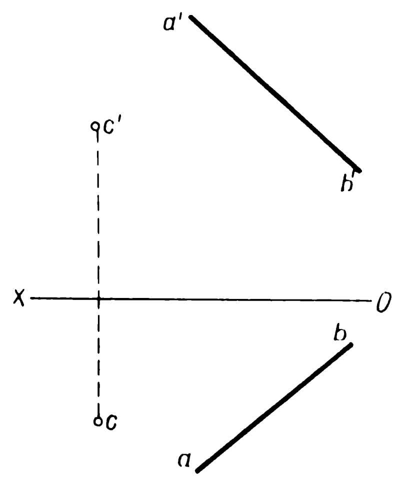



FIG. 1207.

634. Through the point $C$ pass a plane $P$ distant 20 mm from the line $AB$ (Fig. 1206).

635. Through the line $AB$ pass a plane $P$ distant 20 mm from the point $C$ (Fig. 1207).

*Printed in the Union of Soviet Socialist Republics*

Along with problems and typical solutions the book gives brief theoretical information, which is particularly valuable for students working without the guidance of a teacher.

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In addition to the basic course the book sets out the principles of the theory of shadows in orthogonal projections, perspective and axonometry.

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DESCRIPTIVE GEOMETRY by N. Krylov et al. * This textbook is a complete course of descriptive geometry for students of civil engineering higher colleges, covering the following subjects: orthogonal projections, axonometry, linear perspective and projections with numerical markings. The book also sets out the principles of the theory of shadows in orthogonal projections, perspective, axonometry, and contains problems for independent work. Besides students, this book will also be useful for the engineering personnel of design offices, planning civil engineering organizations, mechanical engineering and other establishments.

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